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1. Applications of J. Radon' s inequality in triangle

D.M. Bătineţu-Giurgiu and Neculai Stanciu

Theorem 1. If $x, y, z \in R_+^*$, $m \in R_+$, then in any triangle ABC holds

$$\frac{\cos^{2m+2}\frac{A}{2}}{(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2})^m}+\frac{\cos^{2m+2}\frac{B}{2}}{(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}+z\cos^2\frac{A}{2})^m}+$$
$$+\frac{\cos^{2m+2}\frac{C}{2}}{(x\sin^2\frac{C}{2}+y\sin^2\frac{A}{2}+z\cos^2\frac{B}{2})^m}\geq\frac{(4R+r)^{m+1}}{2R(2(x+2y+2z)R+(2z-x-y)r)^m}$$

Proof. By J.Radon's inequality we have

$$\sum\frac{\cos^{2m+2}\frac{A}{2}}{(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2})^m}=\sum\frac{\left(\cos^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}\right)^m}\geq$$
$$\geq\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left(\sum\left(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}\right)\right)^m}=\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left((x+y)\sum\sin^2\frac{A}{2}+z\sum\cos^2\frac{A}{2}\right)^m}$$

Because,

$$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R},$$

and

$$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

we obtain the conclusion.

Theorem 2. If $x, y \in R_+^*$, $m \in R_+$, then in any triangle ABC holds

$$\frac{\cos^{2m+2}\frac{A}{2}}{(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}+\frac{\cos^{2m+2}\frac{B}{2}}{(x\cos^2\frac{C}{2}+y\cos^2\frac{A}{2})^m}+\frac{\cos^{2m+2}\frac{C}{2}}{(x\cos^2\frac{A}{2}+y\cos^2\frac{B}{2})^m}\geq\geq\frac{4R+r}{2R(x+y)^m}.$$

Proof. By J.Radon's inequality we have

$$\sum\frac{\cos^{2m+2}\frac{A}{2}}{(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}=\sum\frac{\left(\cos^2\frac{A}{2}\right)^{m+1}}{\left(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)^m}\geq$$

$$\geq\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum\cos^2\frac{A}{2}\right)^m}=\frac{\sum\cos^2\frac{A}{2}}{(x+y)^m}$$

Since,

$$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

we get the desired conclusion.

Theorem 3. If $x,y\in R_+^*$, $m\in R_+$, then in any triangle ABC holds

$$\frac{\cos^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}+\frac{\cos^{2m+2}\frac{B}{2}}{(x\sin^2\frac{C}{2}+y\cos^2\frac{A}{2})^m}+\frac{\cos^{2m+2}\frac{C}{2}}{(x\sin^2\frac{A}{2}+y\cos^2\frac{B}{2})^m}\geq$$

$$\geq\frac{(4R+r)^{m+1}}{2R(2(x+2y)R+(y-x)r)^m}.$$

Proof. By J.Radon's inequality we have

$$\sum\frac{\cos^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}=\sum\frac{\left(\cos^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)^m}\geq$$

$$\geq\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left(\sum\left(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)\right)^m}=\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left(x\sum\sin^2\frac{A}{2}+y\sum\cos^2\frac{A}{2}\right)^m}$$

Since,

$$\sum \sin^2 \frac{A}{2} = \frac{2R-r}{2R},$$

and

$$\sum \cos^2 \frac{A}{2} = \frac{4R+r}{2R},$$

we obtain the conclusion.

Theorem 4. If $x, y \in R_+^*$, $m \in R_+$, then in any triangle ABC holds

$$\frac{\cos^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2})^m}+\frac{\cos^{2m+2}\frac{B}{2}}{(x\sin^2\frac{C}{2}+y\sin^2\frac{A}{2})^m}+\frac{\cos^{2m+2}\frac{C}{2}}{(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2})^m}\geq$$
$$\geq \frac{(4R+r)^{m+1}}{2R(x+y)^m(2R-r)^m}.$$

Proof. By J.Radon's inequality we have

$$\sum \frac{\cos^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2})^m}=\sum\frac{\left(\cos^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}\right)^m}\geq$$
$$\geq \frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left(\sum\left(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}\right)\right)^m}=\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left((x+y)\cdot\sum\sin^2\frac{A}{2}\right)^m}=\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum\sin^2\frac{A}{2}\right)^m}$$

Since,

$$\sum \sin^2 \frac{A}{2} = \frac{2R-r}{2R},$$

and

$$\sum \cos^2 \frac{A}{2} = \frac{4R+r}{2R},$$

we deduce the result.

