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COORDONATOR:
ANDREI OCTAVIAN DOBRE
REDACTORI PRINCIPALI ȘI SUSȚINĂTORI PERMANENȚI AI REVISTEI
NECULAI STANCIU
MARIN CHIRCIU
ROXANA MIHAELA STANCIU
NELA CICEU

ARTICOLE

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1. Asupra unei probleme din Crux Mathematicorum

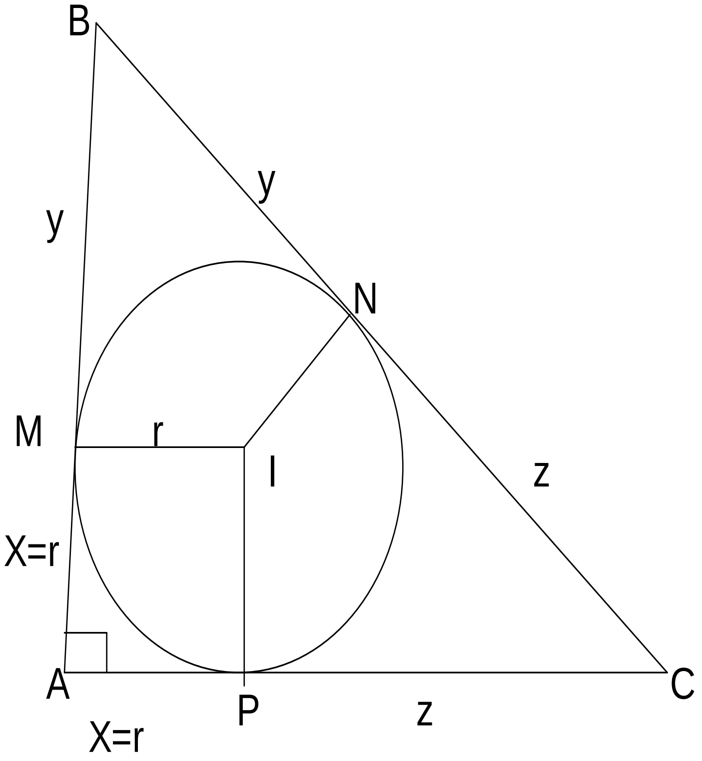
de Neculai Stanciu, Buzău

În Crux Mathematicorum, Vol. 48, Nr. 3, Martie, 2022 – în articolul Pythagorean Triples, pp. 128-132 - *Shawn Godin* propune spre rezolvare următoarea:

Problemă (de *Neculai Stanciu*, Buzău şi *Titu Zvonaru*, Comăneşti**).** Determinaţi lungimile laturilor unui triunghi dreptunghic pentru care $r = 2013$ (unde r este raza cercului înscris) ştiind că acestea sunt numere naturale, prime între ele două câte două.

Scopul acestei note este de a prezenta o soluţie la această problemă şi alte trei probleme similare.

Soluţie.



Avem: $\begin{cases} y+z=a \\ z+r=b \\ y+r=c \end{cases} \Rightarrow 2a+2r=a+b+c \Rightarrow (1) r=\dfrac{b+c-a}{2}$, unde b,c sunt lungimile catetelor iar a este lungimea ipotenuzei. Se ştie că $(2) b=2mn, c=m^2-n^2, a=m^2+n^2$, unde $(m,n)=1$. Din (1) şi (2) avem: (3) $r=n(m-n)$. Dar $r=2013=3\cdot 11\cdot 61$. Deci $n(m-n)=3\cdot 11\cdot 61$.

Rezultă că avem următoarele triunghiuri care verifică ipoteza:

1) $n=1, m=2014, b=4028, c=4056195, a=4055197$;

2) $n=3, m=674, b=4044, c=454267, a=454285$

3) $n=11, m=194, b=4268, c=37515, a=37757$

4) $n=33, m=94, b=6204, c=7747, a=9925$

5) $n=61, m=94, b=11468, c=5115, a=12557$

6) $n=183, m=194, b=71004, c=4147, a=71125$

7) $n=671, m=674, b=904508, c=4035, a=904517$

8) $n=2013, m=2014, b=8108364, c=4027, a=8108365$.

Problema 1. Determinaţi lungimile laturilor unui triunghi dreptunghic ştiind că acestea sunt numere naturale prime între ele, iar raza cercului înscris triunghiului este 2021.

Soluţie. $r=2021=43\cdot 47$. Deci $n(m-n)=2\cdot 3\cdot 337$. Rezultă că avem următoarele triunghiuri care verifică ipoteza

1) $n=1, m=2022, b=4044, c=4088483, a=4088485$;

2) $n=43, m=90, b=7740, c=6251, a=9949$;

3) $n=47, m=90, b=8460, c=5891, a=10309$;

4) $n=2021, m=2022, b=8172924, c=4043, a=8172925$.

Problema 2. Determinaţi lungimile laturilor unui triunghi dreptunghic ştiind că acestea sunt numere naturale, iar raza cercului înscris triunghiului este 2022.

Soluţie. $r=2022=2\cdot 3\cdot 337$. Deci $n(m-n)=2\cdot 3\cdot 337$. Rezultă că avem următoarele triunghiuri care verifică ipoteza

1) $n=1, m=2023, b=4046, c=4092528, a=4092530$;

2) $n=2, m=1013, b=4052, c=1026165, a=1026173$;

3) $n=3, m=677, b=4062, c=458320, a=458338$;

4) $n=6, m=343, b=4116, c=117613, a=117685$;

5) $n=337, m=343, b=231182, c=4080, a=231218$;

6) $n=674, m=677, b=912596, c=4053, a=912605$;

7) $n=1011, m=1013, b=2048286, c=4048, a=2048290$;

8) $n=2022, m=2023, b=8181012, c=4045, a=8181013$.

Problema 3. Determinaţi lungimile laturilor unui triunghi dreptunghic ştiind că acestea sunt numere naturale prime între ele, iar raza cercului înscris triunghiului este 2022.

Soluţie. $r = 2022 = 2 \cdot 3 \cdot 337$. Deci $n(m-n) = 2 \cdot 3 \cdot 337$. Rezultă că avem următoarele triunghiuri care verifică ipoteza lungimile laturilor numere naturale:

1) $n=1, m=2023, b=4046, c=4092528, a=4092530$;

2) $n=2, m=1013, b=4052, c=1026165, a=1026173$;

3) $n=3, m=677, b=4062, c=458320, a=458338$;

4) $n=6, m=343, b=4116, c=117613, a=117685$;

5) $n=337, m=343, b=231182, c=4080, a=231218$;

6) $n=674, m=677, b=912596, c=4053, a=912605$;

7) $n=1011, m=1013, b=2048286, c=4048, a=2048290$;

8) $n=2022, m=2023, b=8181012, c=4045, a=8181013$.

Apoi avem următoarele triunghiuri care verifică ipoteza lungimile laturilor numere naturale prime între ele

I) $n=2, m=1013, b=4052, c=1026165, a=1026173$;

II) $n=6, m=343, b=4116, c=117613, a=117685$;

III) $n=674, m=677, b=912596, c=4053, a=912605$;

IV) $n=2022, m=2023, b=8181012, c=4045, a=8181013$.

2. Inegalitatea lui Panaitopol
Aplicații

Marin Chirciu[1]
Art 3645

În acest articol se prezintă inegalitatea Profesorului Laurențiu Panaitopol , Universitatea București, propusă în Gazeta Matematică 11/1982, C:260.
Este dată soluția problemei și sunt selectate probleme din publicațiile recente care folosesc în rezolvare această inegalitate.

Lema (Panaitopol).
In ΔABC

$$\frac{m_a}{h_a} \le \frac{R}{2r}.$$

Laurențiu Panaitopol, GM 11/1982, C:260.

Demonstrație.
Folosim rezultatul cunoscut în triunghi:
În orice triunghi are loc inegalitatea normală:

$$Max(m_a \sin A, m_b \sin B, m_c \sin C) \le \frac{p}{2}.$$

Într-adevăr:
Fie A' mijlocul laturii (BC), iar M, N proiecțiile lui A' pe AB, AC respectiv.
Fie apoi P, Q simetricele lui A' față de AB, AC respectiv.
Deoarece $m(AMA') = m(ANA') = 90^\circ$ rezultă că penctele A, M, A', N se află pe cercul de diametru $[AA']$.
Obținem:
$MN = \frac{1}{2}PQ = AA'\sin A = m_a \sin A$, de unde $PQ = 2m_a \sin A$.
Dacă B', C' sunt respectiv mijloacele laturilor $[AC], [AB]$ lungimea liniei poligonale $PC'B'Q$ este p și ea majorează lungimea segmentului $[PQ]$.
De aici rezultă că $2m_a \sin A \le p$ și deci $Max(m_a \sin A, m_b \sin B, m_c \sin C) \le \frac{p}{2}$.
Din $2m_a \sin A \le p$ rezultă :
$m_a \sin A \le \frac{p}{2} \Leftrightarrow m_a \cdot \frac{a}{2R} \le \frac{p}{2} \Leftrightarrow m_a \cdot a \le Rp \Leftrightarrow \frac{m_a}{h_a} \le \frac{R}{2r}$.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația 1
In ΔABC

$$m_a \le \frac{Rp}{a}.$$

Laurențiu Panaitopol

Soluție.

[1] *Profesor, Colegiul Național „Zinca Golescu", Pitești*

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ și $h_a = \frac{2S}{a} = \frac{2pr}{a}$ rezultă $m_a \le \frac{Rp}{a}$.

Aplicația 2

In ΔABC

$$m_a \le \frac{(b+c)^2}{16r}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$m_a \overset{Panaitopol}{\le} \frac{Rp}{a} = \frac{4bc}{16r} \overset{AM-GM}{\le} \frac{(b+c)^2}{16r}.$$

Aplicația 3

In ΔABC

$$m_a \le \frac{bc}{4r}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$m_a \overset{Panaitopol}{\le} \frac{Rp}{a} = \frac{bc}{4r}.$$

Aplicația 4

In ΔABC

$$\sum \frac{m_a}{r_a} \le \frac{p^2 + r^2 - 8Rr}{4r^2}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$, obținem: $\frac{m_a}{r_a} \le \frac{h_a \cdot \frac{R}{2r}}{r_a} = \frac{R}{2r} \cdot \frac{h_a}{r_a}$,

de unde $\sum \frac{m_a}{r_a} \le \sum \frac{R}{2r} \cdot \frac{h_a}{r_a} = \frac{R}{2r} \sum \frac{h_a}{r_a} = \frac{R}{2r} \cdot \frac{p^2 + r^2 - 8Rr}{2Rr} = \frac{p^2 + r^2 - 8Rr}{4r^2}$.

Aplicația 5

In ΔABC

$$\sum \frac{m_a^2 + m_b^2}{h_a^2 + h_b^2} \le 6\left(\frac{R}{2r}\right)^3 - 3.$$

George Apostolopoulos, Greece, RMM 2022

Soluție.

Lema

In ΔABC

$$\sum \frac{m_a^2 + m_b^2}{h_a^2 + h_b^2} \le 3\left(\frac{R}{2r}\right)^2.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\sum \frac{m_a^2+m_b^2}{h_a^2+h_b^2} \le \sum \frac{\left(\frac{R}{2r}h_a\right)^2+\left(\frac{R}{2r}h_b\right)^2}{h_a^2+h_b^2} = \sum \left(\frac{R}{2r}\right)^2 = 3\left(\frac{R}{2r}\right)^2.$$

Folosind **Lema** obținem:

$$LHS = \sum \frac{m_a^2+m_b^2}{h_a^2+h_b^2} \overset{Lema}{\le} 3\left(\frac{R}{2r}\right)^2 \overset{Euler}{\le} 6\left(\frac{R}{2r}\right)^3 - 3 = RHS.$$

Aplicația 6

In ΔABC

$$3 \le \sum \frac{m_a^2+m_b^2}{h_a^2+h_b^2} \le 3\left(\frac{R}{2r}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\sum \frac{m_a^2+m_b^2}{h_a^2+h_b^2} \le \sum \frac{\left(\frac{R}{2r}h_a\right)^2+\left(\frac{R}{2r}h_b\right)^2}{h_a^2+h_b^2} = \sum \left(\frac{R}{2r}\right)^2 = 3\left(\frac{R}{2r}\right)^2.$$

Inegalitatea din stânga.

Folosind $m_a \ge h_a$ obținem concluzia.

Aplicația 7

In ΔABC

$$3 \le \sum \frac{m_a^2+m_b^2}{w_a^2+w_b^2} \le 3\left(\frac{R}{2r}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\sum \frac{m_a^2+m_b^2}{w_a^2+w_b^2} \le \sum \frac{m_a^2+m_b^2}{h_a^2+h_b^2} \le \sum \frac{\left(\frac{R}{2r}h_a\right)^2+\left(\frac{R}{2r}h_b\right)^2}{h_a^2+h_b^2} = \sum \left(\frac{R}{2r}\right)^2 = 3\left(\frac{R}{2r}\right)^2.$$

Inegalitatea din stânga.

Folosind $m_a \ge w_a$ obținem concluzia.

Aplicația 8

In ΔABC

$$3\left(\frac{2r}{R}\right)^2 \le \sum \frac{s_a^2+s_b^2}{w_a^2+w_b^2} \le 3\left(\frac{R}{2r}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\sum\frac{s_a^2+s_b^2}{w_a^2+w_b^2}\overset{s_a\le m_a}{\underset{h_a\le w_a}{\le}}\sum\frac{m_a^2+m_b^2}{h_a^2+h_b^2}\le\sum\frac{\left(\frac{R}{2r}h_a\right)^2+\left(\frac{R}{2r}h_b\right)^2}{h_a^2+h_b^2}=\sum\left(\frac{R}{2r}\right)^2=3\left(\frac{R}{2r}\right)^2.$$

Inegalitatea din stânga.

$$\sum\frac{s_a^2+s_b^2}{w_a^2+w_b^2}\overset{s_a\ge h_a}{\underset{w_a\le m_a}{\ge}}\sum\frac{h_a^2+h_b^2}{m_a^2+m_b^2}\overset{Panaitopol}{\ge}\sum\frac{h_a^2+h_b^2}{\left(\frac{R}{2r}h_a\right)^2+\left(\frac{R}{2r}h_b\right)^2}=\sum\frac{1}{\left(\frac{R}{2r}\right)^2}=3\left(\frac{2r}{R}\right)^2.$$

Aplicația 9

In ΔABC

$$\sum aw_am_ar_a\le Rp^3.$$

Soumava Chakraborty, India

Soluție

Folosind inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ și $w_ar_a\le\frac{ap}{2}$ obținem concluzia.

Aplicația 10

In ΔABC

$$\sum a^2w_am_ar_a\le\frac{9R^3p^2}{2}.$$

Marin Chirciu

Soluție

Folosind inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ și $w_ar_a\le\frac{ap}{2}$ obținem :

$$\sum a^2w_am_ar_a\le\sum a^2\cdot\frac{Rp}{a}\cdot\frac{ap}{2}=\frac{Rp^2}{2}\sum a^2\overset{Leibniz}{\le}\frac{Rp^2}{2}\cdot 9R^2=\frac{9R^3p^2}{2}$$

Aplicația 11

In ΔABC

$$\sum a^3w_am_ar_a\le\frac{3R^4p^3}{2r}.$$

Marin Chirciu

Soluție

Folosind inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ și $w_ar_a\le\frac{ap}{2}$ obținem :

$$\sum a^3w_am_ar_a\le\sum a^3\cdot\frac{Rp}{a}\cdot\frac{ap}{2}=\frac{Rp^2}{2}\sum a^3=\frac{Rp^2}{2}\cdot 2p\left(p^2-3r^2-6Rr\right)\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{Rp^2}{2}\cdot 2p\left(4R^2+4Rr+3r^2-3r^2-6Rr\right)=\frac{Rp^2}{2}\cdot 2p\left(4R^2-2Rr\right)\overset{Euler}{\le}\frac{Rp^2}{2}\cdot\frac{3R^3p}{r}=\frac{3R^4p^3}{2r}.$$

Aplicația 12

In ΔABC

$$\sum a^4w_am_ar_a\le\frac{R^4\left(4R+r\right)^2p^2}{3r}.$$

Marin Chirciu

Soluție

Folosind inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ și $w_ar_a\le\frac{ap}{2}$ obținem :

$$\sum a^4 w_a m_a r_a \le \sum a^4 \cdot \frac{Rp}{a} \cdot \frac{ap}{2} = \frac{Rp^2}{2} \sum a^4 = \frac{Rp^2}{2} \cdot 2\left[p^2\left(p^2 - 6r^2 - 8Rr\right) + r^2\left(4R+r\right)^2 \right] \overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le} \frac{Rp^2}{2} \cdot 2\left[\frac{R\left(4R+r\right)^2}{2\left(2R-r\right)} \left(4R^2 + 4Rr + 3r^2 - 6r^2 - 8Rr\right) + r^2\left(4R+r\right)^2 \right] =$$

$$= \frac{Rp^2}{2} \cdot 2\left(4R+r\right)^2 \left[\frac{R\left(4R^2 - 4Rr - 3r^2\right)}{2\left(2R-r\right)} + r^2 \right] = Rp^2 \cdot \left(4R+r\right)^2 \left[\frac{R\left(4R^2 - 4Rr - 3r^2\right)}{2\left(2R-r\right)} + r^2 \right] =$$

$$= Rp^2 \cdot \left(4R+r\right)^2 \cdot \frac{4R^3 - 4R^2 r + Rr^2 - 2r^3}{2\left(2R-r\right)} \overset{Euler}{\le} Rp^2 \cdot \left(4R+r\right)^2 \cdot \frac{R^4}{3r} = \frac{R^5\left(4R+r\right)^2 p^2}{3r}.$$

Aplicația 13

In ΔABC,

$$\frac{2r}{3R^2} \sum \frac{m_b m_c}{\sin A} \le p.$$

Eldeniz Hesenov, Georgia

Soluție

Folosind $m_a \le \dfrac{Rp}{a}$ obținem:

$$\frac{2r}{3R^2} \sum \frac{m_b m_c}{\sin A} \overset{Panaitopol}{\le} \frac{2r}{3R^2} \sum \frac{\frac{Rp}{b} \cdot \frac{Rp}{c}}{\frac{a}{2R}} = \frac{2r}{3R^2} \sum \frac{Rp \cdot Rp \cdot 2R}{abc} = \frac{2r}{3R^2} \sum \frac{Rp \cdot Rp \cdot 2R}{4Rrp} = \frac{1}{3} \sum p.$$

Aplicația 14

In ΔABC

$$\frac{2r^2}{R^2} \sum \frac{m_a}{\sin A} \le p.$$

Marin Chirciu

Soluție

Folosind $m_a \le \dfrac{Rp}{a}$ obținem:

$$LHS = \frac{2r^2}{R^2} \sum \frac{m_a}{\sin A} \overset{Panaitopol}{\le} \frac{2r^2}{R^2} \sum \frac{\frac{Rp}{a}}{\frac{a}{2R}} = \frac{2r^2}{R^2} \sum \frac{Rp \cdot 2R}{a^2} = 4r^2 p \sum \frac{1}{a^2} \overset{Steinig}{\le} 4r^2 p \cdot \frac{1}{4r^2} = p = RHS$$

Aplicația 15

In ΔABC

$$\frac{r}{2R} \sum \frac{m_b + m_c}{\sin A} \le p.$$

Marin Chirciu

Folosind obținem:

$$LHS = \frac{r}{2R} \sum \frac{m_b + m_c}{\sin A} \overset{Panaitopol}{\le} \frac{r}{2R} \sum \frac{\frac{Rp}{b} + \frac{Rp}{c}}{\frac{a}{2R}} = \frac{r}{2R} \cdot Rp \cdot 2R \sum \frac{b+c}{abc} = Rrp \sum \frac{b+c}{4Rrp} =$$

$$= \frac{Rrp}{4Rrp} \cdot 2\sum a = \frac{1}{4} \cdot 2 \cdot 2p = p = RHS.$$

Aplicația 16

In ΔABC

$$\sum\left(\frac{m_b+m_c}{\sin A}\right)^2\le\left(\frac{3R^2}{r}\right)^2.$$

Marin Chirciu

Soluție

Folosind $m_a\le\frac{Rp}{a}$ obținem:

$$LHS=\sum\left(\frac{m_b+m_c}{\sin A}\right)^2\overset{Panaitopol}{\le}\sum\left(\frac{\frac{Rp}{b}+\frac{Rp}{c}}{\frac{a}{2R}}\right)^2=R^2p^2\cdot 4R^2\sum\left(\frac{b+c}{abc}\right)^2=$$

$$=4R^4p^2\sum\left(\frac{b+c}{abc}\right)^2=4R^4p^2\cdot\frac{1}{16R^2r^2p^2}\sum(b+c)^2=\frac{R^2}{4r^2}\cdot 2\left(3p^2-r^2-4Rr\right)\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{R^2}{2r^2}\cdot\left(3\left(4R^2+4Rr+3r^2\right)-r^2-4Rr\right)=\frac{R^2}{2r^2}\cdot\left(12R^2+8Rr+8r^2\right)=$$

$$=\frac{2R^2}{r^2}\cdot\left(3R^2+2Rr+2r^2\right)\overset{Euler}{\le}\frac{2R^2}{r^2}\cdot\frac{9R^2}{2}=\frac{9R^4}{r^2}.$$

Aplicația 17

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\sum\left(\frac{m_a}{AI\cdot AI_a}\right)^2\le\frac{3}{16}\sum\frac{1}{h_a}\sum\frac{1}{r_a}.$$

Eldeniz Hesenov, Georgia, SP.481,RMM-33,2022

Soluție.

Folosind $AI_a=\frac{bc}{AI}$ și $\sum\frac{1}{h_a}=\frac{1}{r}=\sum\frac{1}{r_a}$ obținem:

$$Ms=\sum\left(\frac{m_a}{AI\cdot AI_a}\right)^2=\sum\left(\frac{m_a}{bc}\right)^2\overset{Panaitopol}{\le}\sum\left(\frac{\frac{Rp}{a}}{bc}\right)^2=\left(\sum\frac{Rp}{abc}\right)^2=\left(\sum\frac{Rp}{4Rrp}\right)^2=$$

$$=\left(\sum\frac{1}{4r}\right)^2\frac{3}{16r^2}=\frac{3}{16}\sum\frac{1}{h_a}\sum\frac{1}{r_a}=Md.$$

Aplicația 18

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{2R}\le\sum\frac{m_a}{AI\cdot AI_a}\le\frac{3}{4r}.$$

Marin Chirciu

Soluție.

Folosind $AI_a=\frac{bc}{AI}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{m_a}{AI \cdot AI_a} = \sum \frac{m_a}{bc} \overset{Panaitopol}{\leq} \sum \frac{\frac{Rp}{a}}{bc} = \sum \frac{Rp}{abc} = \sum \frac{Rp}{4Rrp} = \frac{3}{4r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

<u>Inegalitatea din stânga.</u>

$$\sum \frac{m_a}{AI \cdot AI_a} = \sum \frac{m_a}{bc} \overset{Tereshin}{\geq} \sum \frac{\frac{b^2+c^2}{4R}}{bc} \overset{AGM}{\geq} \frac{1}{4R} \sum \frac{2bc}{bc} = \frac{1}{4R} \cdot 6 = \frac{3}{2R}.$$

<u>Aplicaţia 19</u>

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{2R} \leq \sum \frac{s_a}{AI \cdot AI_a} \leq \frac{3}{4r}.$$

Marin Chirciu

<u>Soluţie.</u>

Folosind $AI_a = \frac{bc}{AI}$ obţinem:

<u>Inegalitatea din dreapta.</u>

$$\sum \frac{s_a}{AI \cdot AI_a} = \sum \frac{s_a}{bc} \overset{s_a \leq m_a}{\leq} \sum \frac{m_a}{bc} \overset{Panaitopol}{\leq} \sum \frac{\frac{Rp}{a}}{bc} = \sum \frac{Rp}{abc} = \sum \frac{Rp}{4Rrp} = \frac{3}{4r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

<u>Inegalitatea din stânga.</u>

$$\sum \frac{s_a}{AI \cdot AI_a} = \sum \frac{s_a}{bc} \overset{s_a \geq h_a}{\geq} \sum \frac{h_a}{bc} = \sum \frac{\frac{2F}{a}}{bc} = \sum \frac{2F}{abc} = \sum \frac{2F}{4RF} = \frac{3}{2R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

<u>Aplicaţia 20</u>

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{2R} \leq \sum \frac{w_a}{AI \cdot AI_a} \leq \frac{3}{4r}.$$

Marin Chirciu

<u>Soluţie.</u>

Folosind $AI_a = \frac{bc}{AI}$ obţinem:

<u>Inegalitatea din dreapta.</u>

$$\sum \frac{w_a}{AI \cdot AI_a} = \sum \frac{w_a}{bc} \overset{w_a \leq m_a}{\leq} \sum \frac{m_a}{bc} \overset{Panaitopol}{\leq} \sum \frac{\frac{Rp}{a}}{bc} = \sum \frac{Rp}{abc} = \sum \frac{Rp}{4Rrp} = \frac{3}{4r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

<u>Inegalitatea din stânga.</u>

$$\sum \frac{w_a}{AI \cdot AI_a} = \sum \frac{w_a}{bc} \overset{w_a \geq h_a}{\geq} \sum \frac{h_a}{bc} = \sum \frac{\frac{2F}{a}}{bc} = \sum \frac{2F}{abc} = \sum \frac{2F}{4RF} = \frac{3}{2R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

<u>Aplicaţia 21</u>

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{4r} \le \sum \frac{r_a}{AI \cdot AI_a} \le \frac{1}{r} - \frac{1}{2R}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{r_a}{bc} = \frac{2R-r}{2Rr}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{r_a}{AI \cdot AI_a} = \sum \frac{r_a}{bc} = \frac{2R-r}{2Rr} = \frac{1}{r} - \frac{1}{2R}.$$

Inegalitatea din stânga.

$$\sum \frac{r_a}{AI \cdot AI_a} = \sum \frac{r_a}{bc} = \frac{2R-r}{2Rr} \ge \frac{3}{4r}.$$

Aplicația 22

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{2R} \le \sum \frac{h_a}{AI \cdot AI_a} \le \frac{3}{4r}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{h_a}{bc} = \frac{3}{2R}$ obținem:

Inegalitatea din stânga.

$$\sum \frac{h_a}{AI \cdot AI_a} = \sum \frac{h_a}{bc} = \frac{3}{2R}.$$

Inegalitatea din dreapta.

$$\sum \frac{h_a}{AI \cdot AI_a} = \sum \frac{h_a}{bc} = \frac{3}{2R} \overset{Euler}{\le} \frac{3}{4r}.$$

Aplicația 23

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{1}{4R^2} \le \sum \frac{\cos B \cos C}{AI \cdot AI_a} \le \frac{1}{16r^2}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{\cos B \cos C}{bc} = \frac{1}{4R^2}$ obținem:

Inegalitatea din stânga.

$$\sum \frac{\cos B \cos C}{AI \cdot AI_a} = \sum \frac{\cos B \cos C}{bc} = \frac{1}{4R^2}.$$

Inegalitatea din dreapta.

$$\sum \frac{\cos B \cos C}{AI \cdot AI_a} = \sum \frac{\cos B \cos C}{bc} = \frac{1}{4R^2} \overset{Euler}{\le} \frac{1}{16r^2}.$$

Aplicația 24

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{1}{6Rr} \le \sum \frac{\tan^2 \frac{A}{2}}{AI \cdot AI_a} \le \frac{2R^2 - 2Rr - r^2}{2Rr^2(4R+r)}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{1}{bc}\tan^2\frac{A}{2} = \frac{8R^2+2Rr-p^2}{2p^2Rr}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{\tan^2\frac{A}{2}}{AI\cdot AI_a} = \sum \frac{\tan^2\frac{A}{2}}{bc} = \frac{8R^2+2Rr-p^2}{2p^2Rr} = \frac{1}{2Rr}\left(\frac{2R(4R+r)}{p^2}-1\right) \overset{Gerretsen}{\leq} \frac{1}{2Rr}\left(\frac{2R(4R+r)}{\frac{r(4R+r)^2}{R+r}}-1\right) =$$

$$= \frac{1}{2Rr}\left(\frac{2R(R+r)}{r(4R+r)}-1\right) = \frac{1}{2Rr}\left(\frac{2R^2+2Rr-4Rr-r^2}{r(4R+r)}\right) = \frac{1}{2Rr}\cdot\frac{2R^2-2Rr-r^2}{r(4R+r)} =$$

$$= \frac{2R^2-2Rr-r^2}{2Rr^2(4R+r)}.$$

Inegalitatea din stânga.

$$\sum \frac{\tan^2\frac{A}{2}}{AI\cdot AI_a} = \sum \frac{\tan^2\frac{A}{2}}{bc} = \frac{8R^2+2Rr-p^2}{2p^2Rr} = \frac{1}{2Rr}\left(\frac{2R(4R+r)}{p^2}-1\right) \overset{Gerretsen}{\geq} \frac{1}{2Rr}\left(\frac{2R(4R+r)}{\frac{R(4R+r)^2}{2(2R-r)}}-1\right) =$$

$$= \frac{1}{2Rr}\left(\frac{4(2R-r)}{(4R+r)}-1\right) = \frac{8R-4r-4R-r}{2Rr(4R+r)} = \frac{4R-5r}{2Rr(4R+r)} \overset{Euler}{\geq} \frac{1}{2Rr}\cdot\frac{1}{3} = \frac{1}{6Rr}.$$

Aplicația 25

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{4r^2} \leq \sum \frac{\cot^2\frac{A}{2}}{AI\cdot AI_a} \leq \frac{1}{r^2} - \frac{1}{2Rr}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{1}{bc}\cot^2\frac{A}{2} = \frac{2R-r}{2Rr^2}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{\cot^2\frac{A}{2}}{AI\cdot AI_a} = \sum \frac{\cot^2\frac{A}{2}}{bc} = \frac{2R-r}{2Rr^2} = \frac{1}{r^2} - \frac{1}{2Rr}.$$

Inegalitatea din stânga.

$$\sum \frac{\cot^2\frac{A}{2}}{AI\cdot AI_a} = \sum \frac{\cot^2\frac{A}{2}}{bc} = \frac{2R-r}{2Rr^2} = \frac{1}{r^2} - \frac{1}{2Rr} \overset{Euler}{\geq} \frac{3}{4r^2}.$$

Aplicația 26

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{9r}{2Rp^2} \leq \sum \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{AI\cdot AI_a} \leq \frac{9}{4p^2}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{1}{bc}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{4R+r}{2Rp^2}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{AI\cdot AI_a} = \sum \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{bc} = \frac{4R+r}{2Rp^2} \overset{Euler}{\leq} \frac{4R+\frac{R}{2}}{2Rp^2} = \frac{\frac{9R}{2}}{2Rp^2} = \frac{9}{4p^2}.$$

Inegalitatea din stânga.

$$\sum \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{AI\cdot AI_a} = \sum \frac{\tan\frac{B}{2}\tan\frac{C}{2}}{bc} = \frac{4R+r}{2Rp^2} \overset{Euler}{\geq} \frac{4\cdot 2r+r}{2Rp^2} = \frac{9r}{2Rp^2}.$$

Aplicația 27

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{3}{4r^2} \leq \sum \frac{\cot\frac{B}{2}\cot\frac{C}{2}}{AI\cdot AI_a} \leq \frac{1}{r^2} - \frac{1}{2Rr}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{1}{bc}\cot\frac{B}{2}\cot\frac{C}{2} = \frac{2R-r}{2Rr^2}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{\cot\frac{B}{2}\cot\frac{C}{2}}{AI\cdot AI_a} = \sum \frac{\cot\frac{B}{2}\cot\frac{C}{2}}{bc} = \frac{2R-r}{2Rr^2} = \frac{1}{r^2} - \frac{1}{2Rr}.$$

Inegalitatea din stânga.

$$\sum \frac{\cot\frac{B}{2}\cot\frac{C}{2}}{AI\cdot AI_a} = \sum \frac{\cot\frac{B}{2}\cot\frac{C}{2}}{bc} = \frac{2R-r}{2Rr^2} = \frac{1}{r^2} - \frac{1}{2Rr} \overset{Euler}{\geq} \frac{3}{4r^2}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația 28

In ΔABC, I -incenter, I_a, I_b, I_c -excenters

$$\frac{4}{R^2} \leq \sum \frac{\csc^2\frac{A}{2}}{AI\cdot AI_a} \leq \frac{1}{r^2}.$$

Marin Chirciu

Soluție.

Folosind $AI_a = \frac{bc}{AI}$ și $\sum \frac{\csc^2\frac{A}{2}}{bc} = \frac{1}{r^2}$ obținem:

Inegalitatea din dreapta.

$$\sum \frac{\csc^2\frac{A}{2}}{AI\cdot AI_a} = \sum \frac{\csc^2\frac{A}{2}}{bc} = \frac{1}{r^2}.$$

Inegalitatea din stânga.

$$\sum\frac{\csc^2\frac{A}{2}}{AI\cdot AI_a}=\sum\frac{\csc^2\frac{A}{2}}{bc}=\frac{1}{r^2}\overset{Euler}{\geq}\frac{4}{R^2}.$$

Aplicația 29

In ΔABC

$$\frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1}\leq\sum\frac{m_b+m_c}{a}\leq\frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^{2}.$$

Konstantinos Geronikolas, Greece, Mathematical Inequalites, 2/2022.

Soluție.

Inegalitatea din dreapta.

Lema

In ΔABC

$$\sum\frac{m_b+m_c}{a}\leq\frac{p}{r}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $m_a\leq\frac{R}{2r}h_a$ obținem:

$$\sum\frac{m_b+m_c}{a}\leq\frac{R}{2r}\sum\frac{h_b+h_c}{a}=\frac{R}{2r}\cdot\frac{2p}{R}=\frac{p}{r}.$$

Folosind **Lema** rămâne să arătăm că:

$\frac{p}{r}\leq\frac{3\sqrt{3}}{8}\left(\frac{R}{r}\right)^2$, care rezultă din inegalitatea lui Mitrinovic $p\leq\frac{3\sqrt{3}R}{2}$.

Este sufficient să arătăm că:

$$\frac{\frac{3\sqrt{3}R}{2}}{r}\leq\frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^2\Leftrightarrow\frac{R}{r}\leq\frac{1}{2}\left(\frac{R}{r}\right)^2\Leftrightarrow 1\leq\frac{1}{2}\cdot\frac{R}{r}\Leftrightarrow R\geq 2r,\text{(Euler)}.$$

Inegalitatea din stânga.

Lema

In ΔABC

$$\sum\frac{m_b+m_c}{a}\geq 3\sqrt[3]{\frac{2p}{R}}.$$

Demonstrație.

$$\sum\frac{m_b+m_c}{a}\geq 3\sqrt[3]{\prod\frac{m_b+m_c}{a}}=3\sqrt[3]{\frac{\prod(m_b+m_c)}{abc}}\geq 3\sqrt[3]{\frac{\prod 2\left(\sqrt{m_bm_c}\right)}{abc}}=3\sqrt[3]{\frac{8m_am_bm_c}{abc}}\geq 3\sqrt[3]{\frac{8r_ar_br_c}{abc}}=$$

$$=3\sqrt[3]{\frac{8\cdot rp^2}{4Rrp}}=3\sqrt[3]{\frac{2p}{R}}.$$

Folosind **Lema** rămâne să arătăm că:

$3\sqrt[3]{\frac{2p}{R}}\geq\frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1}\Leftrightarrow 3\sqrt[3]{\frac{2p}{R}}\geq\frac{18}{\sqrt{3}}\cdot\frac{r}{R}\Leftrightarrow\sqrt[3]{\frac{2p}{R}}\geq 2\sqrt{3}\cdot\frac{r}{R}\Leftrightarrow\frac{2p}{R}\geq 24\sqrt{3}\cdot\frac{r^3}{R^3}$, care rezultă din inegalitatea lui Mitrinovic $p\geq 3\sqrt{3}r$.

Este sufficient să arătăm că:

$$\frac{2\cdot 3\sqrt{3}r}{R}\geq 24\sqrt{3}\cdot\frac{r^3}{R^3}\Leftrightarrow\frac{r}{R}\geq 4\cdot\frac{r^3}{R^3}\Leftrightarrow 1\geq 4\cdot\frac{r^2}{R^2}\Leftrightarrow R\geq 2r,\text{(Euler)}.$$

Aplicația 30

In ΔABC

$$\frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \le \sum \frac{w_b + w_c}{a} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^2 .$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Lema

In ΔABC

$$\sum \frac{w_b + w_c}{a} \le \frac{p}{r} .$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{R}{2r} h_a$ obținem:

$$\sum \frac{w_b + w_c}{a} \overset{w_a \le m_a}{\le} \sum \frac{m_b + m_c}{a} \overset{Panaitopol}{\le} \frac{R}{2r} \sum \frac{h_b + h_c}{a} = \frac{R}{2r} \cdot \frac{2p}{R} = \frac{p}{r} .$$

Folosind **Lema** rămâne să arătăm că:

$\frac{p}{r} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^2$, care rezultă din inegalitatea lui Mitrinovic $p \le \frac{3\sqrt{3}R}{2}$.

Este sufficient să arătăm că:

$$\frac{\frac{3\sqrt{3}R}{2}}{r} \le \frac{3\sqrt{3}}{8}\left(\frac{R}{r}\right)^2 \Leftrightarrow \frac{R}{r} \le \frac{1}{2}\left(\frac{R}{r}\right)^2 \Leftrightarrow 1 \le \frac{1}{2} \cdot \frac{R}{r} \Leftrightarrow R \ge 2r \text{ ,(Euler).}$$

Inegalitatea din stânga.

Lema

In ΔABC

$$\sum \frac{w_b + w_c}{a} \ge 9\sqrt[3]{\frac{r^2}{4Rp}} .$$

Demonstrație.

$$\sum \frac{w_b + w_c}{a} \ge 3\sqrt[3]{\prod \frac{w_b + w_c}{a}} = 3\sqrt[3]{\frac{\prod (w_b + w_c)}{abc}} \ge 3\sqrt[3]{\frac{\prod 2\left(\sqrt{w_b w_c}\right)}{abc}} = 3\sqrt[3]{\frac{8 w_a w_b w_c}{abc}} \overset{w_a w_b w_c \ge 27 r^3}{\ge}$$

$$\overset{w_a w_b w_c \ge 27 r^3}{\ge} 3\sqrt[3]{\frac{8 \cdot 27 r^3}{4Rrp}} = 9\sqrt[3]{\frac{2r^2}{Rp}} .$$

Folosind **Lema** rămâne să arătăm că:

$9\sqrt[3]{\frac{2r^2}{Rp}} \ge \frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \Leftrightarrow 9\sqrt[3]{\frac{2r^2}{Rp}} \ge \frac{18}{\sqrt{3}} \cdot \frac{r}{R} \Leftrightarrow \sqrt[3]{\frac{2r^2}{Rp}} \ge \frac{2}{\sqrt{3}} \cdot \frac{r}{R} \Leftrightarrow \frac{2r^2}{Rp} \ge \frac{8}{3\sqrt{3}} \cdot \frac{r^3}{R^3}$, care rezultă din inegalitatea lui Mitrinovic $p \le \frac{3\sqrt{3}R}{2}$.

Este sufficient să arătăm că:

$$\frac{2r^2}{R \cdot \frac{3\sqrt{3}R}{2}} \ge \frac{8}{3\sqrt{3}} \cdot \frac{r^3}{R^3} \Leftrightarrow 1 \ge 2 \cdot \frac{r}{R} \Leftrightarrow R \ge 2r \text{ ,(Euler).}$$

Aplicația 31

In ΔABC

$$\frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \le \sum \frac{s_b+s_c}{a} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^{2}.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Lema

In ΔABC

$$\sum \frac{s_b+s_c}{a} \le \frac{p}{r}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{R}{2r}h_a$ obținem:

$$\sum \frac{s_b+s_c}{a} \overset{s_a\le m_a}{\le} \sum \frac{m_b+m_c}{a} \overset{Panaitopol}{\le} \frac{R}{2r}\sum \frac{h_b+h_c}{a} = \frac{R}{2r}\cdot\frac{2p}{R} = \frac{p}{r}.$$

Folosind **Lema** rămâne să arătăm că:

$\frac{p}{r} \le \frac{3\sqrt{3}}{8}\left(\frac{R}{r}\right)^2$, care rezultă din inegalitatea lui Mitrinovic $p \le \frac{3\sqrt{3}R}{2}$.

Este sufficient să arătăm că:

$$\frac{\frac{3\sqrt{3}R}{2}}{r} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^2 \Leftrightarrow \frac{R}{r} \le \frac{1}{2}\left(\frac{R}{r}\right)^2 \Leftrightarrow 1 \le \frac{1}{2}\cdot\frac{R}{r} \Leftrightarrow R \ge 2r, \text{(Euler)}.$$

Inegalitatea din stânga.

Lema

In ΔABC

$$\sum \frac{s_b+s_c}{a} \ge 3\sqrt[3]{\frac{4rp}{R^2}}.$$

Demonstrație.

$$\sum \frac{s_b+s_c}{a} \ge 3\sqrt[3]{\prod \frac{s_b+s_c}{a}} = 3\sqrt[3]{\frac{\prod (s_b+s_c)}{abc}} \ge 3\sqrt[3]{\frac{\prod 2\left(\sqrt{s_b s_c}\right)}{abc}} = 3\sqrt[3]{\frac{8 s_a s_b s_c}{abc}} \overset{s_a s_b s_c \ge \frac{2r^2p^2}{R}}{\ge}$$

$$\overset{s_a s_b s_c \ge \frac{2r^2p^2}{R}}{\ge} 3\sqrt[3]{\frac{8\cdot\frac{2r^2p^2}{R}}{4Rrp}} = 3\sqrt[3]{\frac{4rp}{R^2}}.$$

Folosind **Lema** rămâne să arătăm că:

$3\sqrt[3]{\frac{4rp}{R^2}} \ge \frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \Leftrightarrow 3\sqrt[3]{\frac{4rp}{R^2}} \ge \frac{18}{\sqrt{3}}\cdot\frac{r}{R} \Leftrightarrow \sqrt[3]{\frac{4rp}{R^2}} \ge \frac{6}{\sqrt{3}}\cdot\frac{r}{R} \Leftrightarrow \frac{4rp}{R^2} \ge \frac{216}{3\sqrt{3}}\cdot\frac{r^3}{R^3}$, care rezultă din inegalitatea lui Mitrinovic $p \ge 3\sqrt{3}r$.

Este sufficient să arătăm că:

$$\frac{4r\cdot 3\sqrt{3}r}{R^2} \ge \frac{216}{3\sqrt{3}}\cdot\frac{r^3}{R^3} \Leftrightarrow 1 \ge 2\cdot\frac{r}{R} \Leftrightarrow R \ge 2r, \text{(Euler)}.$$

Aplicația 32

In ΔABC

$$\frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \le \sum \frac{h_b+h_c}{a} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^{2}.$$

Marin Chirciu

Soluție.
Lema
In ΔABC

$$\sum \frac{h_b+h_c}{a} = \frac{2p}{R}.$$

Demonstrație.
Folosind $h_a = \frac{2S}{a}$ obținem:

$$\sum \frac{h_b+h_c}{a} = \sum \frac{\frac{2S}{b}+\frac{2S}{c}}{a} = 2S\sum \frac{\frac{1}{b}+\frac{1}{c}}{a} = \frac{2S}{abc}\sum(b+c) = \frac{2S}{4RS}\cdot 2\sum a = \frac{1}{R}\cdot 2p = \frac{2p}{R}.$$

Inegalitatea din dreapta.
Folosind **Lema** rămâne să arătăm că:

$\frac{2p}{R} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^{2}$, care rezultă din inegalitatea lui Mitrinovic $p \le \frac{3\sqrt{3}R}{2}$.

Este sufficient să arătăm că:

$$\frac{2\cdot\frac{3\sqrt{3}R}{2}}{R} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^{2} \Leftrightarrow 1 \le \frac{1}{4}\left(\frac{R}{r}\right)^{2} \Leftrightarrow 1 \le \frac{1}{2}\cdot\frac{R}{r} \Leftrightarrow R \ge 2r, \text{(Euler)}.$$

Inegalitatea din stânga.
Folosind **Lema** rămâne să arătăm că:

$\frac{2p}{R} \ge \frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \Leftrightarrow \frac{2p}{R} \ge \frac{18}{\sqrt{3}}\cdot\frac{r}{R}$, care rezultă din inegalitatea lui Mitrinovic $p \ge 3\sqrt{3}r$.

Este sufficient să arătăm că: $\frac{2\cdot 3\sqrt{3}r}{R} \ge \frac{18}{\sqrt{3}}\cdot\frac{r}{R} \Leftrightarrow \sqrt{3} \ge \frac{3}{\sqrt{3}}$, evident cu egalitate.

Aplicația 33
In ΔABC

$$\frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \le \sum \frac{r_b+r_c}{a} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^{2}.$$

Marin Chirciu

Soluție.
Lema
In ΔABC

$$\sum \frac{r_b+r_c}{a} = \frac{p}{r}.$$

Demonstrație.
Folosind $r_a = \frac{S}{p-a}$ obținem:

$$\sum \frac{r_b+r_c}{a} = \sum \frac{\frac{S}{p-b}+\frac{S}{p-c}}{a} = S\sum \frac{\frac{1}{p-b}+\frac{1}{p-c}}{a} = S\sum \frac{1}{(p-b)(p-c)} = pr\cdot\frac{1}{r^2} = \frac{p}{r}.$$

Inegalitatea din dreapta.

Folosind **Lema** rămâne să arătăm că:

$\frac{p}{r} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^2$, care rezultă din inegalitatea lui Mitrinovic $p \le \frac{3\sqrt{3}R}{2}$.

Este sufficient să arătăm că:

$$\frac{\frac{3\sqrt{3}R}{2}}{r} \le \frac{3\sqrt{3}}{4}\left(\frac{R}{r}\right)^2 \Leftrightarrow \frac{R}{2r} \le \frac{1}{4}\left(\frac{R}{r}\right)^2 \Leftrightarrow 1 \le \frac{1}{2}\cdot\frac{R}{r} \Leftrightarrow R \ge 2r \text{,(Euler).}$$

<u>Inegalitatea din stânga.</u>

Folosind **Lema** rămâne să arătăm că:

$\frac{p}{r} \ge \frac{18}{\sqrt{3}}\left(\frac{R}{r}\right)^{-1} \Leftrightarrow \frac{p}{r} \ge \frac{18}{\sqrt{3}}\cdot\frac{r}{R}$, care rezultă din inegalitatea lui Mitrinovic $p \ge 3\sqrt{3}r$.

Este sufficient să arătăm că:

$$\frac{3\sqrt{3}r}{r} \ge \frac{18}{\sqrt{3}}\cdot\frac{r}{R} \Leftrightarrow 1 \ge 2\cdot\frac{r}{R} \Leftrightarrow R \ge 2r \text{,(Euler).}$$

<u>Aplicația 34</u>

In ΔABC

$$\sum\frac{h_b+h_c}{a} \le \sum\frac{r_b+r_c}{a}.$$

Marin Chirciu

<u>Soluție.</u>

<u>Lema</u>

In ΔABC

$$\sum\frac{h_b+h_c}{a} = \frac{2p}{R}.$$

<u>Demonstrație.</u>

Folosind $h_a = \frac{2S}{a}$ obținem:

$$\sum\frac{h_b+h_c}{a} = \sum\frac{\frac{2S}{b}+\frac{2S}{c}}{a} = 2S\sum\frac{\frac{1}{b}+\frac{1}{c}}{a} = \frac{2S}{abc}\sum(b+c) = \frac{2S}{4RS}\cdot 2\sum a = \frac{1}{R}\cdot 2p = \frac{2p}{R}.$$

<u>Lema</u>

In ΔABC

$$\sum\frac{r_b+r_c}{a} = \frac{p}{r}.$$

<u>Demonstrație.</u>

Folosind $r_a = \frac{S}{p-a}$ obținem:

$$\sum\frac{r_b+r_c}{a} = \sum\frac{\frac{S}{p-b}+\frac{S}{p-c}}{a} = S\sum\frac{\frac{1}{p-b}+\frac{1}{p-c}}{a} = S\sum\frac{1}{(p-b)(p-c)} = pr\cdot\frac{1}{r^2} = \frac{p}{r}.$$

Folosind **Lemele** de mai sus avem sumele: $\sum\frac{h_b+h_c}{a} = \frac{2p}{R}$ și $\sum\frac{r_b+r_c}{a} = \frac{p}{r}$.

Inegalitatea se scrie: $\frac{2p}{R} \le \frac{p}{r} \Leftrightarrow R \ge 2r$,(Euler).

<u>Aplicația 35</u>

In ΔABC

$$\frac{9}{8}\left(\frac{R}{2r}\right)^{-2} \le \sum \frac{w_a^2}{b^2+c^2} \le \frac{9}{8}\left(\frac{R}{2r}\right).$$

George Apostolopoulos, Greece

Soluție.

Inegalitatea din dreapta.

Folosind $w_a \le \sqrt{p(p-a)}$ obținem:

$$\sum \frac{w_a^2}{b^2+c^2} \le \sum \frac{p(p-a)}{2bc} = \frac{4R+r}{4R} \overset{(1)}{\le} \frac{9}{8}\left(\frac{R}{2r}\right),$$

unde (1) $\Leftrightarrow \frac{4R+r}{4R} \le \frac{9}{8}\left(\frac{R}{2r}\right) \Leftrightarrow 9R^2 - 16Rr - 4r^2 \ge 0 \Leftrightarrow (R-2r)(9R-2r) \ge 0$, evident din inegalitatea lui Euler $R \ge 2r$.

Inegalitatea din stânga.

Folosind inegalitatea lui Bergstrőm obținem:

$$\sum \frac{w_a^2}{b^2+c^2} \ge \frac{\left(\sum w_a\right)^2}{\sum\left(b^2+c^2\right)} \ge \frac{(9r)^2}{2\sum a^2} \overset{Leibniz}{\ge} \frac{81r^2}{2\cdot 9R^2} = \frac{9r^2}{2R^2} = \frac{9}{8}\left(\frac{R}{2r}\right)^{-2}.$$

Aplicația 36

In ΔABC

$$\frac{9r^2}{2R^2} \le \sum \frac{w_a^2}{b^2+c^2} \le 1 + \frac{r}{4R}.$$

Marin Chirciu

Inegalitatea din dreapta.

Folosind $w_a \le \sqrt{p(p-a)}$ obținem: $\sum \frac{w_a^2}{b^2+c^2} \le \sum \frac{p(p-a)}{2bc} = \frac{4R+r}{4R} = 1 + \frac{r}{4R}$,

Inegalitatea din stânga.

Folosind inegalitatea lui Bergstrőm obținem:

$$\sum \frac{w_a^2}{b^2+c^2} \ge \frac{\left(\sum w_a\right)^2}{\sum\left(b^2+c^2\right)} \ge \frac{(9r)^2}{2\sum a^2} \overset{Leibniz}{\ge} \frac{81r^2}{2\cdot 9R^2} = \frac{9r^2}{2R^2}.$$

Aplicația 37

In ΔABC

$$\frac{9r^2}{2R^2} \le \sum \frac{m_a^2}{b^2+c^2} \le \frac{1}{2}\left(\frac{3R}{4r}\right)^2.$$

Marin Chirciu

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$\sum \frac{m_a^2}{b^2+c^2} \le \sum \frac{\left(\frac{Rp}{a}\right)^2}{2bc} = \frac{R^2p^2}{2abc}\sum \frac{1}{a} \le \frac{R^2p^2}{2\cdot 4Rrp}\cdot\frac{p}{3Rr} = \frac{9R^2}{32r^2} = \frac{1}{2}\left(\frac{3R}{4r}\right)^2,$$

Inegalitatea din stânga.

Folosind inegalitatea lui Bergstrőm obținem:

$$\sum\frac{m_a^2}{b^2+c^2}\ge\frac{\left(\sum m_a\right)^2}{\sum\left(b^2+c^2\right)}\ge\frac{(9r)^2}{2\sum a^2}\overset{Leibniz}{\ge}\frac{81r^2}{2\cdot 9R^2}=\frac{9r^2}{2R^2}.$$

Aplicația 38

In ΔABC

$$\frac{9r^2}{2R^2}\le\sum\frac{s_a^2}{b^2+c^2}\le\frac{1}{2}\left(\frac{3R}{4r}\right)^2.$$

Marin Chirciu

Inegalitatea din dreapta.

Folosind $s_a\le m_a$ și inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ obținem:

$$\sum\frac{s_a^2}{b^2+c^2}\le\sum\frac{m_a^2}{b^2+c^2}\le\sum\frac{\left(\frac{Rp}{a}\right)^2}{2bc}=\frac{R^2p^2}{2abc}\sum\frac{1}{a}\le\frac{R^2p^2}{2\cdot 4Rrp}\cdot\frac{p}{3Rr}=\frac{9R^2}{32r^2}=\frac{1}{2}\left(\frac{3R}{4r}\right)^2,$$

Inegalitatea din stânga.

Folosind $s_a\ge h_a$ și inegalitatea lui Bergstrőm obținem:

$$\sum\frac{s_a^2}{b^2+c^2}\ge\sum\frac{h_a^2}{b^2+c^2}\ge\frac{(9r)^2}{2\sum a^2}\overset{Leibniz}{\ge}\frac{81r^2}{2\cdot 9R^2}=\frac{9r^2}{2R^2}.$$

Aplicația 39

In ΔABC

$$\frac{9r^2}{2R^2}\le\sum\frac{h_a^2}{b^2+c^2}\le\frac{1}{2}\left(\frac{3R}{4r}\right)^2.$$

Marin Chirciu

Inegalitatea din dreapta.

Folosind $h_a\le m_a$ și inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ obținem:

$$\sum\frac{h_a^2}{b^2+c^2}\le\sum\frac{m_a^2}{b^2+c^2}\le\sum\frac{\left(\frac{Rp}{a}\right)^2}{2bc}=\frac{R^2p^2}{2abc}\sum\frac{1}{a}\le\frac{R^2p^2}{2\cdot 4Rrp}\cdot\frac{p}{3Rr}=\frac{9R^2}{32r^2}=\frac{1}{2}\left(\frac{3R}{4r}\right)^2.$$

Inegalitatea din stânga.

Folosind inegalitatea lui Bergstrőm obținem:

Aplicația 40

In ΔABC

$$\frac{9r^2}{2R^2}\le\sum\frac{r_a^2}{b^2+c^2}\le\frac{2R}{r}+\frac{5r}{4R}-\frac{7}{2}.$$

Marin Chirciu

Inegalitatea din dreapta.

$$\sum\frac{r_a^2}{b^2+c^2}\le\sum\frac{r_a^2}{2bc}=\frac{8R^2+2Rr-p^2}{4Rr}\overset{Gerretsen}{\le}\frac{8R^2+2Rr-16Rr+5r^2}{4Rr}=\frac{8R^2-14Rr+5r^2}{4Rr}=$$
$$=\frac{2R}{r}+\frac{5r}{4R}-\frac{7}{2}.$$

Inegalitatea din stânga.

Folosind inegalitatea lui Bergstrőm obținem:

$$\sum\frac{r_a^2}{b^2+c^2}\geq\frac{\left(\sum r_a\right)^2}{\sum\left(b^2+c^2\right)}\geq\frac{(9r)^2}{2\sum a^2}\overset{Leibniz}{\geq}\frac{81r^2}{2\cdot 9R^2}=\frac{9r^2}{2R^2}.$$

Aplicația 41

In ΔABC

$$\frac{1}{m_aw_a}+\frac{1}{m_bw_b}+\frac{1}{m_cw_c}\leq\frac{4R+r}{rp^2}.$$

Mihaly Bencze, Brașov, Octogon Magazine, 2022

Soluție.

Folosind inegalitatea $m_aw_a\geq p(p-a)$ obținem:

$$Ms=\sum\frac{1}{m_aw_a}\leq\sum\frac{1}{p(p-a)}=\frac{4R+r}{rp^2}=Md.$$

Aplicația 42

In ΔABC

$$\frac{1}{m_aw_a}+\frac{1}{m_bw_b}+\frac{1}{m_cw_c}\geq\left(\frac{6r}{Rp}\right)^2.$$

Marin Chirciu

Folosind inegalitatea $m_a\geq w_a$ obținem:

$$\sum\frac{1}{m_aw_a}\geq\sum\frac{1}{m_a^2}\overset{Panaitopol}{\geq}\sum\frac{1}{\left(\frac{Rp}{a}\right)^2}=\frac{1}{R^2p^2}\sum a^2\overset{Neuberg}{\geq}\frac{1}{R^2p^2}\cdot 36r^2=\left(\frac{6r}{Rp}\right)^2.$$

Aplicația 43

In ΔABC

$$\left(\frac{6r}{Rp}\right)^2\leq\frac{1}{m_aw_a}+\frac{1}{m_bw_b}+\frac{1}{m_cw_c}\leq\frac{1}{27r^2}\left(1+\frac{4R}{r}\right).$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea $m_aw_a\geq p(p-a)$ obținem:

$$\sum\frac{1}{m_aw_a}\leq\sum\frac{1}{p(p-a)}=\frac{4R+r}{rp^2}\overset{Mitrinovic}{\leq}\frac{4R+r}{r\cdot 27r^2}=\frac{1}{27r^2}\left(1+\frac{4R}{r}\right).$$

Inegalitatea din stânga.

Folosind inegalitatea $m_a\geq w_a$ obținem:

$$\sum\frac{1}{m_aw_a}\geq\sum\frac{1}{m_a^2}\overset{Panaitopol}{\geq}\sum\frac{1}{\left(\frac{Rp}{a}\right)^2}=\frac{1}{R^2p^2}\sum a^2\overset{Neuberg}{\geq}\frac{1}{R^2p^2}\cdot 36r^2=\left(\frac{6r}{Rp}\right)^2.$$

Aplicația 44

In ΔABC

$$\left(\frac{6r}{Rp}\right)^2\leq\frac{1}{m_ah_a}+\frac{1}{m_bh_b}+\frac{1}{m_ch_c}\leq\left(\frac{3R}{2S}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea $m_a \geq h_a$ obținem:

$$Ms = \sum \frac{1}{m_a h_a} \leq \sum \frac{1}{h_a^2} = \sum \frac{1}{\left(\frac{2S}{a}\right)^2} = \sum \frac{a^2}{4S^2} \overset{Leibniz}{\leq} \frac{9R^2}{4S^2} = \left(\frac{3R}{2S}\right)^2 = Md .$$

Inegalitatea din stânga.

Folosind inegalitatea $m_a \geq h_a$ obținem:

$$Ms = \sum \frac{1}{m_a h_a} \geq \sum \frac{1}{m_a^2} \overset{Panaitopol}{\geq} \sum \frac{1}{\left(\frac{Rp}{a}\right)^2} = \frac{1}{R^2 p^2} \sum a^2 \overset{Neuberg}{\geq} \frac{1}{R^2 p^2} \cdot 36r^2 = \left(\frac{6r}{Rp}\right)^2 = Md .$$

Aplicația 45

In ΔABC

$$\left(\frac{6r}{Rp}\right)^2 \leq \frac{1}{m_a s_a} + \frac{1}{m_b s_b} + \frac{1}{m_c s_c} \leq \left(\frac{3R}{2S}\right)^2 .$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea $m_a \geq s_a \geq h_a$ obținem:

$$Ms = \sum \frac{1}{m_a s_a} \leq \sum \frac{1}{h_a^2} = \sum \frac{1}{\left(\frac{2S}{a}\right)^2} = \sum \frac{a^2}{4S^2} \overset{Leibniz}{\leq} \frac{9R^2}{4S^2} = \left(\frac{3R}{2S}\right)^2 = Md .$$

Inegalitatea din stânga.

Folosind inegalitatea $m_a \geq s_a$ obținem:

$$Ms = \sum \frac{1}{m_a s_a} \geq \sum \frac{1}{m_a^2} \overset{Panaitopol}{\geq} \sum \frac{1}{\left(\frac{Rp}{a}\right)^2} = \frac{1}{R^2 p^2} \sum a^2 \overset{Neuberg}{\geq} \frac{1}{R^2 p^2} \cdot 36r^2 = \left(\frac{6r}{Rp}\right)^2 = Md .$$

Aplicația 46

In ΔABC

$$\frac{2(4R+r)}{Rp^2} \leq \frac{1}{m_a r_a} + \frac{1}{m_b r_b} + \frac{1}{m_c r_c} \leq \frac{4R+r}{rp^2} .$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea $m_a \geq h_a$ și $\sum \frac{1}{h_a r_a} = \frac{4R+r}{rp^2}$ obținem: $\sum \frac{1}{m_a r_a} \leq \sum \frac{1}{h_a r_a} = \frac{4R+r}{rp^2}$.

Inegalitatea din stânga.

Folosind inegalitatea Panaitopol $m_a \leq \frac{Rp}{a}$ și $\sum \frac{a}{r_a} = \frac{2(4R+r)}{p}$ obținem:

$$Ms = \sum \frac{1}{m_a r_a} \overset{Panaitopol}{\geq} \sum \frac{1}{\frac{Rp}{a} \cdot r_a} = \frac{1}{Rp} \sum \frac{a}{r_a} \overset{Neuberg}{\geq} \frac{1}{Rp} \cdot \frac{2(4R+r)}{p} = \frac{2(4R+r)}{Rp^2} = Md .$$

Aplicația 47

In ΔABC

$$\left(\frac{6r}{Rp}\right)^2 \le \frac{1}{m_a^2}+\frac{1}{m_b^2}+\frac{1}{m_c^2} \le \frac{4R+r}{rp^2}.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea $m_a^2 \ge p(p-a)$ obținem: $\sum\frac{1}{m_a^2} \le \sum\frac{1}{p(p-a)} = \frac{4R+r}{rp^2}$.

Inegalitatea din stânga.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$\sum\frac{1}{m_a^2} \overset{Panaitopol}{\ge} \sum\frac{1}{\left(\frac{Rp}{a}\right)^2} = \frac{1}{R^2p^2}\sum a^2 \overset{Neuberg}{\ge} \frac{1}{R^2p^2}\cdot 36r^2 = \left(\frac{6r}{Rp}\right)^2.$$

Aplicația 48

In ΔABC

$$\frac{4R+r}{rp^2} \le \frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2} \le \left(\frac{3R}{2S}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind inegalitatea $w_a \ge h_a$ obținem:

$$\sum\frac{1}{w_a^2} \le \sum\frac{1}{h_a^2} = \sum\frac{1}{\left(\frac{2S}{a}\right)^2} = \sum\frac{a^2}{4S^2} \overset{Leibniz}{\le} \frac{9R^2}{4S^2} = \left(\frac{3R}{2S}\right)^2.$$

Inegalitatea din stânga.

Folosind inegalitatea $w_a^2 \le p(p-a)$ obținem: $\sum\frac{1}{w_a^2} \ge \sum\frac{1}{p(p-a)} = \frac{4R+r}{rp^2}$.

Aplicația 49

In ΔABC

$$\left(\frac{3}{p}\right)^2 \le \frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2} \le \left(\frac{3R}{2S}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta. $\sum\frac{1}{h_a^2} = \sum\frac{1}{\left(\frac{2S}{a}\right)^2} = \sum\frac{a^2}{4S^2} \overset{Leibniz}{\le} \frac{9R^2}{4S^2} = \left(\frac{3R}{2S}\right)^2.$

Inegalitatea din stânga.

$$\sum\frac{1}{h_a^2} = \sum\frac{1}{\left(\frac{2S}{a}\right)^2} = \sum\frac{a^2}{4S^2} = \frac{1}{4S^2}\sum a^2 \overset{Neuberg}{\ge} \frac{1}{4p^2r^2}\cdot 36r^2 = \frac{9}{p^2} = \left(\frac{3}{p}\right)^2.$$

Aplicația 50

In ΔABC

$$\left(\frac{6r}{Rp}\right)^2 \le \frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2} \le \frac{4R+r}{rp^2}.$$

Marin Chirciu

Soluție.

Folosim identitatea $\sum \frac{1}{r_a^2}=\frac{p^2-2r^2-8Rr}{p^2r^2}$.

Inegalitatea din dreapta.

Folosind inegalitatea $m_a^2 \ge p(p-a)$ obținem: $\sum \frac{1}{m_a^2} \le \sum \frac{1}{p(p-a)}=\frac{4R+r}{rp^2}$.

Inegalitatea din stânga.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$\sum \frac{1}{m_a^2} \overset{Panaitopol}{\ge} \sum \frac{1}{\left(\frac{Rp}{a}\right)^2}=\frac{1}{R^2p^2}\sum a^2 \overset{Neuberg}{\ge} \frac{1}{R^2p^2}\cdot 36r^2=\left(\frac{6r}{Rp}\right)^2.$$

Aplicația 51

In ΔABC

$$\frac{\sqrt{m_ar_a}}{w_a}+\frac{\sqrt{m_br_b}}{w_b}+\frac{\sqrt{m_cr_c}}{w_c} \le 1+\frac{R}{r}.$$

Bogdan Fuștei

Soluție.

Demonstrăm

Lema

In ΔABC

$$\frac{\sqrt{m_ar_a}}{w_a} \le \frac{b+c}{4(p-a)}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$\frac{\sqrt{m_ar_a}}{w_a} \le \frac{\sqrt{\frac{Rp}{a}\cdot\frac{rp}{p-a}}}{\frac{2bc}{b+c}\cos\frac{A}{2}}=\frac{\sqrt{\frac{Rp}{a}\cdot\frac{rp}{p-a}}}{\frac{2bc}{b+c}\sqrt{\frac{p(p-a)}{bc}}}=\frac{b+c}{4(p-a)}.$$

Obținem $Ms=\sum \frac{\sqrt{m_ar_a}}{w_a} \le \sum \frac{b+c}{4(p-a)}=\frac{1}{4}\sum \frac{b+c}{p-a}=\frac{1}{4}\cdot 4\left(1+\frac{R}{r}\right)=1+\frac{R}{r}=Md$.

Aplicația 52

In ΔABC

$$\frac{\sqrt{s_ar_a}}{w_a}+\frac{\sqrt{s_br_b}}{w_b}+\frac{\sqrt{s_cr_c}}{w_c} \le 1+\frac{R}{r}.$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\frac{\sqrt{s_a r_a}}{w_a} \le \frac{b+c}{4(p-a)}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ și $s_a = \frac{2bc}{b^2+c^2} m_a \le m_a$ obținem:

$$\frac{\sqrt{s_a r_a}}{w_a} \le \frac{\sqrt{m_a r_a}}{w_a} \le \frac{\sqrt{\frac{Rp}{a} \cdot \frac{rp}{p-a}}}{\frac{2bc}{b+c}\cos\frac{A}{2}} = \frac{\sqrt{\frac{Rp}{a} \cdot \frac{rp}{p-a}}}{\frac{2bc}{b+c}\sqrt{\frac{p(p-a)}{bc}}} = \frac{b+c}{4(p-a)}.$$

Obținem $Ms = \sum \frac{\sqrt{s_a r_a}}{w_a} \le \sum \frac{b+c}{4(p-a)} = \frac{1}{4}\sum \frac{b+c}{p-a} = \frac{1}{4} \cdot 4\left(1+\frac{R}{r}\right) = 1+\frac{R}{r} = Md$.

Aplicația 53

In ΔABC

$$\frac{\sqrt{h_a r_a}}{w_a} + \frac{\sqrt{h_b r_b}}{w_b} + \frac{\sqrt{h_c r_c}}{w_c} \le 1+\frac{R}{r}.$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\frac{\sqrt{h_a r_a}}{w_a} \le \frac{b+c}{4(p-a)}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ și $h_a \le m_a$ obținem:

$$\frac{\sqrt{h_a r_a}}{w_a} \le \frac{\sqrt{m_a r_a}}{w_a} \le \frac{\sqrt{\frac{Rp}{a} \cdot \frac{rp}{p-a}}}{\frac{2bc}{b+c}\cos\frac{A}{2}} = \frac{\sqrt{\frac{Rp}{a} \cdot \frac{rp}{p-a}}}{\frac{2bc}{b+c}\sqrt{\frac{p(p-a)}{bc}}} = \frac{b+c}{4(p-a)}.$$

Obținem $Ms = \sum \frac{\sqrt{h_a r_a}}{w_a} \le \sum \frac{b+c}{4(p-a)} = \frac{1}{4}\sum \frac{b+c}{p-a} = \frac{1}{4} \cdot 4\left(1+\frac{R}{r}\right) = 1+\frac{R}{r} = Md$.

Aplicația 54

In ΔABC

$$\sum m_a h_a \le 3\left(\frac{3R}{2}\right)^2.$$

Eldeniz Hesenov, RMM 7/2020

Solutie.

Folosim Lema lui Panaitopol: $m_a \le \frac{Rp}{a}$.

Aplicația 55

In ΔABC

$$\sum m_a h_a \le p^2 \cdot \frac{R}{2r}.$$

Marin Chirciu

Solutie.

Folosind Lema lui Panaitopol: $m_a \le \frac{Rp}{a}$.

$$\sum m_a h_a \le \sum \frac{Rp}{a} \cdot \frac{2S}{a} = Rp \cdot 2pr \sum \frac{1}{a^2} = 2Rrp^2 \sum \frac{1}{a^2} \le 2Rrp^2 \cdot \frac{1}{4r^2} = p^2 \cdot \frac{R}{2r}$$

Aplicația 56

In ΔABC

$$\sum \frac{m_b^2 + m_c^2}{bc} \ge \frac{9r}{R}.$$

Kostas Geronikolas, RMM 5/2021

Soluție.

Folosind $m_a \ge \sqrt{p(p-a)}$ obținem:

$$Ms = \sum \frac{m_b^2 + m_c^2}{bc} \ge \sum \frac{p(p-b)+p(p-c)}{bc} = \sum \frac{pa}{bc} = p \cdot \frac{a^2+b^2+c^2}{abc} = \frac{2p\left(p^2 - r^2 - 4Rr\right)}{4Rrp} =$$

$$\frac{p^2 - r^2 - 4Rr}{2Rr} \overset{(1)}{\ge} \frac{9r}{R} = Md,$$

unde (1) $\Leftrightarrow \frac{p^2 - r^2 - 4Rr}{2Rr} \ge \frac{9r}{R} \Leftrightarrow p^2 \ge 4Rr + 19r^2$, care rezultă din inegalitatea lui Gerretsen $p^2 \ge 16Rr - 5r^2$.

Rămâne să arătăm că: $16Rr - 5r^2 \ge 4Rr + 19r^2 \Leftrightarrow R \ge 2r$, (Euler).

Aplicația 57

In ΔABC

$$\sum \frac{m_b^2 + m_c^2}{bc} \le \frac{9}{2}\left(\frac{R}{2r}\right)^3.$$

Marin Chirciu

Soluție.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$\sum \frac{m_b^2 + m_c^2}{bc} \le \sum \frac{\left(\frac{Rp}{b}\right)^2 + \left(\frac{Rp}{c}\right)^2}{bc} = R^2p^2 \sum \frac{1}{bc}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) =$$

$$= R^2p^2 \frac{p^4 + p^2\left(2r^2 - 10Rr\right) + r^2(4R+r)(2R+r)}{32R^3r^3p^2} = \frac{p^4 + p^2\left(2r^2 - 10Rr\right) + r^2(4R+r)(2R+r)}{32Rr^3} =$$

$$= \frac{p^2\left(p^2 + 2r^2 - 10Rr\right) + r^2(4R+r)(2R+r)}{32Rr^3} \overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le} \frac{\left(4R^2 + 4Rr + 3r^2\right)\left(4R^2 + 4Rr + 3r^2 + 2r^2 - 10Rr\right) + r^2(4R+r)(2R+r)}{32Rr^3} =$$

$$= \frac{\left(4R^2 + 4Rr + 3r^2\right)\left(4R^2 - 6Rr + 5r^2\right) + r^2(4R+r)(2R+r)}{32Rr^3} = \frac{16R^4 - 8R^3r + 16R^2r^2 + 8Rr^3 + 16r^4}{32Rr^3} =$$

$$= \frac{2R^4 - R^3r + 2R^2r^2 + Rr^3 + 2r^4}{4Rr^3} \overset{(1)}{\le} \frac{9}{2}\left(\frac{R}{2r}\right)^3,$$

unde (1)

$$\Leftrightarrow \frac{2R^4-R^3r+2R^2r^2+Rr^3+2r^4}{4Rr^3}\le\frac{9}{2}\left(\frac{R}{2r}\right)^3 \Leftrightarrow R^4+4R^3r-8R^2r^2-4Rr^3-8r^4\ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(R^3+6R^2r+4Rr^2+4r^3)\ge 0$, evident din inegalitatea lui Euler $R\ge 2r$.

Aplicația 58

In ΔABC

$$6-\frac{3r}{R}\le\sum\frac{m_b^2+m_c^2}{bc}\le\frac{9}{2}\left(\frac{R}{2r}\right)^3.$$

Marin Chirciu

Soluție.

Inegalitatea din stânga.

Folosind $m_a\ge\sqrt{p(p-a)}$ obținem:

$$E=\sum\frac{m_b^2+m_c^2}{bc}\ge\sum\frac{p(p-b)+p(p-c)}{bc}=\sum\frac{pa}{bc}=p\cdot\frac{a^2+b^2+c^2}{abc}=\frac{2p(p^2-r^2-4Rr)}{4Rrp}=$$

$$\frac{p^2-r^2-4Rr}{2Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2-r^2-4Rr}{2Rr}=\frac{12Rr-6r^2}{2Rr}=\frac{6R-3r}{R}=6-\frac{3r}{R}=Md.$$

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ obținem:

$$E=\sum\frac{m_b^2+m_c^2}{bc}\le\sum\frac{\left(\frac{Rp}{b}\right)^2+\left(\frac{Rp}{c}\right)^2}{bc}=R^2p^2\sum\frac{1}{bc}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)=$$

$$=R^2p^2\frac{p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)}{32R^3r^3p^2}=\frac{p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)}{32Rr^3}=$$

$$=\frac{p^2(p^2+2r^2-10Rr)+r^2(4R+r)(2R+r)}{32Rr^3}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-10Rr)+r^2(4R+r)(2R+r)}{32Rr^3}=$$

$$=\frac{(4R^2+4Rr+3r^2)(4R^2-6Rr+5r^2)+r^2(4R+r)(2R+r)}{32Rr^3}=\frac{16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4}{32Rr^3}=$$

$$=\frac{2R^4-R^3r+2R^2r^2+Rr^3+2r^4}{4Rr^3}\overset{(1)}{\le}\frac{9}{2}\left(\frac{R}{2r}\right)^3,$$

unde (1)

$$\Leftrightarrow \frac{2R^4-R^3r+2R^2r^2+Rr^3+2r^4}{4Rr^3}\le\frac{9}{2}\left(\frac{R}{2r}\right)^3 \Leftrightarrow R^4+4R^3r-8R^2r^2-4Rr^3-8r^4\ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(R^3+6R^2r+4Rr^2+4r^3)\ge 0$, evident din inegalitatea lui Euler $R\ge 2r$.

Aplicația 59

In ΔABC

$$\frac{8r}{R} \le \sum \frac{bc}{m_a^2} \le \frac{2R}{r}.$$

Marin Chirciu

Soluție.

Inegalitatea din stânga.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$ obținem:

$$\sum \frac{bc}{m_a^2} \ge \sum \frac{bc}{\left(\frac{Rp}{a}\right)^2} = \frac{1}{R^2p^2}\sum a^2bc = \frac{abc\sum a}{R^2p^2} = \frac{4Rrp \cdot 2p}{R^2p^2} = \frac{8r}{R}.$$

Inegalitatea din dreapta.

Folosind inegalitatea lui Tereshin $m_a \ge \frac{b^2+c^2}{4R} \ge \frac{2bc}{4R} = \frac{bc}{2R}$ obținem:

$$\sum \frac{bc}{m_a^2} \le \sum \frac{bc}{\left(\frac{bc}{2R}\right)^2} = 4R^2 \sum \frac{1}{bc} = 4R^2 \frac{\sum a}{abc} = 4R^2 \cdot \frac{2p}{4Rrp} = \frac{2R}{r}.$$

Aplicația 60

In ΔABC

$$\frac{8r}{R} \le \sum \frac{bc}{s_a^2} \le \frac{R^3}{2r^3}.$$

Marin Chirciu

Soluție.

Inegalitatea din stânga.

Avem $s_a = \frac{2bc}{b^2+c^2} m_a \le m_a$.

Folosind inegalitatea lui Panaitopol $m_a \le \frac{Rp}{a}$, obținem: $s_a \le m_a \le \frac{Rp}{a}$.

Inegalitatea din dreapta.

Folosind inegalitatea lui Tereshin $m_a \ge \frac{b^2+c^2}{4R} \ge \frac{2bc}{4R} = \frac{bc}{2R}$ obținem:

$$\sum \frac{bc}{s_a^2} = \sum \frac{bc}{\left(\frac{2bc}{b^2+c^2} m_a\right)^2} = \frac{1}{4}\sum \frac{b^2+c^2}{bc \cdot m_a^2} \le \frac{1}{4}\sum \frac{\left(b^2+c^2\right)^2}{bc\left(\frac{bc}{2R}\right)^2} = R^2 \sum \frac{\left(b^2+c^2\right)^2}{b^3c^3} =$$

$$= R^2 \cdot \frac{p^6 + p^4\left(3r^2 - 14Rr\right) + p^2r^2\left(48R^2 - 4Rr + 3r^2\right) + r^3\left(4R+r\right)^2\left(2R+r\right)}{32R^3r^3p^2} =$$

$$= \frac{p^6 + p^4\left(3r^2 - 14Rr\right) + p^2r^2\left(48R^2 - 4Rr + 3r^2\right) + r^3\left(4R+r\right)^2\left(2R+r\right)}{32Rr^3p^2} =$$

$$= \frac{1}{32Rr^3}\left[p^4 + p^2\left(3r^2 - 14Rr\right) + r^2\left(48R^2 - 4Rr + 3r^2\right) + \frac{r^3\left(4R+r\right)^2\left(2R+r\right)}{p^2}\right] =$$

$$= \frac{1}{32Rr^3}\left[p^2\left(p^2 + 3r^2 - 14Rr\right) + r^2\left(48R^2 - 4Rr + 3r^2\right) + \frac{r^3\left(4R+r\right)^2\left(2R+r\right)}{p^2}\right] \overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{1}{32Rr^3}\left[p^2\left(4R^2+4Rr+3r^2+3r^2-14Rr\right)+r^2\left(48R^2-4Rr+3r^2\right)+\frac{r^3(4R+r)^2(2R+r)}{p^2}\right]=$$

$$=\frac{1}{32Rr^3}\left[p^2\left(4R^2-10Rr+6r^2\right)+r^2\left(48R^2-4Rr+3r^2\right)+\frac{r^3(4R+r)^2(2R+r)}{p^2}\right]\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{1}{32Rr^3}\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2-10Rr+6r^2\right)+r^2\left(48R^2-4Rr+6r^2\right)+\frac{r^3(4R+r)^2(2R+r)}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{32Rr^3}\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2-10Rr+6r^2\right)+r^2\left(48R^2-4Rr+3r^2\right)+r^2(2R+r)(R+r)\right]=$$

$$=\frac{1}{32Rr^3}\left[16R^4-24R^3r-4R^2r^2-6Rr^3+18r^4+r^2\left(48R^2-4Rr+3r^2\right)+r^2(2R+r)(R+r)\right]=$$

$$=\frac{1}{32Rr^3}\left[16R^4-24R^3r-4R^2r^2-6Rr^3+18r^4+48R^2r^2-4Rr^3+3r^4+2R^2r^2+3Rr^3+r^4\right]=$$

$$=\frac{16R^4-24R^3r+46R^2r^2-7Rr^3+22r^4}{32Rr^3}\overset{Euler}{\le}\frac{16R^4}{32Rr^3}=\frac{R^3}{2r^3}=\frac{1}{2}\left(\frac{R}{r}\right)^2.$$

Aplicația 61

In ΔABC

$$4\le\sum\frac{bc}{w_a^2}\le\frac{2R}{r}.$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\sum\frac{bc}{w_a^2}=\frac{R+2r}{r}.$$

Demonstrație.

Folosind $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$ obținem:

$$\sum\frac{bc}{w_a^2}=\sum\frac{bc}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\sum\frac{bc}{\frac{4b^2c^2}{(b+c)^2}\cdot\frac{p(p-a)}{bc}}=\sum\frac{(b+c)^2}{4p(p-a)}=\frac{1}{4p}\sum\frac{(b+c)^2}{p-a}=$$

$$=\frac{1}{4p}\sum\frac{(b+c)^2}{p-a}=\frac{1}{4p}\cdot\frac{4p(R+2r)}{r}=\frac{R+2r}{r}.$$

Inegalitatea din stânga.

Folosind **Lema** și inegalitatea lui Euler obținem: $\sum\frac{bc}{w_a^2}=\frac{R+2r}{r}\ge 4$.

Inegalitatea din dreapta.

Folosind **Lema** și inegalitatea lui Euler obținem: $\sum\frac{bc}{w_a^2}=\frac{R+2r}{r}\le\frac{2R}{r}$.

Aplicația 62

In ΔABC

$$4 \le \sum \frac{bc}{h_a^2} \le \frac{2R}{r}.$$

Marin Chirciu

Soluție.
Lema

In ΔABC

$$\sum \frac{bc}{h_a^2} = \frac{2R}{r}.$$

Demonstrație.

Folosind $h_a = \frac{2S}{a}$ obținem:

$$\sum \frac{bc}{h_a^2} = \sum \frac{bc}{\left(\frac{2S}{a}\right)^2} = \frac{1}{4S^2}\sum a^2bc = \frac{1}{4p^2r^2}\cdot abc\sum a = \frac{1}{4p^2r^2}\cdot 4Rrp\cdot 2p = \frac{2R}{r}.$$

Folosind **Lema** și inegalitatea lui Euler obținem concluzia

Aplicația 63

In ΔABC

$$4\left(\frac{R}{r}-1\right) \le \sum \frac{bc}{r_a^2} \le 4\left(\frac{R}{r}-1\right)^2.$$

Marin Chirciu

Soluție.
Lema

In ΔABC

$$\sum \frac{bc}{r_a^2} = \frac{p^2+r^2-12Rr}{r^2}.$$

Demonstrație.

Folosind $r_a = \frac{S}{p-a}$ obținem:

$$\sum \frac{bc}{r_a^2} = \sum \frac{bc}{\left(\frac{S}{p-a}\right)^2} = \frac{1}{S^2}\sum bc(p-c)^2 = \frac{1}{p^2r^2}\cdot p^2\left(p^2+r^2-12Rr\right) = \frac{p^2+r^2-12Rr}{r^2}.$$

Inegalitatea din stânga.

Folosind **Lema** și inegalitatea lui Gerretsen și Euler obținem:

$$\sum \frac{bc}{r_a^2} = \frac{p^2+r^2-12Rr}{r^2} \ge \frac{16Rr-5r^2+r^2-12Rr}{r^2} = \frac{4Rr-4r^2}{r^2} = \frac{4(R-r)}{r} \ge 4.$$

Inegalitatea din dreapta.

Folosind **Lema** și inegalitatea lui Gerretsen și Euler obținem:

$$\sum \frac{bc}{r_a^2} = \frac{p^2+r^2-12Rr}{r^2} \le \frac{4R^2+4Rr+3r^2+r^2-12Rr}{r^2} = \frac{4R^2-8Rr+4r^2}{r^2} = \frac{4(R-r)^2}{r^2}.$$

Aplicația 64

In ΔABC

$$\frac{4r}{R^2} \le \sum \frac{r_a}{m_a^2} \le \frac{R-r}{r^2}.$$

George Apostolopoulos, Greece,RMM 4/2021

Soluție.

Inegalitatea din dreapta.

Folosind $m_a \geq \sqrt{p(p-a)}$ și $r_a \leq \frac{a^2}{pr}$ obținem:

$$E=\sum \frac{r_a}{m_a^2} \leq \sum \frac{\frac{a^2}{4r}}{p(p-a)}=\frac{1}{4pr}\sum \frac{a^2}{p-a}=\frac{1}{4pr}\cdot\frac{4p(R-r)}{r}=\frac{R-r}{r^2}=Md .$$

Lema

In ΔABC

$$r_a \leq \frac{a^2}{4r} .$$

Demonstrație.

Folosind $r_a=\frac{S}{p-a}$ și $r=\frac{S}{p}$ avem $\frac{S}{p-a} \leq \frac{a^2}{4\frac{S}{p}} \Leftrightarrow 4S^2 \leq p(p-a)a^2 \Leftrightarrow$

$\Leftrightarrow 4p(p-a)(p-b)(p-c) \leq p(p-a)a^2 \Leftrightarrow 4(p-b)(p-c) \leq a^2 \Leftrightarrow$

$\Leftrightarrow (a+c-b)(a+b-c) \leq a^2 \Leftrightarrow a^2-(b-c)^2 \leq a^2 \Leftrightarrow (b-c)^2 \geq 0 .$

Inegalitatea din stânga.

Folosind inegalitatea lui Panaitopol $m_a \leq \frac{Rp}{a}$ obținem:

$$E=\sum \frac{r_a}{m_a^2} \geq \sum \frac{\frac{S}{p-a}}{\left(\frac{Rp}{a}\right)^2}=\frac{S}{R^2p^2}\sum \frac{a^2}{p-a}=\frac{pr}{R^2p^2}\cdot\frac{4p(R-r)}{r}=\frac{4(R-r)}{R^2} \geq \frac{4r}{R}=Ms .$$

Aplicația 64

In ΔABC

$$\frac{4(R-r)}{R^2} \leq \sum \frac{r_a}{m_a^2} \leq \frac{R-r}{r^2} .$$

Soluție.

Inegalitatea din dreapta.

Folosind $m_a \geq \sqrt{p(p-a)}$ și $r_a=\frac{S}{p-a}$ obținem:

$$E=\sum \frac{r_a}{m_a^2} \leq \sum \frac{\frac{S}{p-a}}{p(p-a)}=\frac{S}{p}\sum \frac{1}{(p-a)^2}=\frac{pr}{p}\cdot\frac{(4R+r)^2-2p^2}{p^2r^2}=\frac{(4R+r)^2-2p^2}{p^2r}=Md .$$

$$=\frac{1}{r}\left[\frac{(4R+r)^2}{p^2}-2\right] \stackrel{Gerretsen}{\leq} \frac{1}{r}\left[\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}-2\right]=\frac{R-r}{r^2}=Md$$

Inegalitatea din stânga.

Folosind inegalitatea lui Panaitopol $m_a \leq \frac{Rp}{a}$ obținem:

$$E=\sum\frac{r_a}{m_a^2}\geq\sum\frac{\frac{S}{p-a}}{\left(\frac{Rp}{a}\right)^2}=\frac{S}{R^2p^2}\sum\frac{a^2}{p-a}=\frac{pr}{R^2p^2}\cdot\frac{4p(R-r)}{r}=\frac{4(R-r)}{R^2}=Ms.$$

Aplicația 65

In ΔABC

$$\sum\frac{m_a^2}{w_bw_c}\geq 3\left(\frac{R}{2r}\right)^{-2}.$$

George Apostolopoulos, Greece,RMM-4/2021

Soluție.

Lema

In ΔABC

$$\sum\frac{m_a^2}{w_bw_c}\geq 3.$$

Demonstrație.

Folosind $m_a\geq\sqrt{p(p-a)}\geq w_a$ obținem:

$$\sum\frac{m_a^2}{w_bw_c}\geq\sum\frac{p(p-a)}{\sqrt{p(p-b)}\sqrt{p(p-c)}}=\sum\frac{(p-a)}{\sqrt{(p-b)(p-c)}}\geq 3\sqrt[3]{\prod\frac{(p-a)}{\sqrt{(p-b)(p-c)}}}=3.$$

Folosind **Lema** și inegalitatea lui Euler $R\geq 2r$ obținem concluzia.

Aplicația 66

In ΔABC

$$\sum\frac{m_a^2}{w_bw_c}\geq 3\geq 3\left(\frac{R}{2r}\right)^{-2}.$$

Soluție.

Vezi mai sus.

Aplicația 67

In ΔABC

$$\sum\frac{m_a^2}{w_bw_c}\leq 3\left(\frac{R}{2r}\right)^{5}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $m_a\leq\frac{Rp}{a}$ obținem:

$$\sum\frac{m_a^2}{w_bw_c}\leq\sum\frac{\left(\frac{Rp}{a}\right)^2}{w_bw_c}=R^2p^2\sum\frac{1}{a^2w_bw_c}\leq R^2p^2\sum\frac{1}{a^2h_bh_c}=$$

$$=R^2p^2\cdot\frac{p^6+p^4\left(3r^2-12Rr\right)+3p^2r^4+r^3\left(4R+r\right)^3}{64R^2p^4r^4}=\frac{p^6+p^4\left(3r^2-12Rr\right)+3p^2r^4+r^3\left(4R+r\right)^3}{64p^2r^4}=$$

$$=\frac{p^2\left[p^4+p^2\left(3r^2-12Rr\right)+3r^4\right]+r^3\left(4R+r\right)^3}{64p^2r^4}=\frac{1}{64r^4}\left[p^4+p^2\left(3r^2-12Rr\right)+3r^4+\frac{r^3\left(4R+r\right)^3}{p^2}\right]=$$

$$\overset{Gerretsen}{\leq}\frac{1}{64r^4}\left[p^2\left(p^2+3r^2-12Rr\right)+3r^4+\frac{r^3(4R+r)^3}{p^2}\right]=$$

$$=\frac{1}{64r^4}\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2+3r^2-12Rr\right)+3r^4+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{64r^4}\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2-8Rr+6r^2\right)+3r^4+r^2(4R+r)(R+r)\right]=$$

$$=\frac{1}{64r^4}\left[16R^4-16R^3r+4R^2r^2+18r^4+3r^4+r^2(4R+r)(R+r)\right]=$$

$$=\frac{1}{64r^4}\left(16R^4-16R^3r+4R^2r^2+18r^4+3r^4+4R^2r^2+5Rr^3+r^4\right)=$$

$$=\frac{1}{64r^4}\left(16R^4-16R^3r+8R^2r^2+5Rr^3+22r^4\right)=\frac{16R^4-16R^3r+8R^2r^2+5Rr^3+22r^4}{64r^4}\overset{Euler}{\leq}$$

$$\overset{Euler}{\leq}\frac{6R^5}{64r^5}=\frac{3R^5}{32r^5}=3\left(\frac{R}{2r}\right)^5.$$

Aplicația 68

In ΔABC

$$1\leq\sum\frac{m_a^2}{w_bw_c}\leq 3\left(\frac{R}{2r}\right)^5.$$

Marin Chirciu

Soluție.

Inegalitatea din stânga.

Folosind $m_a\geq\sqrt{p(p-a)}\geq w_a$ obținem:

$$\sum\frac{m_a^2}{w_bw_c}\geq\sum\frac{p(p-a)}{\sqrt{p(p-b)}\sqrt{p(p-c)}}=\sum\frac{(p-a)}{\sqrt{(p-b)(p-c)}}\geq 3\sqrt[3]{\prod\frac{(p-a)}{\sqrt{(p-b)(p-c)}}}=3.$$

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $m_a\leq\frac{Rp}{a}$ obținem:

$$\sum\frac{m_a^2}{w_bw_c}\leq\sum\frac{\left(\frac{Rp}{a}\right)^2}{w_bw_c}=R^2p^2\sum\frac{1}{a^2w_bw_c}\leq R^2p^2\sum\frac{1}{a^2h_bh_c}=$$

$$=R^2p^2\cdot\frac{p^6+p^4\left(3r^2-12Rr\right)+3p^2r^4+r^3(4R+r)^3}{64R^2p^4r^4}=\frac{p^6+p^4\left(3r^2-12Rr\right)+3p^2r^4+r^3(4R+r)^3}{64p^2r^4}=$$

$$=\frac{p^2\left[p^4+p^2\left(3r^2-12Rr\right)+3r^4\right]+r^3(4R+r)^3}{64p^2r^4}=\frac{1}{64r^4}\left[p^4+p^2\left(3r^2-12Rr\right)+3r^4+\frac{r^3(4R+r)^3}{p^2}\right]=$$

$$\overset{Gerretsen}{\leq}\frac{1}{64r^4}\left[p^2\left(p^2+3r^2-12Rr\right)+3r^4+\frac{r^3(4R+r)^3}{p^2}\right]=$$

$$=\frac{1}{64r^4}\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2+3r^2-12Rr\right)+3r^4+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{64r^4}\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2-8Rr+6r^2\right)+3r^4+r^2(4R+r)(R+r)\right]=$$

$$=\frac{16R^4-16R^3r+8R^2r^2+5Rr^3+22r^4}{64r^4}\overset{Euler}{\le}\frac{3R^5}{32r^5}=3\left(\frac{R}{2r}\right)^5.$$

Aplicația 69

In ΔABC

$$\frac{2}{R}\le\sum\frac{m_a}{w_bw_c}\le\frac{1}{r}\left(\frac{R}{2r}\right)^2.$$

Marin Chirciu

Soluție.

Inegalitatea din stânga.

Folosind $m_a\ge\sqrt{p(p-a)}\ge w_a$ obținem:

$$\sum\frac{m_a}{w_bw_c}\ge\sum\frac{\sqrt{p(p-a)}}{\sqrt{p(p-b)}\sqrt{p(p-c)}}=\sum\frac{\sqrt{(p-a)}}{\sqrt{p(p-b)(p-c)}}\ge3\sqrt[3]{\prod\frac{\sqrt{(p-a)}}{\sqrt{p(p-b)(p-c)}}}=$$

$$=3\sqrt[3]{\frac{1}{p\sqrt{p(p-a)(p-b)(p-c)}}}=3\sqrt[3]{\frac{1}{pS}}=3\sqrt[3]{\frac{1}{p^2r}}\overset{Mitrinovic+Euler}{\ge}3\sqrt[3]{\frac{1}{\frac{27R^2}{4}\cdot\frac{R}{2}}}=\frac{2}{R}.$$

Inegalitatea din dreapta.

Folosind inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ obținem:

$$\sum\frac{m_a^2}{w_bw_c}\le\sum\frac{\frac{Rp}{a}}{w_bw_c}=Rp\sum\frac{1}{aw_bw_c}\le Rp\sum\frac{1}{ah_bh_c}=Rp\cdot\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2(4R+r)^2}{16p^3r^3R}=$$

$$=\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2(4R+r)^2}{16p^2r^3}=\frac{1}{16r^3}\left[\left(p^2+2r^2-8Rr\right)+\frac{r^2(4R+r)^2}{p^2}\right]\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{1}{16r^3}\left[4R^2+4Rr+3r^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{1}{16r^3}\left[4R^2-4Rr+5r^2+r(R+r)\right]=$$

$$=\frac{4R^2-3Rr+6r^2}{16r^3}\overset{Euler}{\le}\frac{4R^2}{16r^3}=\frac{R^2}{4r^3}=\frac{1}{r}\left(\frac{R}{2r}\right)^2.$$

Aplicația 70

In ΔABC

$$\frac{64r^3}{9R^7}\le\sum\frac{\sin^2B+\sin^2C}{m_a^4}\le\frac{1}{2F^2}\left(\frac{2R}{r}-1\right).$$

Marin Chirciu

Soluție.

Inegalitatea din stânga.

$$Ms=\sum\frac{\sin^2 B+\sin^2 C}{m_a^4}=\sum\frac{\frac{b^2}{4R^2}+\frac{c^2}{4R^2}}{m_a^4}=\frac{1}{4R^2}\sum\frac{b^2+c^2}{m_a^4}\overset{(1)}{\geq}\frac{1}{4R^2}\cdot\frac{256r^3}{9R^5}=\frac{64r^3}{9R^7}=Md\ ,$$

unde (1) rezultă din:

Lema1

In ΔABC

$$\sum\frac{b^2+c^2}{m_a^4}\geq\frac{256r^3}{9R^5}.$$

Demonstrație.

Folosind Inegalitatea lui Panaitopol $m_a\leq\frac{Rp}{a}$ obținem:

$$Ms=\sum\frac{b^2+c^2}{m_a^4}\geq\sum\frac{2bc}{\left(\frac{Rp}{a}\right)^4}=\frac{1}{p^4R^4}\sum 2a^4bc=\frac{2abc}{p^4R^4}\sum a^3=\frac{2\cdot 4Rrp}{p^4R^4}\cdot 2p\left(p^2-3r^2-6Rr\right)=$$

$$=\frac{16r\left(p^2-3r^2-6Rr\right)}{p^2R^3}\overset{Gerretsen}{\geq}\frac{16r\left(16Rr-5r^2-3r^2-6Rr\right)}{p^2R^3}=\frac{16r\left(10Rr-8r^2\right)}{p^2R^3}=\frac{32r^2\left(5R-4r\right)}{p^2R^3}=$$

$$\overset{Euler}{\geq}\frac{32r^2\cdot 6r}{p^2R^3}=\frac{192r^3}{p^2R^3}\overset{Mitrinovic}{\geq}\frac{192r^3}{\frac{27R^2}{4}\cdot R^3}=\frac{256r^3}{9R^5}.$$

Inegalitatea din dreapta.

Folosind Teorema sinusurilor obținem:

$$Ms=\sum\frac{\sin^2 B+\sin^2 C}{m_a^4}=\sum\frac{\frac{b^2}{4R^2}+\frac{c^2}{4R^2}}{m_a^4}=\frac{1}{4R^2}\sum\frac{b^2+c^2}{m_a^4}\overset{(1)}{\leq}\frac{1}{4R^2}\cdot\frac{R^2p}{F^3}\cdot\left(2R-r\right)=,$$

$$=\frac{1}{2rF^2}\cdot\left(2R-r\right)=\frac{1}{2F^2}\left(\frac{2R}{r}-1\right)=Md$$ unde (1) rezultă din:

Lema2

In ΔABC

$$\sum\frac{b^2+c^2}{m_a^4}\leq\frac{4R^2p}{F^3}\cdot\left(2R-r\right).$$

Demonstrație.

Folosind inegalitatea lui Tereshin $m_a\geq\frac{b^2+c^2}{4R}$ obținem:

$$Ms=\sum\frac{b^2+c^2}{m_a^4}\leq\sum\frac{b^2+c^2}{\left(\frac{b^2+c^2}{4R}\right)^4}=256R^4\sum\frac{1}{\left(b^2+c^2\right)^3}\overset{AGM}{\leq}256R^4\sum\frac{1}{8b^3c^3}=32R^4\sum\frac{1}{b^3c^3}=$$

$$=32R^4\cdot\frac{\sum a^3}{a^3b^3c^3}\leq\frac{32R^4}{64R^3F^3}\cdot 2p\left(p^2-3r^2-6Rr\right)=\frac{Rp}{F^3}\cdot\left(p^2-3r^2-6Rr\right)\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{Rp}{F^3}\cdot\left(4R^2+4Rr+3r^2-3r^2-6Rr\right)=\frac{Rp}{F^3}\cdot\left(4R^2-2Rr\right)=\frac{2R^2p}{F^3}\cdot\left(2R-r\right)=Md\,.$$

Aplicația 71

In ΔABC

$$\sum m_a\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\geq 7-\frac{2r}{R}.$$

Marian Ursărescu, RMM 1/2021

Soluție.

Lema

$$\frac{1}{r_b}+\frac{1}{r_c}=\frac{a}{S}$$

Demonstrație.

Folosind $r_a=\dfrac{S}{p-a}$ obținem:

$$\frac{1}{\frac{S}{p-b}}+\frac{1}{\frac{S}{p-c}}=\frac{p-b+p-c}{S}=\frac{2p-b-c}{S}=\frac{a}{S}$$

Folosind **Lema** obținem:

$$Ms=\sum m_a\left(\frac{1}{r_b}+\frac{1}{r_c}\right)=\sum m_a\cdot\frac{a}{S}=\frac{1}{S}\sum am_a\overset{Tereshin}{\geq}\frac{1}{pr}\sum a\cdot\frac{b^2+c^2}{4R}=\frac{1}{4Rrp}\sum a\left(b^2+c^2\right)=$$

$$=\frac{1}{4Rrp}\sum a\left(b^2+c^2\right)=\frac{1}{4Rrp}\cdot 2p\left(p^2+r^2-2Rr\right)=\frac{p^2+r^2-2R}{2Rr}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2-2R}{2Rr}=$$

$$=\frac{14Rr-4r^2}{2Rr}=7-\frac{2r}{R}=Md\,.$$

Aplicația 72

In ΔABC

$$\sum m_a\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\leq\frac{3R}{r}.$$

Marin Chirciu

Soluție.

Folosind $\dfrac{1}{r_b}+\dfrac{1}{r_c}=\dfrac{a}{S}$ și inegalitatea lui Panaitopol $m_a\leq\dfrac{Rp}{a}$ obținem:

$$Ms=\sum m_a\left(\frac{1}{r_b}+\frac{1}{r_c}\right)=\sum m_a\cdot\frac{a}{S}=\frac{1}{S}\sum am_a\overset{Panaitopol}{\leq}\frac{1}{pr}\sum a\cdot\frac{Rp}{a}=\frac{1}{rp}3Rp=\frac{3R}{r}=Md\,.$$

Aplicația 73

In ΔABC

$$7-\frac{2r}{R}\leq\sum m_a\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\leq\frac{3R}{r}.$$

Soluție.

Vezi mai sus.

Aplicația 74

In ΔABC

$$\frac{17}{2}-\frac{5r}{R}\le\sum m_a\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\le\frac{3}{2}\left(\frac{R}{r}\right)^2.$$

Marin Chirciu

<u>Soluție</u>.
<u>Lema</u>

$$\frac{1}{h_b}+\frac{1}{h_c}=\frac{b+c}{2S}$$

<u>Demonstrație</u>.
Folosind $h_a=\frac{2S}{a}$ obținem: $\frac{1}{\frac{2S}{b}}+\frac{1}{\frac{2S}{c}}=\frac{b+c}{2S}$

<u>Inegalitatea din stânga.</u>
Folosind **Lema** obținem:

$$Ms=\sum m_a\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\sum m_a\cdot\frac{b+c}{2S}=\frac{1}{2S}\sum(b+c)m_a\overset{Tereshin}{\ge}\frac{1}{2pr}\sum(b+c)\cdot\frac{b^2+c^2}{4R}=$$

$$=\frac{1}{8Rrp}\sum(b+c)\left(b^2+c^2\right)=\frac{1}{8Rrp}\cdot 2p\left(3p^2-5r^2-14Rr\right)=\frac{3p^2-5r^2-14Rr}{4Rr}\overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge}\frac{3\left(16Rr-5r^2\right)-5r^2-14Rr}{24r}=\frac{34Rr-20r^2}{4Rr}=\frac{17}{2}-\frac{5r}{R}=Md.$$

$$\sum(b+c)\left(b^3+c^3\right)=2p\left(3p^2-5r^2-14Rr\right)$$

<u>Inegalitatea din dreapta.</u>

Folosind **Lema** și inegalitatea lui Panaitopol $m_a\le\frac{Rp}{a}$ obținem:

$$Ms=\sum m_a\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\sum m_a\cdot\frac{b+c}{2S}=\frac{1}{2S}\sum(b+c)m_a\overset{Panaitopol}{\le}\frac{1}{2pr}\sum(b+c)\cdot\frac{Rp}{a}=$$

$$=\frac{Rp}{2pr}\sum\frac{b+c}{a}=\frac{R}{2r}\cdot\frac{p^2+r^2-2Rr}{2Rr}=\frac{p^2+r^2-2Rr}{4r^2}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2-2Rr}{4r^2}=$$

$$=\frac{4R^2+2Rr+4r^2}{4r^2}=\frac{2R^2+Rr+2r^2}{2r^2}\overset{Euler}{\le}\frac{3R^2}{2r^2}=\frac{3}{2}\left(\frac{R}{r}\right)^2=Md.$$

<u>Aplicația 75</u>

In ΔABC

$$\frac{\sum m_bm_c}{\sum r_br_c}\le\frac{R}{2r}.$$

Eldeniz Hesenov, Georgia, RMM 1/2021

<u>Soluție</u>.
<u>Lema</u>

In ΔABC

$$\sum m_bm_c\le\frac{R}{2r}\cdot p^2.$$

<u>Demonstrație</u>.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a}\le\frac{R}{2r}\Leftrightarrow m_a\le\frac{R}{2r}h_a$ obținem:

$$\sum m_b m_c \le \sum \frac{R}{2r} h_b \cdot \frac{R}{2r} h_c = \left(\frac{R}{2r}\right)^2 \sum h_b h_c = \left(\frac{R}{2r}\right)^2 \cdot \frac{2rp^2}{R} = \frac{R}{2r} \cdot p^2 .$$

Folosind **Lema** și $\sum r_b r_c = p^2$ obținem concluzia.

<u>Aplicația 76</u>

In ΔABC

$$\frac{\sum m_b m_c}{\sum r_b r_c} \ge \frac{2r}{R} .$$

Marin Chirciu

<u>Soluție</u>.

<u>Lema</u>

In ΔABC

$$\sum m_b m_c \ge 3pr\sqrt{3} .$$

<u>Demonstrație</u>.

Folosind $m_a \ge \sqrt{p(p-a)}$ obținem:

$$\sum m_b m_c \ge \sum \sqrt{p(p-b)}\sqrt{p(p-c)} = p\sum \sqrt{(p-b)(p-c)} \overset{AM-GM}{\ge} p \cdot 3\sqrt[3]{\prod \sqrt{(p-b)(p-c)}} =$$

$$= 3p\sqrt[3]{\prod (p-a)} = 3p\sqrt[3]{pr^2} \overset{Mitrinovic}{\ge} 3p\sqrt[3]{r^2 \cdot 3r\sqrt{3}} = 3pr\sqrt{3} .$$

Să trecem la rezolvarea problemei din enunț

Folosind **Lema** și $\sum r_b r_c = p^2$ obținem :

$$\frac{\sum m_b m_c}{\sum r_b r_c} \ge \frac{3pr\sqrt{3}}{p^2} = \frac{3r\sqrt{3}}{p} \overset{Mitrinovic}{\ge} \frac{3r\sqrt{3}}{\frac{3R\sqrt{3}}{2}} = \frac{2r}{R} .$$

<u>Aplicația 77</u>

In ΔABC

$$\frac{2r}{R} \le \frac{\sum m_b m_c}{\sum r_b r_c} \le \frac{R}{2r} .$$

<u>Soluție</u>.

Vezi mai sus.

<u>Aplicația 78</u>

In ΔABC

$$\frac{8r}{R^2} \le \sum \frac{r_b + r_c}{m_b m_c} \le \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right).$$

George Apostolopoulos, Greece, RMM nr.25-Summer Edition2022

<u>Soluție</u>.

<u>Inegalitatea din dreapta.</u>

Folosind $h_a \le m_a$ obținem:

$$E = \sum \frac{r_b + r_c}{m_b m_c} \overset{h_a \le m_a}{\le} \sum \frac{r_b + r_c}{h_b h_c} = \frac{R}{r^2} \overset{(1)}{\le} \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right) = Md ,$$

unde(1) $\Leftrightarrow \frac{R}{r^2} \le \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right) \Leftrightarrow (R-2r)\left(3R^3 + 3R^2 r + 3Rr^2 + r^3\right) \ge 0$, care rezultă din inegalitatea lui Euler $R \ge 2r$.

Inegalitatea din stânga.

Folosind Panaitopol Inequality: $m_a \le \frac{R}{2r} h_a$ obținem:

$$E = \sum \frac{r_b + r_c}{m_b m_c} \overset{Panaitopol}{\ge} \sum \frac{r_b + r_c}{\frac{R}{2r} h_b \cdot \frac{R}{2r} h_c} \ge \left(\frac{2r}{R}\right)^2 \sum \frac{r_b + r_c}{h_b h_c} = \left(\frac{2r}{R}\right)^2 \cdot \frac{R}{r^2} = \frac{4}{R} \ge \frac{8r}{R^2} = Ms .$$

Aplicația 79

In ΔABC

$$\frac{4}{R} \le \sum \frac{r_b + r_c}{m_b m_c} \le \frac{R}{r^2} .$$

.

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Folosind $h_a \le m_a$ obținem: $E = \sum \frac{r_b + r_c}{m_b m_c} \overset{h_a \le m_a}{\le} \sum \frac{r_b + r_c}{h_b h_c} = \frac{R}{r^2} = Md$,

Inegalitatea din stânga.

Folosind Panaitopol Inequality: $m_a \le \frac{R}{2r} h_a$ obținem:

$$E = \sum \frac{r_b + r_c}{m_b m_c} \overset{Panaitopol}{\ge} \sum \frac{r_b + r_c}{\frac{R}{2r} h_b \cdot \frac{R}{2r} h_c} \ge \left(\frac{2r}{R}\right)^2 \sum \frac{r_b + r_c}{h_b h_c} = \left(\frac{2r}{R}\right)^2 \cdot \frac{R}{r^2} = \frac{4}{R} = Ms .$$

Aplicația 80

In ΔABC

$$\frac{8r}{R^2} \le \frac{4}{R} \le \sum \frac{r_b + r_c}{m_b m_c} \le \frac{R}{r^2} \le \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right).$$

Soluție.

Vezi

i) $\frac{R}{r^2} \le \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right) \Leftrightarrow (R-2r)\left(3R^3 + 3R^2 r + 3Rr^2 + r^3\right) \ge 0$, vezi $R \ge 2r$,(Euler).

ii) $\frac{8r}{R^2} \le \frac{4}{R} \Leftrightarrow R \ge 2r$, (inegalitatea lui Euler).

Aplicația 81

In ΔABC

$$\sum \frac{p-a}{h_a - 2r} \ge 4 \sum \frac{m_a}{b+c} .$$

Bogdan Fuștei, Romania,RMM 12/2020

Soluție.

Lema1

In ΔABC

$$\sum \frac{p-a}{h_a - 2r} = \frac{p}{r} .$$

Soluție.

$$\sum\frac{p-a}{h_a-2r}=\sum\frac{p-a}{\frac{2S}{a}-2r}=\sum\frac{p-a}{\frac{2pr}{a}-2r}=\frac{1}{2r}\sum\frac{p-a}{\frac{p}{a}-1}=\frac{1}{2r}\sum a=\frac{1}{2r}\cdot 2p=\frac{p}{r}.$$

Lema2

In ΔABC

$$\sum\frac{m_a}{b+c}\le\frac{p}{4r}.$$

Soluție.

$$\sum\frac{m_a}{b+c}\overset{Panaitopol}{\le}\sum\frac{\frac{R}{2r}\cdot h_a}{b+c}=\sum\frac{\frac{R}{2r}\cdot\frac{2S}{a}}{b+c}=\sum\frac{\frac{R}{2r}\cdot\frac{2pr}{a}}{b+c}=\sum\frac{\frac{Rp}{a}}{b+c}=Rp\sum\frac{1}{a(b+c)}=$$

$$=Rp\sum\frac{1}{ab+ac}\le Rp\cdot\sum\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{ac}\right)=Rp\cdot\sum\frac{1}{2}\frac{1}{bc}=\frac{Rp}{2}\cdot\frac{\sum a}{abc}=\frac{Rp}{2}\cdot\frac{2p}{4Rrp}=\frac{p}{4r}.$$

Folosind **Lemele** de mai sus obținem: $Ms=\sum\frac{p-a}{h_a-2r}\underset{Lema2}{\overset{Lema1}{\ge}}4\sum\frac{m_a}{b+c}=Md$.

Aplicația 82

In ΔABC

$$\sum\frac{p-a}{r_a-r}\ge 2\sum\frac{m_a}{b+c}.$$

Marin Chirciu

Soluție.

Lema1

In ΔABC

$$\sum\frac{p-a}{r_a-r}=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}.$$

Soluție.

$$\sum\frac{p-a}{r_a-r}=\sum\frac{p-a}{\frac{S}{p-a}-r}=\sum\frac{p-a}{\frac{pr}{p-a}-r}=\frac{1}{r}\sum\frac{p-a}{\frac{p}{p-a}-1}=\frac{1}{r}\sum\frac{(p-a)^2}{a}=\frac{1}{r}\cdot\frac{p\left(p^2+r^2-12Rr\right)}{4Rr}=$$

$$=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}.$$

Lema2

In ΔABC

$$\sum\frac{m_a}{b+c}\le\frac{p}{4r}.$$

Soluție.

$$\sum\frac{m_a}{b+c}\overset{Panaitopol}{\le}\sum\frac{\frac{R}{2r}\cdot h_a}{b+c}=\sum\frac{\frac{R}{2r}\cdot\frac{2S}{a}}{b+c}=\sum\frac{\frac{R}{2r}\cdot\frac{2pr}{a}}{b+c}=\sum\frac{\frac{Rp}{a}}{b+c}=Rp\sum\frac{1}{a(b+c)}=$$

$$=Rp\sum\frac{1}{ab+ac}\le Rp\cdot\sum\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{ac}\right)=Rp\cdot\sum\frac{1}{2}\frac{1}{bc}=\frac{Rp}{2}\cdot\frac{\sum a}{abc}=\frac{Rp}{2}\cdot\frac{2p}{4Rrp}=\frac{p}{4r}.$$

Folosind **Lemele** de mai sus obținem:

$$Ms=\sum\frac{p-a}{r_a-r}\overset{Lema1}{=}\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}\overset{(1)}{\geq}\frac{p}{2r}\overset{Lema2}{\geq}2\sum\frac{m_a}{b+c}=Md\ ,$$

unde(1) $\Leftrightarrow\dfrac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}\geq\dfrac{p}{2r}\Leftrightarrow\left(p^2+r^2-12Rr\right)\geq 2Rr\Leftrightarrow p^2\geq 14Rr-r^2$, care rezultă din inegalitatea lui Gerretsen: $p^2\geq 16Rr-5r^2$.

Rămâne să arătăm că: $16Rr-5r^2\geq 14Rr-r^2\Leftrightarrow R\geq 2r$, (inegalitatea lui Euler).

Aplicația 83

In ΔABC

$$\sum\frac{p-a}{h_a-2r}\leq 2\sum\frac{p-a}{r_a-r}.$$

Marin Chirciu

Soluție.

Folosind **Lemele** de mai sus avem sumele $\sum\dfrac{p-a}{h_a-2r}=\dfrac{p}{r}$ și $\sum\dfrac{p-a}{r_a-r}=\dfrac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}$.

Inegalitatea se scrie:

$$\frac{p}{r}\leq 2\cdot\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}\Leftrightarrow p^2\geq 14Rr-r^2,$$

care rezultă din inegalitatea lui Gerretsen: $p^2\geq 16Rr-5r^2$.

Rămâne să arătăm că:

$16Rr-5r^2\geq 14Rr-r^2\Leftrightarrow R\geq 2r$, (inegalitatea lui Euler).

Aplicația 84

In ΔABC

$$\frac{4\sqrt{3}}{3R}\leq\sum\frac{\csc A}{m_a}\leq\frac{\sqrt{3}R}{3r^2}.$$

George Apostolopoulos, Greece,RMM 12/2020

Soluție.

Inegalitatea din dreapta.

$$E=\sum\frac{\csc A}{m_a}=\sum\frac{\frac{1}{\sin A}}{m_a}=\sum\frac{\frac{2R}{a}}{m_a}=2R\sum\frac{1}{am_a}\overset{h_a\leq m_a}{\leq}2R\sum\frac{1}{ah_a}=2R\sum\frac{1}{2S}=\frac{3R}{S}=$$

$$=\frac{3R}{pr}\overset{Mitrinovic}{\leq}\frac{3R}{3r\sqrt{3}\cdot r}=\frac{\sqrt{3}R}{3r^2}=Md\ .$$

Inegalitatea din stânga.

$$E=\sum\frac{\csc A}{m_a}=\sum\frac{\frac{1}{\sin A}}{m_a}=\sum\frac{\frac{2R}{a}}{m_a}=2R\sum\frac{1}{am_a}\overset{m_a\leq\frac{Rp}{a}}{\geq}2R\sum\frac{1}{a\frac{Rp}{a}}=\frac{6}{p}\overset{Mitrinovic}{\geq}\frac{3R}{\frac{3R\sqrt{3}}{2}\cdot r}=$$

$$=\frac{4\sqrt{3}}{3R}=Ms\ .$$

Aplicația 85

In ΔABC

$$\frac{3}{p}\left(\frac{2r}{R}\right)^{\frac{1}{3}}\leq\sum\frac{\csc A}{m_b+m_c}\leq\frac{R}{S}\left(\frac{R}{r}-\frac{1}{2}\right).$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

$$E=\sum\frac{\csc A}{m_b+m_c}=\sum\frac{\frac{1}{\sin A}}{m_b+m_c}=\sum\frac{\frac{2R}{a}}{m_b+m_c}=2R\sum\frac{1}{a\left(m_b+m_c\right)}\overset{h_a\le m_a}{\le}2R\sum\frac{1}{a\left(h_b+h_c\right)}=$$

$$=2R\sum\frac{1}{a\left(\frac{2S}{b}+\frac{2S}{c}\right)}=\frac{2R}{2S}\sum\frac{bc}{a\left(b+c\right)}=\frac{R}{S}\sum\frac{bc}{a\left(b+c\right)}\overset{(1)}{\le}\frac{R}{S}\cdot\left(\frac{R}{r}-\frac{1}{2}\right)=Md,$$

unde(1) $\Leftrightarrow\sum\frac{bc}{a\left(b+c\right)}\le\frac{R}{r}-\frac{1}{2}$, care rezultă din:

$$\sum\frac{bc}{a\left(b+c\right)}\overset{AM-GM}{\le}\sum\frac{bc}{a\cdot2\sqrt{bc}}=\frac{\sum bc\sqrt{bc}}{2abc}\overset{(1)}{\le}\frac{\sum a^3}{2abc}=\frac{2p\left(p^2-3r^2-6Rr\right)}{2\cdot4Rrp}=$$

$$=\frac{p^2-3r^2-6Rr}{4Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-3r^2-6Rr}{4Rr}=\frac{4R^2-2Rr}{4Rr}=\frac{2R-r}{2r}=\frac{R}{r}-\frac{1}{2}.$$

Inegalitatea din stânga.

$$E=\sum\frac{\csc A}{m_b+m_c}=\sum\frac{\frac{1}{\sin A}}{m_b+m_c}=\sum\frac{\frac{2R}{a}}{m_b+m_c}=2R\sum\frac{1}{a\left(m_b+m_c\right)}\overset{m_a\le\frac{Rp}{a}}{\ge}2R\sum\frac{1}{a\left(\frac{Rp}{b}+\frac{Rp}{c}\right)}=$$

$$=\frac{2}{p}\sum\frac{bc}{a\left(b+c\right)}\ge\frac{2}{p}\cdot3\sqrt[3]{\prod\frac{bc}{a\left(b+c\right)}}=\frac{6}{p}\sqrt[3]{\frac{abc}{\prod\left(b+c\right)}}=\frac{6}{p}\sqrt[3]{\frac{4Rrp}{2p\left(p^2+r^2+2Rr\right)}}=$$

$$=\frac{6}{p}\sqrt[3]{\frac{Rr}{p^2+r^2+2Rr}}\overset{Gerretsen}{\ge}\frac{6}{p}\sqrt[3]{\frac{2Rr}{4R^2+4Rr+3r^2+r^2+2Rr}}=\frac{6}{p}\sqrt[3]{\frac{2Rr}{4R^2+6Rr+4r^2}}\overset{Euler}{\ge}\frac{6}{p}\sqrt[3]{\frac{2Rr}{8R^2}}=$$

$$=\frac{6}{p}\cdot\frac{1}{2}\sqrt[3]{\frac{2r}{R}}=\frac{3}{p}\left(\frac{2r}{R}\right)^{\frac{1}{3}}=Ms.$$

Aplicația 86

In ΔABC

$$\sum\frac{p-a}{h_a-2r}\ge4\sum\frac{m_a}{b+c}.$$

Bogdan Fuștei, Romania, RMM 11/2020

Soluție.

Lema

In ΔABC

$$\sum\frac{p-a}{h_a-2r}=\frac{p}{r}.$$

Demonstrație.

Folosind $h_a=\frac{2S}{a}$ și $r=\frac{S}{p}$ obținem: $\sum\frac{p-a}{h_a-2r}=\sum\frac{p-a}{\frac{2S}{a}-\frac{2S}{p}}=\frac{1}{2S}\sum ap=\frac{1}{2pr}\cdot2p^2=\frac{p}{r}$.

$$Md=4\sum\frac{m_a}{b+c}\overset{Panaitopol}{\leq}4\sum\frac{\frac{Rp}{a}}{b+c}=4Rp\sum\frac{1}{a(b+c)}=4Rp\cdot\frac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{8Rrp^2\left(p^2+r^2+2Rr\right)}=$$

$$=\frac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{2rp\left(p^2+r^2+2Rr\right)}\overset{(1)}{\leq}\frac{p}{r}\overset{Lema}{=}\sum\frac{p-a}{h_a-2r}=Ms,$$

unde (1) $\Leftrightarrow\dfrac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{2rp\left(p^2+r^2+2Rr\right)}\leq\dfrac{p}{r}\Leftrightarrow p^2\left(p^2-12Rr\right)\geq r^2(4R+r)^2$, care

rezultă din inegalitatea lui Gerretsen: $p^2\geq16Rr-5r^2\geq\dfrac{r(4R+r)^2}{R+r}$.

Rămâne să arătăm că:

$$\frac{r(4R+r)^2}{R+r}\left(16Rr-5r^2-12Rr\right)\geq r^2(4R+r)^2\Leftrightarrow\frac{r(4R+r)^2}{R+r}\left(4Rr-5r^2\right)\geq r^2(4R+r)^2\Leftrightarrow$$

$\Leftrightarrow 4R-5r\geq R+r\Leftrightarrow R\geq2r$, (Euler).

Aplicația 87

In ΔABC

$$\sum\frac{p-a}{r_a-r}\geq2\sum\frac{m_a}{b+c}.$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\sum\frac{p-a}{r_a-r}=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}.$$

Demonstrație.

Folosind $r_a=\dfrac{S}{p-a}$ și $r=\dfrac{S}{p}$ obținem:

$$\sum\frac{p-a}{r_a-r}=\sum\frac{p-a}{\frac{S}{p-a}-\frac{S}{p}}=\frac{p}{S}\sum\frac{(p-a)^2}{a}=\frac{p}{pr}\cdot\frac{p\left(p^2+r^2-8Rr\right)}{4Rr}=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}.$$

$$Md=4\sum\frac{m_a}{b+c}\overset{Panaitopol}{\leq}2\sum\frac{\frac{Rp}{a}}{b+c}=4Rp\sum\frac{1}{a(b+c)}=2Rp\cdot\frac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{8Rrp^2\left(p^2+r^2+2Rr\right)}=$$

$$=\frac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{4rp\left(p^2+r^2+2Rr\right)}\overset{(1)}{\leq}\frac{p^2+r^2-8Rr}{4pRr^2}\overset{Lema}{=}\sum\frac{p-a}{r_a-r}=Ms,$$

unde (1) $\Leftrightarrow\dfrac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{4rp\left(p^2+r^2+2Rr\right)}\leq\dfrac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}\Leftrightarrow$

$$\Leftrightarrow p^2\left(p^2+r^2+2Rr\right)\left(p^2+r^2-12Rr\right)\geq Rr\left[p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2\right]\Leftrightarrow$$

$$\Leftrightarrow p^2\left(p^4+p^2r^2-12Rrp^2+p^2r^2+r^4-12Rr^3+2Rrp^2+2Rr^3-24R^2r^2\right)\geq$$

$$\geq Rr\left[p^4+p^2\left(16Rr+2r^2\right)+r^2\left(4R+r\right)^2\right]\Leftrightarrow$$
$$p^6+2p^4r^2-10Rrp^4+p^2r^4-10p^2Rr^3-24p^2R^2r^2\geq p^4Rr+p^2Rr\left(16Rr+2r^2\right)+Rr^3\left(4R+r\right)^2$$
$$p^6+p^4\left(2r^2-11Rr\right)+p^2\left(r^4-10Rr^3-24R^2r^2-16R^2r^2-2Rr^3\right)\geq Rr^3\left(4R+r\right)^2$$
$$p^2\left[p^2\left(p^2+2r^2-11Rr\right)+r^2\left(r^2-12Rr-40R^2\right)\right]\geq Rr^3\left(4R+r\right)^2,$$

care rezultă din inegalitatea lui Gerretsen: $p^2\geq 16Rr-5r^2\geq \dfrac{r\left(4R+r\right)^2}{R+r}$.

Rămâne să arătăm că:

$$\frac{r\left(4R+r\right)^2}{R+r}\left[\left(16Rr-5r^2\right)\left(16Rr-5r^2+2r^2-11Rr\right)+r^2\left(r^2-12Rr-40R^2\right)\right]\geq Rr^3\left(4R+r\right)^2\Leftrightarrow$$
$$\Leftrightarrow\left[\left(16Rr-5r^2\right)\left(16Rr-5r^2+2r^2-11Rr\right)+r^2\left(r^2-12Rr-40R^2\right)\right]\geq Rr^2\left(R+r\right)\Leftrightarrow$$
$$\Leftrightarrow\left(16R-5r\right)\left(5R-3r\right)+\left(r^2-12Rr-40R^2\right)\geq Rr\left(R+r\right)\Leftrightarrow$$
$$\Leftrightarrow 80R^2-48Rr-25Rr+15r^2+r^2-12Rr-40R^2\geq R^2+Rr\Leftrightarrow$$

$\Leftrightarrow 39R^2-86Rr+16r^2\geq 0\Leftrightarrow\left(R-2r\right)\left(39R-8r\right)\geq 0$, evident din inegalitatea $R\geq 2r$, (Euler).

Aplicația 88

In ΔABC

$$\sum\frac{p-a}{h_a-2r}\leq 2\sum\frac{p-a}{r_a-r}.$$

Marin Chirciu

Soluție.

Folosind **Lemele** de mai sus avem sumele:

$$\sum\frac{p-a}{h_a-2r}=\frac{p}{r}\text{ și }\sum\frac{p-a}{r_a-r}=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}.$$

Inegalitatea se scrie: $\dfrac{p}{r}\leq 2\cdot\dfrac{p\left(p^2+r^2-12Rr\right)}{4Rr^2}\Leftrightarrow p^2\geq 14Rr-r^2$,

care rezultă din inegalitatea lui Gerretsen: $p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 89

In ΔABC

$$\frac{R}{r}\geq\frac{4\left(m_a^2+m_b^2\right)}{\left(m_a+m_b\right)^2}.$$

Adil Abdullayev, Baku, Azerbaijan, RMM 8/2020

Soluție.

Demonstrăm

Lema1.

In ΔABC

In ΔABC

$$\frac{R}{r}\geq\frac{m_a}{m_b}+\frac{m_b}{m_a}.$$

Adil Abdullayev, Baku, Azerbaijan

Soluție.

Folosim rezultatele:

R 1.

In ΔABC

$$\frac{m_a}{m_b}+\frac{m_b}{m_a}\le 2\cdot\frac{m_c}{h_c}.$$

R2.

In ΔABC

$$\frac{m_a}{h_a}\le\frac{R}{2r}.$$

GM 11/1982, Laurențiu Panaitopol

$$M_s=\frac{m_a}{m_b}+\frac{m_b}{m_a}\overset{R1}{\le}2\cdot\frac{m_c}{h_c}\overset{R2}{\le}2\cdot\frac{R}{2r}=M_d,$$

Lema2.

Dacă $x,y>0$ atunci

$$\frac{x}{y}+\frac{y}{x}\ge\frac{4\left(x^2+y^2\right)}{(x+y)^2}.$$

Demonstrație.

Avem $\frac{x}{y}+\frac{y}{x}\ge\frac{4\left(x^2+y^2\right)}{(x+y)^2}\Leftrightarrow\frac{x^2+y^2}{xy}\ge\frac{4\left(x^2+y^2\right)}{(x+y)^2}\Leftrightarrow(x+y)^2\ge 4xy\Leftrightarrow(x-y)^2\ge 0$,

evident cu egalitate pentru $x=y$.

Punem $x=m_a, y=m_b$ și obținem $\frac{m_a}{m_b}+\frac{m_b}{m_a}\ge\frac{4\left(m_a^2+m_b^2\right)}{(m_a+m_b)^2}$

Să trecem la rezolvarea problemei din enunț.

Folosind **Lemele** de mai sus obținem: $\frac{R}{r}\overset{Lema1}{\ge}\frac{m_a}{m_b}+\frac{m_b}{m_a}\overset{Lema2}{\ge}\frac{4\left(m_a^2+m_b^2\right)}{(m_a+m_b)^2}$.

Aplicația 90

In ΔABC

$$\frac{R}{2r}\ge\sqrt{\frac{\sum m_b m_c}{\sum w_b w_c}}.$$

Adil Abdullayev, Baku, Azerbaijan, RMM 6/2020

Soluție.

Cu $\frac{R}{2r}\ge\frac{m_a}{h_a}$ și inegalitatea $w_a\ge h_a$ rezultă $\frac{R}{2r}\ge\frac{m_a}{h_a}\ge\frac{m_a}{w_a}$, de unde $m_a\le\frac{R}{2r}w_a$.

Obținem $\frac{\sum m_b m_c}{\sum w_b w_c}\le\frac{\sum\frac{R}{2r}w_b\cdot\frac{R}{2r}w_c}{\sum w_b w_c}=\frac{\left(\frac{R}{2r}\right)^2\sum w_b w_c}{\sum w_b w_c}=\left(\frac{R}{2r}\right)^2$, de unde rezultă concluzia.

Aplicația 91

In ΔABC

$$\frac{R}{2r}\ge\sqrt[3]{\frac{\sum m_b m_c}{\sum w_b w_c}}.$$

Marin Chirciu

Soluție.

Folosim inegalitățile :

$w_a \ge h_a$, $\sum x^2 \ge \sum yz$, $\sum m_a^2 = \frac{3}{4}\sum a^2 \overset{Leibniz}{\le} \frac{3}{4}\cdot 9R^2 = \frac{27R^2}{4}$ și $p^2 \overset{Mitrinovic}{\ge} 27r^2$.

Obținem $\frac{\sum m_b m_c}{\sum w_b w_c} \le \frac{\sum m_a^2}{\sum h_b h_c} \le \frac{\frac{27R^2}{4}}{\frac{2rp^2}{R}} = \frac{27R^3}{8rp^2} \le \frac{27R^3}{8r\cdot 27r^2} = \left(\frac{R}{2r}\right)^3$, de unde rezultă concluzia.

Aplicația 92

In ΔABC

$$\frac{R}{2r} \ge \sqrt[3]{\frac{\sum m_b m_c}{\sum w_b w_c}} \ge \sqrt{\frac{\sum m_b m_c}{\sum w_b w_c}}.$$

Marin Chirciu

Soluție.

Vezi $\sqrt[3]{\frac{\sum m_b m_c}{\sum w_b w_c}} \ge \sqrt{\frac{\sum m_b m_c}{\sum w_b w_c}} \Leftrightarrow \frac{\sum m_b m_c}{\sum w_b w_c} \ge 1$, evident din $m_a \ge w_a$.

Aplicația 93

In ΔABC

$$\frac{R}{2r} \ge \sqrt{\frac{m_a m_b + m_b m_c + m_c m_a}{w_a w_b + w_b w_c + w_c w_a}}.$$

Adil Abdullayev, Baku, Azerbaijan, RMM 6/2020

Soluție.

Folosind $h_a \le w_a$ și inegalitatea lui Panaitopol $\frac{h_a}{m_a} \ge \frac{2r}{R}$ obținem:

$\sum m_b m_c \le \sum \frac{R}{2r}h_b \cdot \frac{R}{2r}h_c = \left(\frac{R}{2r}\right)^2 \sum h_b h_c \le \left(\frac{R}{2r}\right)^2 \sum w_b w_c$, de unde:

$$\sum m_b m_c \le \left(\frac{R}{2r}\right)^2 \sum w_b w_c \Leftrightarrow \frac{\sum m_b m_c}{\sum w_b w_c} \le \left(\frac{R}{2r}\right)^2 \Leftrightarrow \sqrt{\frac{\sum m_b m_c}{\sum w_b w_c}} \le \frac{R}{2r} \Leftrightarrow \frac{R}{2r} \ge \sqrt{\frac{\sum m_b m_c}{\sum w_b w_c}}.$$

Aplicația 94

In ΔABC

$$6\left(\frac{R}{r}\right)^{-1} \le \sum \frac{w_a}{m_a} \le 3\sqrt{\frac{R}{2r}}.$$

Kostas Geronikolas, RMM-6/2020

Soluție.

Inegalitatea din dreapta.

$$\sum \frac{w_a}{m_a} \overset{w_a \le m_a}{\le} \sum \frac{m_a}{m_a} = 3 \overset{Euler}{\le} 3\sqrt{\frac{R}{2r}}.$$

Inegalitatea din stânga.

$$\sum \frac{w_a}{m_a} \overset{w_a \ge h_a}{\ge} \sum \frac{h_a}{m_a} \overset{Panaitopol}{\ge} \sum \frac{2r}{R} = \frac{6r}{R} = 6\left(\frac{R}{r}\right)^{-1}.$$

Aplicația 95

In ΔABC

$$\frac{6r}{R} \le \sum \frac{w_a}{m_a} \le 3.$$

Soluție.
Vezi mai sus.
Aplicația 96
In ΔABC

$$\frac{1}{2}\sum \cot\frac{A}{4} \ge \sum \sqrt{\frac{m_a}{r_a}} + \frac{a+b+c}{4r}.$$

Bogdan Fuștei, RMM 3/2020

Soluție.
Lemă.
In ΔABC

$$\sqrt{\frac{m_a}{r_a}} \le \frac{1}{2\sin\frac{A}{2}}.$$

Demonstrație.
Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\frac{m_a}{r_a} \le \frac{R}{2r}\cdot\frac{h_a}{r_a} = \frac{R}{2r}\cdot\frac{\frac{2S}{a}}{\frac{S}{p-a}} = \frac{R}{r}\cdot\frac{p-a}{a} = \frac{1}{4\sin^2\frac{A}{2}}, \text{ de unde } \sqrt{\frac{m_a}{r_a}} \le \frac{1}{2\sin\frac{A}{2}}.$$

Folosind formula $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ rezultă $\cot\frac{A}{4} = \frac{1+\cos\frac{A}{2}}{\sin\frac{A}{2}} = \frac{1}{\sin\frac{A}{2}} + \cot\frac{A}{2}$.

Din identitatea $\sum \cot\frac{A}{2} = \frac{p}{r}$ și relația de mai sus avem:

$$M_s = \frac{1}{2}\sum \cot\frac{A}{4} = \frac{1}{2}\sum\left(\frac{1}{\sin\frac{A}{2}} + \cot\frac{A}{2}\right) = \frac{1}{2}\left(\sum\frac{1}{\sin\frac{A}{2}} + \sum\cot\frac{A}{2}\right) = \frac{1}{2}\left(\sum\frac{1}{\sin\frac{A}{2}} + \frac{p}{r}\right).$$

$$M_d = \sum\sqrt{\frac{m_a}{r_a}} + \frac{a+b+c}{4r} = \sum\sqrt{\frac{m_a}{r_a}} + \frac{p}{2r} \overset{Lema}{\le} \sum\frac{1}{2\sin\frac{A}{2}} + \frac{p}{2r} = \frac{1}{2}\left(\sum\frac{1}{\sin\frac{A}{2}} + \frac{p}{r}\right) = M_x.$$

Aplicația 97
In ΔABC

$$\frac{1}{2}\sum \tan\frac{A}{4} \ge \sum \sqrt{\frac{m_a}{r_a}} - \frac{a+b+c}{4r}.$$

Bogdan Fuștei, RMM 6/2020

Soluție.
Lemă.
In ΔABC

$$\sqrt{\frac{m_a}{r_a}} \le \frac{1}{2\sin\frac{A}{2}}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\frac{m_a}{r_a} \le \frac{R}{2r}\cdot\frac{h_a}{r_a} = \frac{R}{2r}\cdot\frac{\frac{2S}{a}}{\frac{S}{p-a}} = \frac{R}{r}\cdot\frac{p-a}{a} = \frac{1}{4\sin^2\frac{A}{2}}, \text{ de unde } \sqrt{\frac{m_a}{r_a}} \le \frac{1}{2\sin\frac{A}{2}}.$$

Folosind formula $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$ rezultă $\tan\frac{A}{4} = \frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}} = \frac{1}{\sin\frac{A}{2}} - \cot\frac{A}{2}$.

Din identitatea $\sum\cot\frac{A}{2} = \frac{p}{r}$ și relația de mai sus avem:

$$M_s = \frac{1}{2}\sum\tan\frac{A}{4} = \frac{1}{2}\sum\left(\frac{1}{\sin\frac{A}{2}} - \cot\frac{A}{2}\right) = \frac{1}{2}\left(\sum\frac{1}{\sin\frac{A}{2}} - \sum\cot\frac{A}{2}\right) = \frac{1}{2}\left(\sum\frac{1}{\sin\frac{A}{2}} - \frac{p}{r}\right).$$

$$M_d = \sum\sqrt{\frac{m_a}{r_a}} - \frac{a+b+c}{4r} = \sum\sqrt{\frac{m_a}{r_a}} - \frac{p}{2r} \overset{Lema}{\le} \sum\frac{1}{2\sin\frac{A}{2}} - \frac{p}{2r} = \frac{1}{2}\left(\sum\frac{1}{\sin\frac{A}{2}} - \frac{p}{r}\right) = M_x.$$

Aplicația 98

In ΔABC

$$\sum\sqrt{\frac{r_a}{m_a}} \ge 3\sqrt[3]{\frac{2r}{R}}.$$

Marin Chirciu

Soluție.
Lemă.

In ΔABC

$$\sqrt{\frac{r_a}{m_a}} \ge 2\sin\frac{A}{2}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\frac{r_a}{m_a} \ge \frac{2r}{R}\cdot\frac{r_a}{h_a} = \frac{2r}{R}\cdot\frac{\frac{S}{p-a}}{\frac{2S}{a}} = \frac{r}{R}\cdot\frac{a}{p-a} = 4\sin^2\frac{A}{2}, \text{ de unde } \sqrt{\frac{r_a}{m_a}} \ge 2\sin\frac{A}{2}.$$

Folosind **Lema** obținem:

$$\sum\sqrt{\frac{r_a}{m_a}} \ge 2\sum\sin\frac{A}{2} \ge 2\cdot 3\sqrt[3]{\prod\sin\frac{A}{2}} = 2\cdot 3\sqrt[3]{\frac{r}{4R}} = 3\sqrt[3]{\frac{2r}{R}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația 99

In ΔABC

$$\frac{r_a r_b r_c}{m_a m_b m_c} \ge \left(\frac{2r}{R}\right)^2.$$

Marin Chirciu

Soluție.

Lemă.

In ΔABC

$$\frac{r_a}{m_a} \ge 4\sin^2\frac{A}{2}.$$

Demonstrație.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem:

$$\frac{r_a}{m_a} \ge \frac{2r}{R}\cdot\frac{r_a}{h_a} = \frac{2r}{R}\cdot\frac{\frac{S}{p-a}}{\frac{2S}{a}} = \frac{r}{R}\cdot\frac{a}{p-a} = 4\sin^2\frac{A}{2}, \text{ de unde } \frac{r_a}{m_a} \ge 4\sin^2\frac{A}{2}.$$

Folosind **Lema** obținem: $\frac{r_a r_b r_c}{m_a m_b m_c} \ge \prod 4\sin^2\frac{A}{2} = 64\left(\frac{r}{4R}\right)^2 = \left(\frac{2r}{R}\right)^2.$

Aplicația 100

In ΔABC

$$\sum\frac{1}{1-\cos A} \ge 2\sum\frac{m_a}{r_a}.$$

Bogdan Fuștei, Romania, RMM 2/2020

Soluție.

Lema 1.

In ΔABC

$$\sum\frac{1}{1-\cos A} = \frac{p^2+r^2-8Rr}{2r^2}.$$

Soluție.

Folosind formula $\cos x = 1-2\sin^2 x$ obținem:

$$\sum\frac{1}{1-\cos A} = \sum\frac{1}{2\sin^2\frac{A}{2}} = \frac{1}{2}\sum\frac{1}{\sin^2\frac{A}{2}} = \frac{1}{2}\cdot\frac{p^2+r^2-8Rr}{r^2} = \frac{p^2+r^2-8Rr}{2r^2}.$$

Lema 2.

In ΔABC

$$\sum\frac{m_a}{r_a} \le \frac{p^2+r^2-8Rr}{4r^2}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$, (vezi GM 11/1982), obținem:

$$\frac{m_a}{r_a} \le \frac{h_a\cdot\frac{R}{2r}}{r_a} = \frac{R}{2r}\cdot\frac{h_a}{r_a}, \text{ de unde}$$

$$\sum \frac{m_a}{r_a} \le \sum \frac{R}{2r} \cdot \frac{h_a}{r_a} = \frac{R}{2r} \sum \frac{h_a}{r_a} = \frac{R}{2r} \cdot \frac{p^2 + r^2 - 8Rr}{2Rr} = \frac{p^2 + r^2 - 8Rr}{4r^2}.$$

Folosind **Lemele** obținem:

$$M_s = \sum \frac{1}{1 - \cos A} \overset{Lema1}{=} \frac{p^2 + r^2 - 8Rr}{2r^2} \overset{Lema2}{\ge} 2\sum \frac{m_a}{r_a} = M_d.$$

Aplicația 101

In ΔABC

$$\sum \frac{1}{1 - \cos A} \ge 2\sum \frac{m_a}{h_a}.$$

Marin Chirciu, Romania

Soluție.

Lema 1.

In ΔABC

$$\sum \frac{1}{1 - \cos A} = \frac{p^2 + r^2 - 8Rr}{2r^2}.$$

Soluție.

Folosind formula $\cos x = 1 - 2\sin^2 x$ obținem:

$$\sum \frac{1}{1 - \cos A} = \sum \frac{1}{2\sin^2 \frac{A}{2}} = \frac{1}{2} \sum \frac{1}{\sin^2 \frac{A}{2}} = \frac{1}{2} \cdot \frac{p^2 + r^2 - 8Rr}{r^2} = \frac{p^2 + r^2 - 8Rr}{2r^2}.$$

Lema 2.

In ΔABC

$$\sum \frac{m_a}{h_a} \le \frac{3R}{2r}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$ obținem: $\sum \frac{m_a}{h_a} \le \sum \frac{R}{2r} = \frac{3R}{2r}$.

Folosind **Lemele** obținem:

$$M_s = \sum \frac{1}{1 - \cos A} \overset{Lema1}{=} \frac{p^2 + r^2 - 8Rr}{2r^2} \overset{(1)}{\ge} \frac{3R}{r} \overset{Lema2}{\ge} 2\sum \frac{m_a}{h_a} = M_d,$$

unde (1) $\Leftrightarrow \frac{p^2 + r^2 - 8Rr}{2r^2} \ge \frac{3R}{r} \Leftrightarrow p^2 \ge 14Rr - r^2$, care rezultă din inegalitatea lui Gerretsen $p^2 \ge 16Rr - 5r^2$.

Rămâne să arătăm că: $16Rr - 5r^2 \ge 14Rr - r^2 \Leftrightarrow R \ge 2r$, (Euler).

Aplicația 102

In ΔABC

$$\left(\frac{r_a}{s_a}\right)^2 + \left(\frac{r_b}{s_b}\right)^2 + \left(\frac{r_c}{s_c}\right)^2 + n \cdot \frac{2r}{R} \ge n + 3, \text{ unde } n \le 5.$$

Marin Chirciu

Soluție.

Folosind $s_a = \frac{2bc}{b^2 + c^2} m_a \le m_a$ și inegalitatea $\frac{m_a}{h_a} \le \frac{R}{2r}$, (L. Panaitopol, 1982),

obținem $s_a \le \frac{R}{2r} \cdot h_a$, de unde $\frac{r_a}{s_a} \ge \frac{2r}{R} \cdot \frac{r_a}{h_a}$ și de aici:

$$\sum\left(\frac{r_a}{s_a}\right)^2 \geq \sum\left(\frac{2r}{R}\cdot\frac{r_a}{h_a}\right)^2 = \left(\frac{2r}{R}\right)^2\sum\left(\frac{r_a}{h_a}\right)^2 = \frac{4r^2}{R^2}\cdot\frac{8R^2+r^2-p^2}{2r^2} = \frac{2\left(8R^2+r^2-p^2\right)}{R^2}$$, vezi

Lemă.

In ΔABC

$$\sum\left(\frac{r_a}{h_a}\right)^2 = \frac{8R^2+r^2-p^2}{2r^2}.$$

Demonstrație.

Folosind $h_a = \frac{2S}{a}$ și $r_a = \frac{S}{p-a}$ obținem:

$$\sum\left(\frac{r_a}{h_a}\right)^2 = \sum\left(\frac{\frac{S}{p-a}}{\frac{2S}{a}}\right)^2 = \frac{1}{4}\sum\left(\frac{a}{p-a}\right)^2 = \frac{1}{4}\cdot\frac{2\left(8R^2+r^2-p^2\right)}{r^2} = \frac{8R^2+r^2-p^2}{2r^2}, (1).$$

Folosind (1) este suficient să arătăm că:

$\frac{2\left(8R^2+r^2-p^2\right)}{R^2} + n\cdot\frac{2r}{R} \geq n+3 \Leftrightarrow (13-n)R^2+2nRr+2r^2 \geq 2p^2$, care rezultă din inegalitatea lui Gerretsen: $p^2 \leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$(13-n)R^2+2nRr+2r^2 \geq 2\left(4R^2+4Rr+3r^2\right) \Leftrightarrow (5-n)R^2+(2n-8)Rr-4r^2 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)\left[(5-n)R+2r\right] \geq 0$, evident din inegalitatea lui Euler

$R \geq 2r$ și condiția din ipoteză $n \leq 5$, care asigură $\left[(5-n)R+2r\right] \geq 0$.

Aplicația 104

In ΔABC

$$\left(\frac{r_a}{s_a}\right)^2+\left(\frac{r_b}{s_b}\right)^2+\left(\frac{r_c}{s_c}\right)^2+\frac{10r}{R} \geq 8.$$

Soluție.

Folosind inegalitatea (1) de mai sus $\sum\left(\frac{r_a}{s_a}\right)^2 \geq \frac{2\left(8R^2+r^2-p^2\right)}{R^2}$ este suficient să arătăm că:

$\frac{2\left(8R^2+r^2-p^2\right)}{R^2} + \frac{10r}{R} \geq 8 \Leftrightarrow 4R^2+5Rr+r^2 \geq p^2$, care rezultă din inegalitatea lui Gerretsen: $p^2 \leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că: $4R^2+5Rr+r^2 \geq 4R^2+4Rr+3r^2 \Leftrightarrow R \geq 2r$,(inegalitatea lui Euler).

Aplicația 105

In ΔABC

$$\frac{r_a}{s_a}+\frac{r_b}{s_b}+\frac{r_c}{s_c}+n\cdot\frac{2r}{R} \geq n+3, \text{ unde } n \leq 1.$$

Marin Chirciu

Soluție.

Folosind identitatea $s_a = \frac{2bc}{b^2+c^2} m_a$ și inegalitatea $\frac{m_a}{h_a} \le \frac{R}{2r}$, (L. Panaitopol, 1982),

obținem $m_a \le \frac{R}{2r} \cdot h_a \Leftrightarrow s_a \le \frac{bc}{b^2+c^2} \cdot \frac{R}{r} \cdot h_a$, de unde $\frac{r_a}{s_a} \ge \frac{b^2+c^2}{bc} \cdot \frac{r}{R} \cdot \frac{r_a}{h_a}$ și de aici:

$$\sum \frac{r_a}{s_a} \ge \sum \frac{b^2+c^2}{bc} \cdot \frac{r}{R} \cdot \frac{r_a}{h_a} = \frac{r}{R} \sum \frac{b^2+c^2}{bc} \cdot \frac{\frac{S}{p-a}}{\frac{2S}{a}} = \frac{r}{R} \sum \frac{b^2+c^2}{bc} \cdot \frac{a}{p-a} = \frac{r}{R} \cdot \frac{p^2+r^2-8Rr}{r^2} =$$

$= \frac{p^2+r^2-8Rr}{2Rr}$, care rezultă din: $\sum \frac{b^2+c^2}{bc} \cdot \frac{a}{p-a} = \frac{p^2+r^2-8Rr}{2Rr}$

Este suficient să arătăm că:

$\frac{16Rr-5r^2+r^2-8Rr}{2Rr} + n \cdot \frac{2r}{R} \ge n+3$,vezi inegalitatea lui Gerretsen $p^2 \ge 16Rr - 5r^2$.

Rămâne să arătăm că:

$\frac{\left(16Rr-5r^2\right)+r^2-8Rr}{2Rr} + n \cdot \frac{2r}{R} \ge n+3 \Leftrightarrow 8Rr + 4r^2(n-1) \ge 2Rr(n+3) \Leftrightarrow$

$\Leftrightarrow (R-2r)(1-n) \ge 0$, evident din inegalitatea lui Euler $R \ge 2r$ și condiția din ipoteză $n \le 1$.

<u>Aplicația 106</u>

In ΔABC

$$\frac{r_a}{s_a} + \frac{r_b}{s_b} + \frac{r_c}{s_c} + \frac{2r}{R} \ge 4 .$$

<u>Soluție.</u>

Folosind inegalitatea de mai sus $\sum \frac{r_a}{s_a} \ge \frac{p^2+r^2-8Rr}{2Rr}$ este suficient să arătăm că:

$\frac{p^2+r^2-8Rr}{2Rr} + \frac{2r}{R} \ge 4 \Leftrightarrow p^2 \ge 16Rr - 5r^2$,(inegalitatea lui Gerretsen).

<u>Remarcă.</u>

În toate cazurile de mai sus egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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Art 3646
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3. Asupra problemei 28089 din G.M.B nr.6-7-8/2021

George-Florin Şerban
Colegiul Naţional ”D.P.Perpessicius”,Brăila

În Gazeta Matematică Seria B numărul 6-7-8,domnul profesor Neculai Stanciu,propune următoarea problemă:
28089.Demonstraţi că pentru orice $a,b,c>0$ are loc

inegalitatea: $8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^2\geq 162+81\cdot\sum_{ciclică}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right).$

Vom prezenta în continuare o metodă de rezolvare în care vom folosi formula lui Taylor,pentru un polinom cu două variabile şi o generalizare a acestei probleme.

Notăm $\frac{a}{b}=x,\frac{b}{c}=y,\frac{c}{b}=z,$

$x,y,z>0,\ xyz=1,\ p=x+y+z,\ q=xy+yz+xz,\ r=xyz=1.$ Atunci inegalitatea devine

$$8(x+y+z)^2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\geq 162+81\cdot\sum_{ciclică}\left(x^2+\frac{1}{x^2}\right),$$

$$\frac{8(x+y+z)^2(xy+yz+xz)^2}{(xyz)^2}\geq 162+81\sum_{ciclică}x^2+81\sum_{ciclică}y^2z^2,$$

$8p^2q^2\geq 162+81(p^2-2q)+81(q^2-2pr),\ 8p^2q^2\geq 162+81p^2-162q+81q^2-162p.$ Trebuie să demonstrăm inegalitatea $8p^2q^2-81p^2-81q^2+162p+162q-162\geq 0.$ Fie polinomul $f(p,q)=8p^2q^2-81p^2-81q^2+162p+162q-162.$ Aplicăm formula lui Taylor pentru un polinom de două variabile.

$$\frac{\partial f}{\partial p}=16pq^2-162p+162,\ \frac{\partial f}{\partial p}(3,3)=16\cdot 27-162\cdot 2=432-324=108,$$

$$\frac{\partial f}{\partial q}=16p^2q-162q+162,\ \frac{\partial f}{\partial q}(3,3)=108,\ \frac{\partial^2 f}{\partial p^2}=16q^2-162,\ \frac{\partial^2 f}{\partial p^2}(3,3)=144-162=-18,$$

$$\frac{\partial^2 f}{\partial q^2}=16p^2-162,\ \frac{\partial^2 f}{\partial q^2}(3,3)=-18,\ \frac{\partial^2 f}{\partial p\partial q}=32pq,\ \frac{\partial^2 f}{\partial p\partial q}(3,3)=288,$$

$$\frac{\partial^3 f}{\partial p^3}=\frac{\partial^3 f}{\partial q^3}=\frac{\partial^4 f}{\partial p^4}=\frac{\partial^4 f}{\partial q^4}=0,\ \frac{\partial^3 f}{\partial p^2\partial q}=32q,\ \frac{\partial^3 f}{\partial p^2\partial q}(3,3)=96,\ \frac{\partial^3 f}{\partial p\partial q^2}=32p,\ \frac{\partial^3 f}{\partial p\partial q^2}(3,3)=96,$$

$$\frac{\partial^4 f}{\partial p\partial q^3}=\frac{\partial^4 f}{\partial p^3\partial q}=0,\ \frac{\partial^4 f}{\partial p^2\partial q^2}=32,\ \frac{\partial^4 f}{\partial p^2\partial q^2}(3,3)=32,\ f(3,3)=648-1458+972-162=0,$$

$$f(p,q)=f(3,3)+\frac{1}{1!}\left[(p-3)\frac{\partial f}{\partial p}(3,3)+(q-3)\frac{\partial f}{\partial q}(3,3)\right]+\frac{1}{2!}\left[(p-3)^2\frac{\partial^2 f}{\partial p^2}(3,3)+2(p-3)(q-3)\frac{\partial^2 f}{\partial p\partial q}(3,3)+\right.$$

$$\left.+(q-3)^2\frac{\partial^2 f}{\partial q^2}(3,3)\right]+\frac{1}{3!}\left[(p-3)^3\frac{\partial^3 f}{\partial p^3}(3,3)+3(p-3)^2(q-3)\frac{\partial^3 f}{\partial p^2\partial q}(3,3)+3(p-3)(q-3)^2\frac{\partial^3 f}{\partial p\partial q^2}(3,3)+\right.$$

$$\left.+(q-3)^3\frac{\partial^3 f}{\partial q^3}(3,3)\right]+\frac{1}{4!}\left[(p-3)^4\frac{\partial^4 f}{\partial p^4}(3,3)+4(p-3)^3(q-3)\frac{\partial^4 f}{\partial p^3\partial q}(3,3)+6(p-3)^2(q-3)^2\frac{\partial^4 f}{\partial p^2\partial q^2}(3,3)+\right.$$

$+4(p-3)(q-3)^3\dfrac{\partial^4 f}{\partial p\partial q^3}(3,3)+(q-3)^4\dfrac{\partial^4 f}{\partial q^4}(3,3)].$

$f(p,q)=108(p-3)+108(q-3)-9(p-3)^2+288(p-3)(q-3)-9(q-3)^2+48(p-3)^2(q-3)+$
$+48(p-3)(q-3)^2+8(p-3)^2(q-3)^2.$ Vom demonstra inegalitatea 1,

$108(p-3)-9(p-3)^2+144(p-3)(q-3)+24(p-3)^2(q-3)+24(p-3)(q-3)^2+4(p-3)^2(q-3)^2\geq 0.$

$(p-3)[108-9p+27+144q-432+24pq-72p-72q+216+24q^2-144q+216+(4p-12)(q^2-6q+9)]\geq 0,$

$(p-3)[135-81p+24pq-72q+24q^2+4pq^2-24pq+36p-12q^2+72q-108]\geq 0,$

$(p-3)[12q^2+4pq^2-45p+27]\geq 0.$ Aplicăm inegalitatea mediilor, $M_a\geq M_g$, $p=x+y+z\geq 3\sqrt[3]{xyz}=3$, $q=xy+yz+xz\geq 3\sqrt[3]{xyyzxz}=3\sqrt[3]{(xyz)^2}=3$, rez ultă $p,q\geq 3$. Aplicăm consecinţele inegalităţii lui Schur, $p^2\geq 3q$, $q^2\geq 3pr$, $r=1$ deci $q^2\geq 3p$, evident $p-3\geq 0$, iar

$12q^2+4pq^2-45p+27\geq 12\cdot 3p+4p\cdot 9-45p+27=36p+36p-45p+27=27p+27>0,$ re zultă inegalitatea 1. Vom demonstra inegalitatea 2,

$108(q-3)-9(q-3)^2+144(p-3)(q-3)+24(p-3)^2(q-3)+24(p-3)(q-3)^2+4(p-3)^2(q-3)^2\geq 0.$

$(q-3)[108-9q+27+144p-432+24p^2-144p+216+24pq-72p-72q+216+(4q-12)(p^2-6p+9)]\geq 0,$

$(q-3)[135-81q+24p^2+24pq-72p+4p^2q-24pq+36q-12p^2+72p-108]\geq 0,$

$(q-3)[12p^2+4p^2q-45q+27]\geq 0,$ evident $q-3\geq 0,$ iar

$12p^2+4p^2q-45q+27\geq 12\cdot 3q+4\cdot 3q\cdot 3-45q+27=36q+36q-45q+27=27q+27>0,$

rezultă inegalitatea 2. Adunăm inegalităţile 1 şi 2 şi se obţine inegalitatea cerută, $f(p,q)\geq 0.$
Avem egalitate dacă şi numai dacă $p=q=3,$ dacă şi numai dacă $a=b=c.$

Generalizare. Demonstraţi că pentru orice $a,b,c,\alpha,\beta>0$ cu $81\alpha=\beta+6\gamma$, $\gamma\in R$, are loc inegalitatea: $\alpha(\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a})^2(\dfrac{b}{a}+\dfrac{c}{b}+\dfrac{a}{c})^2\geq\beta+\gamma\cdot\sum_{ciclică}(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}).$ Cu aceleaşi notaţii ca mai sus, obţinem $\alpha p^2q^2-\gamma(p^2-2q+q^2-2p)-\beta\geq 0.$

$$\frac{\alpha p^2(q^2-9)}{2}+\frac{\alpha q^2(p^2-9)}{2}+\frac{9\alpha p^2}{2}+\frac{9\alpha q^2}{2}-\gamma p^2-\gamma q^2+2p\gamma+2q\gamma-\beta\geq 0,$$

$\dfrac{\alpha p^2(q^2-9)}{2}+\dfrac{\alpha q^2(p^2-9)}{2}-[(\gamma-\dfrac{9\alpha}{2})p^2-2p\gamma+\dfrac{\beta}{2}]-[(\gamma-\dfrac{9\alpha}{2})q^2-2q\gamma+\dfrac{\beta}{2}]\geq 0.$ Arătăm că $(\gamma-\dfrac{9\alpha}{2})p^2-2p\gamma+\dfrac{\beta}{2}=(p-3)[(\gamma-\dfrac{9\alpha}{2})p-\dfrac{\beta}{6}]=(\gamma-\dfrac{9\alpha}{2})p^2+p(-3\gamma+\dfrac{27\alpha}{2}-\dfrac{\beta}{6})+\dfrac{\beta}{2}=$

$=(\gamma-\dfrac{9\alpha}{2})p^2+p(-3\gamma+\dfrac{81\alpha-\beta}{6})+\dfrac{\beta}{2}=(\gamma-\dfrac{9\alpha}{2})p^2+p(-3\gamma+\gamma)+\dfrac{\beta}{2}=(\gamma-\dfrac{9\alpha}{2})p^2-2p\gamma+\dfrac{\beta}{2},$

adevărat. Analog arătăm că

$(\gamma-\dfrac{9\alpha}{2})q^2-2q\gamma+\dfrac{\beta}{2}=(q-3)[(\gamma-\dfrac{9\alpha}{2})q-\dfrac{\beta}{6}]=$

$=(\gamma-\dfrac{9\alpha}{2})q^2+q(-3\gamma+\dfrac{27\alpha}{2}-\dfrac{\beta}{6})+\dfrac{\beta}{2}=(\gamma-\dfrac{9\alpha}{2})q^2-2q\gamma+\dfrac{\beta}{2},$ adevărat. Rezultă

$$\frac{\alpha p^2(q-3)(q+3)}{2}+\frac{\alpha q^2(p-3)(p+3)}{2}-(p-3)[(\gamma-\frac{9\alpha}{2})p-\frac{\beta}{6}]-(q-3)[(\gamma-\frac{9\alpha}{2})q-\frac{\beta}{6}]\geq 0,$$

$$(p-3)[\frac{\alpha pq^2}{2}+\frac{3\alpha q^2}{2}-(\gamma-\frac{9\alpha}{2})p+\frac{\beta}{6}]+(q-3)[\frac{\alpha p^2q}{2}+\frac{3\alpha p^2}{2}-(\gamma-\frac{9\alpha}{2})q+\frac{\beta}{6}]\geq 0.$$

$$\frac{\alpha pq^2}{2}+\frac{3\alpha q^2}{2}-(\gamma-\frac{9\alpha}{2})p+\frac{\beta}{6}\geq\frac{9\alpha p}{2}+\frac{3\alpha\cdot 3p}{2}-(\gamma-\frac{9\alpha}{2})p+\frac{\beta}{6}=(\frac{27\alpha}{2}-\gamma)p+\frac{\beta}{6}=\frac{\beta p}{6}+\frac{\beta}{6}>0,$$

iar $p-3\geq 0$ rezultă $(p-3)[\frac{\alpha pq^2}{2}+\frac{3\alpha q^2}{2}-(\gamma-\frac{9\alpha}{2})p+\frac{\beta}{6}]\geq 0.$ Analog $q-3\geq 0$ şi

$$\frac{\alpha p^2 q}{2}+\frac{3\alpha p^2}{2}-(\gamma-\frac{9\alpha}{2})\mathrm{q}+\frac{\beta}{6}\geq\frac{9\alpha q}{2}+\frac{3\alpha\cdot 3q}{2}-(\gamma-\frac{9\alpha}{2})\mathrm{q}+\frac{\beta}{6}=(\frac{27\alpha}{2}-\gamma)\mathrm{q}+\frac{\beta}{6}=\frac{\beta q}{6}+\frac{\beta}{6}>0,$$

rezultă $(\mathrm{q}-3)[\frac{\alpha p^2 q}{2}+\frac{3\alpha p^2}{2}-(\gamma-\frac{9\alpha}{2})\mathrm{q}+\frac{\beta}{6}]\geq 0.$ Adunăm cele două inegaltăţi şi obţinem

$$(p-3)[\frac{\alpha pq^2}{2}+\frac{3\alpha q^2}{2}-(\gamma-\frac{9\alpha}{2})p+\frac{\beta}{6}]+(\mathrm{q}-3)[\frac{\alpha p^2 q}{2}+\frac{3\alpha p^2}{2}-(\gamma-\frac{9\alpha}{2})\mathrm{q}+\frac{\beta}{6}]\geq 0.$$

Problema 28089. se obţine pentru $\alpha=\frac{8}{81}>0, \beta=2>0, \gamma=1\in\mathrm{R}$ şi $81\alpha=8=2+6\cdot 1=\beta+6\gamma.$

Avem egalitate dacă şi numai dacă $p=q=3,$ dacă şi numai dacă $a=b=c.$

4. About Martin Lukarevski double inequality for excentral triangle

By Marian Dincă, Bucharest, România

In the mathematical note [1] Martin Lukarevski proves
the following double inequality:
$4R+10r \le AI_a+BI_b+CI_c \le 8R+2r$

First, I remind readers how we determine the elements of the triangle $I_aI_bI_c$
from the cyclic quadrilateral AI_cBI

$$\sphericalangle AI_cB=\sphericalangle AI_cI+\sphericalangle BI_cI=\sphericalangle ABI+\sphericalangle BAI=\frac{A}{2}+\frac{B}{2}=\frac{A+B}{2}=\sphericalangle I_bI_cI_a$$

in the same way we determine the angles:

$$\sphericalangle I_bI_aI_c=\frac{B+C}{2},\sphericalangle I_aI_bI_c=\frac{A+C}{2}$$

we intend to find the radius of the circle circumscribed by the triangle

the quadrilateral ABI_aI_b is cyclic,

the circumscribed circle has a diameter I_aI_b

$$\sphericalangle AIB = \frac{AB + I_aI_b}{2} = 180 - \frac{A+B}{2}$$

because $I_aI_b = 180$, result: $\dfrac{\overline{AB}}{2} = 90 - \dfrac{A+B}{2}$

$$AB = I_aI_b \sin\left(\frac{\overline{AB}}{2}\right) = I_aI_b \cos\left(\frac{A+B}{2}\right)$$

but $AB = 2R\sin C = 2R\sin(A+B) = 4R\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A+B}{2}\right)$

we obtain: $I_aI_b = 4R\sin\left(\dfrac{A+B}{2}\right)$

this proves that the radius of the circumscribed circle of the triangle $I_aI_bI_c$ is equal to $2R$

the segment AI_a is height in the triangle $I_aI_bI_c$

$$AI_a = 4R\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right) = 4R\cos\frac{C}{2}\cos\frac{B}{2}$$

$$BI_b = 4R\cos\frac{C}{2}\cos\frac{A}{2}, CI_c = 4R\cos\frac{B}{2}\cos\frac{A}{2}$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

double inequality will be equivalent to:

$$4R + 10\cdot 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le 4R\cos\frac{C}{2}\cos\frac{B}{2} + 4R\cos\frac{C}{2}\cos\frac{A}{2} + 4R\cos\frac{B}{2}\cos\frac{A}{2} \le$$

$$\le 8R + 2\cdot 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \text{or}$$

$$1 + 10\cdot\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \cos\frac{C}{2}\cos\frac{B}{2} + \cos\frac{C}{2}\cos\frac{A}{2} + \cos\frac{B}{2}\cos\frac{A}{2} \le$$

$$\le 2 + 2\cdot\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \qquad (2)$$

in inequality (2) making the following changes:

$$\frac{\pi}{2} - \frac{A}{2} = \alpha, \frac{\pi}{2} - \frac{B}{2} = \beta, \frac{\pi}{2} - \frac{C}{2} = \gamma$$

where α, β, γ is angles of acute triangle

the inequality is echivalent to:

$1+10\cos\alpha\cos\beta\cos\gamma \le \sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha \le$

$\le 2+2\cos\alpha\cos\beta\cos\gamma \qquad (3)$

I will prove that the double inequality (3) which is for the acute triangle

it is true for any triangle

use the identity:

$$\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$$

$$\sin A\sin B+\sin B\sin C+\sin C\sin A = \frac{ab+bc+ca}{4R^2} = \frac{s^2+r^2+4rR}{4R^2}$$

the inequality on the left will be written:

$$1+10\cdot\frac{s^2-(2R+r)^2}{4R^2} \le \frac{s^2+r^2+4Rr}{4R^2}, \text{or}$$

$$9s^2 \le 10(2R+r)^2+r^2+4Rr-4R^2 = 36R^2+44Rr+11r^2$$

$$s^2 \le 4R^2+\frac{44Rr+11r^2}{9}$$

use Gerretsen inequality:

$$s^2 \le 4R^2+4Rr+3r^2$$

and: $4R^2+4Rr+3r^2 \le 4R^2+\dfrac{44Rr+11r^2}{9}$, is echivalent to:

$9(4Rr+3r^2) \le 44Rr+11r^2$, or $16r^2 \le 8Rr, 2r \le R$, Euler's inequality

the inequality on the right will be written:

$$\frac{s^2+r^2+4Rr}{4R^2} \le 2+2\cdot\frac{s^2-(2R+r)^2}{4R^2}, \text{or}$$

$$r^2+4Rr \le 8R^2+s^2-2(2R+r)^2$$

$$12Rr+3r^2 \le s^2$$

use Gerretsen inequality:

$$s^2 \ge 16Rr-5r^2$$

$16Rr-5r^2 \ge 12Rr+3r^2$, echivalent to:

$4Rr \ge 8r^2,\ \ R \ge 2r$, Euler's inequality

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5. O generalizare şi aplicaţii ale inegalităţii lui Nesbitt

de Gheorghe Ghiţă, Buzău

Articolul prezintă o generalizare a inegalităţii lui Nesbitt, precum şi utilizarea acesteia în demonstrarea unor inegalităţi, în special pentru triunghi.

Vom utiliza următoarele identităţi şi inegalităţi(cu notaţiile uzuale în triunghi):
(1) $abc = 4RS = 4Rpr$; (2) $\sum ab = p^2 + r^2 + 4Rr$; (3) $\sum a^2 = 2p^2 - 2r^2 - 8Rr$; (4) $\sum \frac{1}{a} = \frac{p^2+r^2+4Rr}{4RS}$; (5) $\sum r_a = 4R + r$; (6) $\sum \frac{1}{h_a} = \sum \frac{1}{r_a} = \frac{1}{r}$; (7) $\sum h_a = \frac{p^2+r^2+4Rr}{2R}$; (8) $\sum r_a r_b = p^2$ (9) $\sum a\, r_a = 2p(2R - r)$; (10) $\sum sinA sinB = \frac{p^2+r^2+4Rr}{4R^2}$; (11) $\sum acosA = \frac{2pr}{R}$; (12) $\sum tg\frac{A}{2} = \frac{4R+r}{p}$; (13) $\sum ctg\frac{A}{2} = \frac{p}{r}$; (14) $\frac{1}{r_b} + \frac{1}{r_c} = \frac{2}{h_a}$; (15) $\sum \frac{a}{p-a} = \frac{2(2R-r)}{r}$; (16) $\sum \frac{(p-a)(p-b)}{c} = \frac{r(p^2+(4R+r)^2)}{4pR}$;

(C)– Cebâşev: $(a_1, a_2, \ldots, a_n)$ şi $(x_1, x_2, \ldots, x_n)$ pozitive, la fel ordonate, atunci $\sum_{k=1}^n a_k x_k \geq \frac{1}{n}\sum_{k=1}^n a_k \sum_{k=1}^n x_k$;

(R)- Radon: $m \geq 0, a_1, a_2, \ldots, a_n \geq 0; x_1, x_2, \ldots, x_n > 0 \Rightarrow \sum_{k=1}^n \frac{a_k^m}{x_k^{m-1}} \geq \frac{(\sum_{k=1}^n a_k)^m}{(\sum_{k=1}^n x_k)^{m-1}}$;

(B) – Bergström: $a_1, a_2, \ldots, a_n \geq 0; x_1, x_2, \ldots, x_n > 0$, $\sum_{k=1}^n \frac{a_k^2}{x_k} \geq \frac{(\sum_{k=1}^n a_k)^2}{\sum_{k=1}^n x_k}$;

(G) - Gerretsen: $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$; (M)-Mitrinovic: $3\sqrt{3}r \leq p \leq \frac{3\sqrt{3}R}{2}$ (E) - Euler: $R \geq 2r$;

(P) – Panaitopol: $\frac{m_a}{h_a} \leq \frac{R}{2r}$;

(17) $\sum sin\frac{A}{2} \geq \frac{3}{2}$ (18) $\sum m_a w_a \geq p^2$; (19) $\sum asin\frac{A}{2} \geq p$; (20) $\sum am_a \geq 3Rp$; (21) $m_a \geq \sqrt{p(p-a)}$;

Lemă.$\Delta ABC \Rightarrow p^2 + r^2 + 4Rr \geq 18Rr$.

Demonstraţie. Inegalitatea rezultă din

$p^2 + r^2 + 4Rr \overset{(G)}{\geq} 16Rr - 5r^2 + r^2 + 4Rr = 20Rr - 4r^2$ şi $20Rr - 4r^2 \geq 18Rr \Leftrightarrow R \overset{(E)}{\geq} 2r$.

Propoziţie. Dacă $a_1, a_2, \ldots, a_n, m, p \geq 0$ atunci: $\sum \frac{a_1^m}{a_2^p + a_3^p + \ldots + a_n^p} \geq \frac{(\sum a_1)^m}{n^{m-2}(n-1)(\sum a_1^p)}$, unde $n \in N, n \geq 2$.

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Demonstraţie. n-upletele $(a_1^m, a_2^m, \ldots, a_n^m)$ şi $\left(\frac{1}{\sum a_1^p - a_1^p}, \frac{1}{\sum a_1^p - a_2^p}, \ldots, \frac{1}{\sum a_1^p - a_n^p}\right)$ sunt la fel ordonate, şi atunci: $\sum \frac{a_1^m}{\sum a_1^p - a_1^p} = \sum a_1^m \cdot \frac{1}{\sum a_1^p - a_1^p} \overset{(C)}{\geq} \frac{1}{n}\sum a_1^m \sum \frac{1}{\sum a_1^p - a_1^p} = \frac{1}{n}\sum \frac{a_1^m}{1^{m-1}} \sum \frac{1^2}{\sum a_1^p - a_1^p}$

$\overset{(R,B)}{\geq} \frac{1}{n} \cdot \frac{(\sum a_1)^m}{(\sum 1)^{m-1}} \cdot \frac{(\sum 1)^2}{\sum(\sum a_1^p - a_1^p)} = \frac{1}{n} \cdot \frac{(\sum a_1)^m}{n^{m-1}} \cdot \frac{n^2}{n\sum a_1^p - \sum a_1^p} = \frac{(\sum a_1)^m}{n^{m-2}(n-1)\sum a_1^p}$.

Egalitatea are loc dacă şi numai dacă $a_1 = a_2 = \ldots = a_n$.

Observaţii.

1) Pentru $n=3, m=p=1$ avem inegalitatea clasică a lui Nesbit: $x,y,z>0$ atunci:

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2};$$

2) Pentru $m=p=1$, obţinem inegalitatea lui Nesbitt pentru n variabile:

$$\sum\frac{a_1}{a_2+a_3+\dots+a_n}\geq\frac{n}{n-1};$$

3) Pentru $n=3, p=1$ atunci: $\frac{a_1^m}{a_2+a_3}+\frac{a_2^m}{a_3+a_1}+\frac{a_3^m}{a_1+a_2}\geq\frac{3}{2}\left(\frac{\sum a_1}{3}\right)^{m-1}, m\geq 0.$ (∗)

Notă. In inegalităţile referitoare la triunghi, egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Aplicaţii.

A1) Let a, b, c be three positive numbers such that ab+bc+ca = 6abc. For all positive integers n ≥ 2, show that

$$\frac{bc}{a^n(b+c)}+\frac{ca}{b^n(c+a)}+\frac{ab}{c^n(a+b)}\geq 3\cdot 2^{n-2},$$

problema 3981, Crux Mathematicorum,Vol 40,No 9/2014,by Jose Luis Diaz-Barrero.

Soluţie. $\sum\frac{bc}{a^n(b+c)}=\sum\frac{\left(\frac{1}{a}\right)^n}{\frac{1}{b}+\frac{1}{c}}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum\frac{1}{a}}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{\frac{\sum ab}{abc}}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{6}{3}\right)^{n-1}=3\cdot 2^{n-2}$.

A2) Let ABC be a triangle and m_a, m_b, m_c the lengths of the medians. Then following inequality holds:

$\frac{m_a^3}{m_b^2+m_c^2}+\frac{m_a^3}{m_b^2+m_c^2}+\frac{m_a^3}{m_b^2+m_c^2}\geq\frac{\sqrt{3}}{2}s$, where s is the semiperimeter of ΔABC. When do we have equality?

problema IX. 58. , RMM, nr. 27/2020 de Gheorghe Alexe şi George-Florin Şerban-Romania

Soluţie. $\sum\frac{m_a^3}{m_b^2+m_c^2}=\sum\frac{(m_a^2)^{\frac{3}{2}}}{m_b^2+m_c^2}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum m_a^2}{3}\right)^{\frac{1}{2}}=\frac{\sqrt{3}}{2}\sqrt{\sum m_a^2}=\frac{\sqrt{3}}{2}\sqrt{\sum\frac{2b^2+2c^2-a^2}{4}}=\frac{\sqrt{3}}{2}\sqrt{\frac{3}{4}\sum a^2}$

$\geq\frac{\sqrt{3}}{2}s\Leftrightarrow\frac{3}{4}\sum a^2\geq s^2\Leftrightarrow 3\sum a^2\geq\left(\sum a\right)^2\Leftrightarrow 3\sum a^2\geq\sum a^2+2\sum ab\Leftrightarrow\sum a^2\geq\sum ab.$

Adevărată. Egalitatea are loc pentru triunghiul echilateral.

A3) ΔABC$\Rightarrow\sum\frac{r_a^m}{r_b+r_c}\geq\frac{(4R+r)^{m-1}}{2\cdot 3^{m-2}}, m\geq 0.$

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Soluţie. $\sum\frac{r_a^m}{r_b+r_c}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum r_a}{3}\right)^{m-1}\overset{(5)}{=}\frac{3}{2}\left(\frac{4R+r}{3}\right)^{m-1}=\frac{(4R+r)^{m-1}}{2\cdot 3^{m-2}}$.

Notă. Pentru $m=3$ rezultă $\sum\frac{r_a^m}{r_b+r_c}\geq\frac{(4R+r)^2}{6}$, care este o rafinare a inegalităţii din problema X.2., RMM, nr.21/2018, de D.M.Bătineţu-Giurgiu; Neculai Stanciu- Romania: $\frac{r_a^3}{r_b+r_c}+\frac{r_b^3}{r_c+r_a}+\frac{r_c^3}{r_a+r_b}\geq\frac{\left((4R+r)^2-2p^2\right)^2}{2p^2}$, pentru că $\frac{(4R+r)^2}{6}\geq\frac{\left((4R+r)^2-2p^2\right)^2}{2p^2}$, adevărată, din $\sqrt{3}(4R+r)^2-p(4R+r)-2\sqrt{3}p^2\leq 0\Leftrightarrow\sqrt{3}\left(4R+r-\sqrt{3}p\right)\left(4R+r+\frac{2\sqrt{3}p}{3}\right)\leq 0\Leftrightarrow$

$4R+r\leq\sqrt{3}p$ şi $4R+r\overset{(E)}{\leq}\frac{9R}{2}\overset{(M)}{\leq}\sqrt{3}p.$

A4) $\Delta ABC\Rightarrow\sum\frac{a^n(ka+b+c)}{b+c}\geq\frac{2^{n-1}(k+2)p^n}{3^{n-1}}, k\geq 1, n\geq 0.$

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Soluţie. Avem: $\sum\frac{a^n(ka+b+c)}{b+c}=\sum\frac{a^n}{b+c}\cdot(ka+b+c)$; tripletele $\left(\frac{a^n}{b+c},\frac{b^n}{c+a},\frac{c^n}{a+b}\right)$ şi $(ka+b+c,a+kb+c,a+b+kc)$ sunt la fel ordonate, şi atunci $\sum\frac{a^n(ka+b+c)}{b+c}\overset{(C)}{\geq}$ $\frac{1}{3}\sum\frac{a^n}{b+c}\sum(ka+b+c)\overset{(*)}{\geq}\frac{1}{3}\cdot\frac{3}{2}\left(\frac{\sum a}{3}\right)^{n-1}(k+2)\sum a=\frac{1}{2}(k+2)\left(\frac{2p}{3}\right)^n=\frac{2^{n-1}(k+2)p^n}{3^{n-1}}$.

A5) $\Delta ABC\Rightarrow\sum\frac{r_a^n(a+b)(a+c)}{r_b+r_c}\geq\ 8(3r)^{n+1}$, n$\geq 0$.

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Soluţie.Tripletele $\left(\frac{r_a^n}{r_b+r_c},\frac{r_a^n}{r_b+r_c},\frac{r_a^n}{r_b+r_c}\right)$ şi $\big((a+b)(a+c),(b+c)(b+a),(c+a)(c+b)\big)$ sunt la fel ordonate, şi atunci: $\sum\frac{r_a^n(a+b)(a+c)}{r_b+r_c}\overset{(C)}{\geq}\frac{1}{3}\sum\frac{r_a^n}{r_b+r_c}\sum(a+b)(a+c)\overset{(*)}{\geq}$ $\frac{1}{3}\cdot\frac{3}{2}\left(\frac{\sum r_a}{3}\right)^{n-1}\sum(a^2+ac+ab+bc)\overset{(5)}{=}\frac{1}{2}\left(\frac{4R+r}{3}\right)^{n-1}(\sum a^2+3\sum ab)\overset{(E)}{\geq}\frac{1}{2}$ $(3r)^{n-1}(\sum a^2+3\sum ab)\ \overset{(2,3)}{=}\frac{1}{2}(3r)^{n-1}(p^2+r^2+4Rr+4p^2)\ \overset{Lemă}{\geq}\ \frac{1}{2}$ $(18Rr+4p^2)(3r)^{n-1}\overset{(M)}{\geq}\ \frac{1}{2}(18Rr+27R^2)\ (3r)^{n-1}\overset{(E)}{\geq}\ 72r^2(3r)^{n-1}=\ 8(3r)^{n+1}$.

A6) $\Delta ABC\Rightarrow\ \sum\frac{a^m}{b^2+c^2}\geq\frac{2^{m-1}\sqrt{3^m}r^m}{R^2}$, $m\geq 0$.

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Soluţie. Din propoziţie pentru $p=2$ avem: $\sum\frac{a^m}{b^2+c^2}\geq\frac{(\sum a)^m}{2\cdot 3^{m-2}\sum a^2}\overset{(3)}{=}\frac{(2p)^m}{4\cdot 3^{m-2}(p^2-r^2-4Rr)}$ $=\frac{2^{m-2}p^m}{3^{m-2}(p^2-r^2-4Rr)}\overset{(M)}{\geq}\frac{2^{m-2}3^m\sqrt{3^m}r^m}{3^{m-2}(p^2-r^2-4Rr)}=\frac{2^{m-2}(\sqrt{3})^{m+4}}{p^2-r^2-4Rr}\geq\frac{2^{m-1}\sqrt{3^m}r^m}{R^2}$, ultima inegalitate, fiind echivalentă cu $p^2-r^2-4Rr\leq\frac{9R^2}{2}\Leftrightarrow 2p^2\leq 9R^2+8Rr+2r^2$, adevărată, pentru că $2p^2\overset{(G)}{\leq}8R^2+8Rr+6r^2$ şi $8R^2+8Rr+6r^2\leq 9R^2+8Rr+2r^2\Leftrightarrow R\overset{(E)}{\geq}2r$.

A7) $\Delta ABC\Rightarrow\sum\frac{\sqrt{r_b r_c}}{r_a^n}\geq 3\left(\frac{1}{3r}\right)^{n-1}$, n ≥ 0.

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Soluţie. $\sum\frac{\sqrt{r_b r_c}}{r_a^n}=\sum\frac{r_b r_c}{r_a^n\sqrt{r_b r_c}}\overset{MG\leq MA}{\geq}2\sum\frac{r_b r_c}{r_a^n(r_b+r_c)}=2\sum\frac{\left(\frac{1}{r_a}\right)^n}{\frac{1}{r_b}+\frac{1}{r_c}}$ $\overset{(*)}{\geq}2\cdot\frac{3}{2}\left(\frac{\sum\frac{1}{r_a}}{3}\right)^{n-1}\overset{(6)}{=}3\left(\frac{1}{3r}\right)^{n-1}$.

A8) $\Delta ABC\Rightarrow\sum\frac{(b+c)h_a^n}{h_b+h_c}\geq 2\sqrt{3}(3r)^n$, n$\geq 0$.

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Soluţie. Tripletele $(b+c,c+a,a+b)$ şi $\left(\frac{h_a^n}{h_b+h_c},\frac{h_b^n}{h_c+h_a},\frac{h_c^n}{h_a+h_b}\right)$ sunt la fel ordonate, şi atunci: $\sum\frac{(b+c)h_a^n}{h_b+h_c}\overset{(C)}{\geq}\frac{1}{3}\sum(b+c)\sum\frac{h_a^n}{h_b+h_c}\overset{(*)}{\geq}\frac{1}{3}\cdot\frac{3}{2}(2\sum a)\left(\frac{\sum h_a}{3}\right)^{n-1}=2\text{p}\left(\frac{\sum h_a}{3}\right)^{n-1}\overset{(M)}{\geq}$ $6\sqrt{3}r\left(\frac{\sum h_a}{3}\right)^{n-1}\overset{(7)}{=}6\sqrt{3}r\left(\frac{p^2+r^2+4Rr}{6R}\right)^{n-1}\overset{Lema}{\geq}6\sqrt{3}r(3r)^{n-1}=2\sqrt{3}(3r)^n$.

A9) $\Delta ABC\Rightarrow\min\left\{\sum\frac{m_a^n}{m_b+m_c},\sum\frac{s_a^n}{s_b+s_c},\sum\frac{w_a^n}{w_b+w_c}\right\}\geq\frac{3}{2}(3r)^{n-1}$, n$\geq 0$.

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Soluţie. Avem $\sum\frac{m_a^n}{m_b+m_c}\overset{(*)}{\leq}\frac{3}{2}\left(\frac{\sum m_a}{3}\right)^{n-1}\overset{m_a\geq h_a}{\geq}\frac{3}{2}\left(\frac{\sum h_a}{3}\right)^{n-1}\overset{(7)}{=}\frac{3}{2}\left(\frac{\frac{p^2+r^2+4Rr}{2R}}{3}\right)^{n-1}\overset{Lema}{\geq}$ $\frac{3}{2}\left(\frac{18\mathrm{Rr}}{6R}\right)^{n-1}=\frac{3}{2}(3r)^{n-1}$. Analog pentru celelalte două sume.

A10) $\Delta ABC\Rightarrow\sum\frac{\sin^n\frac{A}{2}}{\sqrt{\sin\frac{B}{2}\sin\frac{C}{2}}}\geq\frac{3}{2^{n-1}}$, n ≥ 0.

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Soluţie. $\sum\frac{\sin^n\frac{A}{2}}{\sqrt{\sin\frac{B}{2}\sin\frac{C}{2}}}\overset{MG\leq MA}{\geq}2\sum\frac{\sin^n\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}\overset{(*)}{\geq}2\cdot\frac{3}{2}\left(\frac{\sum\sin\frac{A}{2}}{3}\right)^{n-1}\overset{(17)}{\geq}3\left(\frac{\frac{3}{2}}{3}\right)^{n-1}=\frac{3}{2^{n-1}}$.

A11) $\Delta ABC\Rightarrow\sum\frac{(r_ar_b)^n}{r_br_c+r_cr_a}\geq\frac{3}{2}(3r)^{2n-2}$, n ≥ 0.

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Soluţie. $\sum\frac{(r_ar_b)^n}{r_br_c+r_cr_a}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum r_ar_b}{3}\right)^{n-1}\overset{(8)}{\geq}\frac{3}{2}\left(\frac{p^2}{3}\right)^{n-1}\overset{(M)}{\geq}\frac{3}{2}(9r^2)^{n-1}=\frac{3}{2}(3r)^{2n-2}$.

A12) $\Delta ABC\Rightarrow\sum\frac{(m_aw_a)^n}{m_bw_b+m_cw_c}\geq\frac{3}{2}(3r)^{2n-2}$, n ≥ 0.

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Soluţie. $\sum\frac{(m_aw_a)^n}{m_bw_b+m_cw_c}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum m_aw_a}{3}\right)^{n-1}\overset{(18)}{\geq}\frac{3}{2}\left(\frac{p^2}{3}\right)^{n-1}\overset{(M)}{\geq}\frac{3}{2}.(9r^2)^{n-1}=\frac{3}{2}(3r)^{2n-2}$.

A13) Fie $a,b,c>0$. Să se arate că: $\sum\frac{a^nb^n}{c(a+b)}\geq 2^{2n-3}3^nr^{2n-2}$, n ≥ 0.

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Soluţie. $\sum\frac{a^nb^n}{c(a+b)}\geq\sum\frac{a^nb^n}{bc+ca}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum ab}{3}\right)^{n-1}\overset{(2)}{=}\frac{3}{2}\left(\frac{p^2+r^2+4Rr}{3}\right)^{n-1}\overset{Lemă}{\geq}\frac{3}{2}\left(\frac{18Rr}{3}\right)^{n-1}=$ $\frac{3}{2}(6Rr)^{n-1}\overset{(E)}{\geq}\frac{3}{2}(12r^2)^{n-1}=2^{2n-3}3^nr^{2n-2}$.

A14) $\Delta ABC\Rightarrow\sum\frac{a^nr_a^n}{br_b+cr_c}\geq\frac{3}{2}\left(3\sqrt{3}Rr\right)^{n-1}$, n ≥ 0.

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Soluţie. $\sum\frac{a^nr_a^n}{br_b+cr_c}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum ar_a}{3}\right)^{n-1}\overset{(9)}{\geq}\frac{3}{2}\left(\frac{2p(2R-r)}{3}\right)^{n-1}\geq\frac{3}{2}\left(3\sqrt{3}Rr\right)^{n-1}$, pentru că $2p(2R-r)\geq 9\sqrt{3}Rr\Leftrightarrow 4pR\geq 2pr+9\sqrt{3}Rr$, adevărată, pentru că $4pR\overset{(M)}{\geq}12\sqrt{3}Rr$, şi că $12\sqrt{3}Rr\geq 2pr+9\sqrt{3}Rr\Leftrightarrow p\overset{(M)}{\leq}\frac{3\sqrt{3}R}{2}$.

A15) $\Delta ABC\Rightarrow\sum\frac{\sin^nA\sin^nB}{\sin C(\sin A+\sin B)}\geq\frac{3}{2}\left(\frac{3r}{2R}\right)^{n-1}$, n ≥ 0.

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Soluţie. $\sum\frac{\sin^nA\sin^nB}{\sin C(\sin A+\sin B)}=\sum\frac{(\sin A\sin B)^n}{\sin B\sin C+\sin C\sin A}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum\sin A\sin B}{3}\right)^{n-1}\overset{(10)}{=}$ $\frac{3}{2}\left(\frac{\frac{p^2+r^2+4Rr}{4R^2}}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{p^2+r^2+4Rr}{12R^2}\right)^{n-1}\overset{Lemă}{\geq}\frac{3}{2}\left(\frac{3r}{2R}\right)^{n-1}$.

A16) $\Delta ABC\Rightarrow\sum\frac{a^n\sin^n\frac{A}{2}}{b\sin\frac{B}{2}+c\sin\frac{C}{2}}\geq\frac{3}{2}\left(\sqrt{3}r\right)^{n-1}$, n ≥ 0.

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Soluţie. $\sum\frac{a^n sin^n\frac{A}{2}}{bsin\frac{B}{2}+csin\frac{C}{2}}=\sum\frac{\left(asin\frac{A}{2}\right)^n}{bsin\frac{B}{2}+csin\frac{C}{2}}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum asin\frac{A}{2}}{3}\right)^{n-1}\overset{(19)}{\geq}\frac{3}{2}\left(\frac{p}{3}\right)^{n-1}\overset{(M)}{\geq}\frac{3}{2}\left(\sqrt{3}r\right)^{n-1}$.

A17) ΔABC ascuţitunghic $\Rightarrow\sum\frac{a^n cos^n A}{bcosB+ccosC}\geq\frac{3}{2}\left(\frac{2\sqrt{3}r^2}{R}\right)^{n-1}$, $n\geq 0$.

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Soluţie. $\sum\frac{a^n cos^n A}{bcosB+ccosC}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum acosA}{3}\right)^{n-1}\overset{(11)}{=}\frac{3}{2}\left(\frac{2pr}{3R}\right)^{n-1}\overset{(M)}{\geq}\frac{3}{2}\left(\frac{2\sqrt{3}r^2}{R}\right)^{n-1}$.

A18) $\Delta ABC\Rightarrow\sum\frac{tg^n\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}}\geq\frac{3}{2}\left(\frac{2r}{R\sqrt{3}}\right)^{n-1}$, $n\geq 0$.

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Soluţie. $\sum\frac{tg^n\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum tg\frac{A}{2}}{3}\right)^{n-1}\overset{(12)}{=}\frac{3}{2}\left(\frac{4R+r}{3p}\right)^{n-1}\overset{(E)}{\geq}\left(\frac{3r}{p}\right)^{n-1}\overset{(M)}{\geq}\frac{3}{2}\left(\frac{2r}{R\sqrt{3}}\right)^{n-1}$.

A19) $\Delta ABC\Rightarrow\sum ctg^n\frac{A}{2}\sqrt{tg\frac{B}{2}tg\frac{C}{2}}\geq\left(\sqrt{3}\right)^{n+1}$, $n\geq 0$.

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Soluţie. $\sum ctg^n\frac{A}{2}\sqrt{tg\frac{B}{2}tg\frac{C}{2}}=\sum\frac{ctg^n\frac{A}{2}}{\sqrt{ctg\frac{B}{2}ctg\frac{C}{2}}}\overset{MG\leq MA}{\geq}2\sum\frac{ctg^n\frac{A}{2}}{ctg\frac{B}{2}+ctg\frac{C}{2}}\overset{(*)}{\geq}2\cdot\frac{3}{2}\left(\frac{\sum ctg\frac{A}{2}}{3}\right)^{n-1}\overset{(13)}{=}$
$3\left(\frac{p}{3r}\right)^{n-1}\overset{(M)}{\geq}\left(\sqrt{3}\right)^{n+1}$.

A20) $\Delta ABC\Rightarrow\sum\frac{a^n m_a^n}{bm_b+cm_c}\geq\frac{3}{2}\left(6\sqrt{3}r^2\right)^{n-1}$, $n\geq 0$.

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Soluţie. $\sum\frac{a^n m_a^n}{bm_b+cm_c}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum am_a}{3}\right)^{n-1}\overset{(20)}{\geq}\frac{3}{2}(pR)^{n-1}\overset{(M)}{\geq}\frac{3}{2}\left(3\sqrt{3}Rr\right)^{n-1}\overset{(E)}{\geq}$
$\frac{3}{2}\left(6\sqrt{3}r^2\right)^{n-1}$.

A21) $\Delta ABC\Rightarrow\sum\frac{(p-a)^n}{\sqrt{(p-b)(p-c)}}\geq 3\left(\frac{p}{3}\right)^{n-1}$, $n\geq 0$

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Soluţie. $\sum\frac{(p-a)^n}{\sqrt{(p-b)(p-c)}}\overset{MG\leq MA}{\geq}2\sum\frac{(p-a)^n}{(p-b)+(p-c)}\overset{(*)}{\geq}2\cdot\frac{3}{2}\left(\frac{\sum(p-a)}{3}\right)^{n-1}=3\left(\frac{3p-\sum a}{3}\right)^{n-1}=$
$3\left(\frac{3p-2p}{3}\right)^{n-1}=3\left(\frac{p}{3}\right)^{n-1}$.

A22) $\Delta ABC\Rightarrow\sum tg^n\frac{A}{2}\sqrt{ctg\frac{B}{2}ctg\frac{C}{2}}\geq 3\left(\frac{3r}{p}\right)^{n-1}$, $n\geq 0$.

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Soluţie. $\sum tg^n\frac{A}{2}\sqrt{ctg\frac{B}{2}ctg\frac{C}{2}}=\sum\frac{tg^n\frac{A}{2}}{\sqrt{tg\frac{B}{2}tg\frac{C}{2}}}\overset{MG\leq MA}{\geq}2\sum\frac{tg^n\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}}\overset{(*)}{\geq}3\left(\frac{\sum tg\frac{A}{2}}{3}\right)^{n-1}\overset{(12)}{=}$
$3\left(\frac{4R+r}{3p}\right)^{n-1}\overset{(E)}{\geq}3\left(\frac{3r}{p}\right)^{n-1}$.

A23) $\Delta ABC\Rightarrow\sum\frac{(2p-c)^n}{2p+c}\geq\frac{2^{2n-3}p^{n-1}}{3^{n-2}}$, $n\geq 0$.

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Soluţie. $\sum\frac{(2p-c)^n}{2p+c}=\sum\frac{(a+b)^n}{(b+c)+(c+a)}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum(a+b)}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{2\sum a}{3}\right)^{n-1}=\frac{2^{2n-3}p^{n-1}}{3^{n-2}}$.

A24) $\Delta ABC\Rightarrow\sum\frac{r_a}{h_a^m(r_a+r)}\geq\frac{1}{4\cdot 3^{m-2}r^m}$, $m\geq 0$

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Soluţie. $\sum\frac{r_a}{h_a^m(r_a+r)}=\frac{1}{r}\sum\frac{1}{h_a^m\left(\frac{1}{r}+\frac{1}{r_a}\right)}\overset{(6)}{\cong}\frac{1}{r}\sum\frac{1}{h_a^m\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_a}\right)}=\frac{1}{r}\sum\frac{1}{h_a^m\left(\left(\frac{1}{r_a}+\frac{1}{r_c}\right)+\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\right)}\overset{(14)}{\cong}$

$\frac{1}{r}\sum\frac{1}{h_a^m\left(\frac{2}{h_b}+\frac{2}{h_c}\right)}=\frac{1}{2r}\sum\frac{\left(\frac{1}{h_a}\right)^m}{\frac{1}{h_b}+\frac{1}{h_c}}\overset{(*)}{\geq}\frac{1}{2r}\cdot\frac{3}{2}\left(\frac{\sum\frac{1}{h_a}}{3}\right)^{m-1}\overset{(6)}{\cong}\frac{3}{4r}\left(\frac{1}{3r}\right)^{m-1}=\frac{1}{4\cdot 3^{m-2}r^m}.$

A25) $\Delta ABC\Rightarrow\sum\frac{h_a}{r_a^m}\geq\frac{1}{3^{m-2}r^{m-1}},m\geq 0.$

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Soluţie. $\sum\frac{h_a}{r_a^m}=\sum\frac{\left(\frac{1}{r_a}\right)^m}{\frac{1}{h_a}}=2\sum\frac{\left(\frac{1}{r_a}\right)^m}{\frac{2}{h_a}}\overset{(14)}{\cong}2\sum\frac{\left(\frac{1}{r_a}\right)^m}{\frac{1}{r_b}+\frac{1}{r_c}}\overset{(*)}{\geq}2\cdot\frac{3}{2}\left(\frac{\sum\frac{1}{r_a}}{3}\right)^{m-1}\overset{(6)}{\cong}\frac{\left(\frac{1}{r}\right)^{m-1}}{3^{m-2}}=$

$\frac{1}{3^{m-2}r^{m-1}}.$

A26) $\Delta ABC\Rightarrow\sum\frac{a^m(b+c)^m}{ab+2bc+ca}\geq 2^{3m-4}\cdot 3^m r^{2m-2},m\geq 0.$

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Soluţie. $\sum\frac{a^m(b+c)^m}{ab+2bc+ca}=\sum\frac{(ab+ac)^m}{(bc+ba)+(ca+cb)}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum(ab+ac)}{3}\right)^{m-1}=\frac{(2\sum ab)^{m-1}}{2\cdot 3^{m-2}}\overset{(2)}{\cong}$

$\frac{2^{m-2}(p^2+r^2+4Rr)^{m-1}}{3^{m-2}}\overset{Lemă}{\geq}\frac{2^{m-2}(18Rr)^{m-1}}{3^{m-2}}\overset{(E)}{\geq}\frac{2^{m-2}(36r^2)^{m-1}}{3^{m-2}}=2^{3m-4}\cdot 3^m r^{2m-2}.$

A27) $\Delta ABC\Rightarrow\sum\frac{\left(\frac{h_a}{m_a}\right)^n}{\frac{h_b}{m_b}+\frac{h_c}{m_c}}\geq\frac{3}{2}\left(\frac{2r}{R}\right)^{n-1},n\geq 0.$

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Soluţie. $\sum\frac{\left(\frac{h_a}{m_a}\right)^n}{\frac{h_b}{m_b}+\frac{h_c}{m_c}}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum\frac{h_a}{m_a}}{3}\right)^{n-1}\overset{(P)}{\geq}\frac{3}{2}\left(\frac{\sum\frac{2r}{R}}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{2r}{R}\right)^{n-1}.$

A28) $\Delta ABC\Rightarrow\sum\frac{\left(\frac{r_a}{h_a}\right)^n}{\frac{r_b}{h_b}+\frac{r_c}{h_c}}\geq\frac{3}{2}.$

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Soluţie. $\sum\frac{\left(\frac{r_a}{h_a}\right)^n}{\frac{r_b}{h_b}+\frac{r_c}{h_c}}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum\frac{r_a}{h_a}}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{\sum\frac{\frac{S}{p-a}}{\frac{2S}{a}}}{3}\right)^{n-1}=\frac{3}{2}\left(\frac{\frac{1}{2}\sum\frac{a}{p-a}}{3}\right)^{n-1}\overset{(15)}{\cong}\frac{3}{2}\left(\frac{\frac{2(2R-r)}{2r}}{3}\right)^{n-1}=$

$\frac{3}{2}\left(\frac{(2R-r)}{3r}\right)^{n-1}\overset{(E)}{\geq}\frac{3}{2}.$

A29) $\Delta ABC\Rightarrow\sum\frac{m_a^n m_b^n}{m_c(m_a+m_b)}\geq\frac{3^{2n-1}r^{2n-2}}{2}.$

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Soluţie. $\sum\frac{m_a^n m_b^n}{m_c(m_a+m_b)}=\sum\frac{m_a^n m_b^n}{m_b m_c+m_c m_a}\overset{(*)}{\geq}\frac{3}{2}\left(\frac{\sum m_a m_b}{3}\right)^{n-1}\overset{(21)}{\geq}\frac{\left(\sum\sqrt{p(p-a)}\sqrt{p(p-b)}\right)^{n-1}}{2\cdot 3^{n-2}}=$

$\frac{1}{2\cdot 3^{n-2}}\left(p\sum\sqrt{(p-a)(p-b)}\right)^{n-1}=\frac{p^{n-1}}{2\cdot 3^{n-2}}\left(\sum\frac{(p-a)(p-b)}{\sqrt{(p-a)(p-b)}}\right)^{n-1}\overset{MG\leq MA}{\geq}$

$\frac{p^{n-1}}{2\cdot 3^{n-2}}\left(2\sum\frac{(p-a)(p-b)}{(p-a)+(p-b)}\right)^{n-1}=\frac{2^{n-2}p^{n-1}}{3^{n-2}}\left(\sum\frac{(p-a)(p-b)}{c}\right)^{n-1}\overset{(16)}{\cong}\frac{2^{n-2}p^{n-1}}{3^{n-2}}\left(\frac{r\left(p^2+(4R+r)^2\right)}{4pR}\right)^{n-1}=$

$$\frac{2^{n-2}p^{n-1}}{3^{n-2}}\left(\frac{r(p^2+r^2+4Rr+16R^2+4Rr)}{4pR}\right)^{n-1}\overset{Lemă}{\geq}\frac{2^{n-2}p^{n-1}}{3^{n-2}}\left(\frac{r(22Rr+16R^2)}{4pR}\right)^{n-1}\overset{(E)}{\geq}$$
$$\frac{2^{n-2}p^{n-1}}{3^{n-2}}\left(\frac{r(54Rr)}{4pR}\right)^{n-1}=\frac{3^{2n-1}r^{2n-2}}{2}.$$

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Profesor Gheorghe Ghiţă, Buzău
ghighibz@gmail.com