3. Generating triangle inequalities from algebraic inequalities (I)

Marin Chirciu [1]

Articolul prezintă inegalităti în triunghi obținute din inegalități algebrice, selectate din diverse publicații de specialitate.

Aplicația1.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{x^7}{2+2\sqrt{yz}}\geq\frac{1}{3^8}\cdot\frac{27}{8}.$$

Konstantinos Geronikolas, Greece, MathTime 8/2023

Remarca.

If $x,y,z>0$, $x+y+z=1$ and $n\in\mathbf{N}$ then

$$\sum\frac{x^n}{1+\sqrt{yz}}\geq\frac{1}{4\cdot3^{n-2}}.$$

Marin Chirciu

Remarca.

In ΔABC

$$\sum\frac{\left(\frac{r}{r_a}\right)^n}{1+\frac{1}{p}\sqrt{rr_a}}\geq\frac{1}{4\cdot3^{n-2}},\ n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

Lema

If $x,y,z>0$, $x+y+z=1$ and $n\in\mathbf{N}$ then

$$\sum\frac{x^n}{1+\sqrt{yz}}\geq\frac{1}{4\cdot3^{n-2}}.$$

Demonstrație.

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[1] *Profesor, Colegiul Național „Zinca Golescu"Pitești*

Pentru $n=0$ avem $\sum\frac{1}{1+\sqrt{yz}}\geq\frac{9}{4}$, vezi $\sum\frac{1}{1+\sqrt{yz}}\overset{CS}{\geq}\frac{9}{\sum\left(1+\sqrt{yz}\right)}=\frac{9}{3+\sum\sqrt{yz}}\overset{(1)}{\geq}\frac{9}{3+1}=\frac{9}{4}$,

unde(1) $\Leftrightarrow\sum\sqrt{yz}\leq 1$, adevărat din $\sum\sqrt{yz}\overset{CBS}{\leq}\sqrt{3\sum yz}\overset{SOS}{\leq}\sqrt{\left(\sum x\right)^2}=1$.

Pentru $n=1$ avem $\sum\frac{x}{1+\sqrt{yz}}\geq\frac{3}{4}$, vezi

$$\sum\frac{x}{1+\sqrt{yz}}=\sum\frac{x^2}{x+x\sqrt{yz}}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum\left(x+x\sqrt{yz}\right)}=\frac{1^2}{\sum x+\sum x\sqrt{yz}}\overset{(1)}{\geq}\frac{1}{1+\frac{1}{3}}=\frac{3}{4},$$

unde(1) $\Leftrightarrow\sum x\sqrt{yz}\leq\frac{1}{3}$, adevărat din

$ab+bc+ca\leq a^2+b^2+c^2$, pentru $(a,b,c)=\left(\sqrt{yz},\sqrt{zx},\sqrt{xy}\right)$ și $yz\leq\frac{1}{3}\left(\sum x\right)^2=\frac{1}{3}$

Pentru $n\geq 2$ se folosește inegalitatea lui Holder.

$$LHS=\sum\frac{x^n}{1+\sqrt{yz}}\overset{Holder}{\geq}\frac{\left(\sum x\right)^n}{3^{n-2}\sum\left(1+\sqrt{yz}\right)}=\frac{1^n}{3^{n-2}\left(3+\sum\sqrt{yz}\right)}\overset{(1)}{\geq}\frac{1}{3^{n-2}(3+\cdot 1)}=\frac{1}{3^{n-2}\cdot 4}=RHS.$$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{3}$.

Se cunoaște identitatea în triunghi $\sum\frac{r}{r_a}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{\left(\frac{r}{r_a}\right)^n}{1+\sqrt{\frac{r}{r_b}\frac{r}{r_c}}}\geq\frac{1}{4\cdot 3^{n-2}}\Leftrightarrow\sum\frac{\left(\frac{r}{r_a}\right)^n}{1+\sqrt{\frac{r^2}{r_br_c}}}\geq\frac{1}{4\cdot 3^{n-2}}\overset{r_ar_br_c=rp^2}{\Leftrightarrow}\sum\frac{\left(\frac{r}{r_a}\right)^n}{1+\sqrt{\frac{rr_a}{p^2}}}\geq\frac{1}{4\cdot 3^{n-2}}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\left(\frac{r}{r_a}\right)^n}{1+\frac{1}{p}\sqrt{rr_a}}\geq\frac{1}{4\cdot 3^{n-2}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

Problema se poate dezvolta.

If $x, y, z > 0$, $x + y + z = 1$ and $n \in \mathbf{N}$, $\lambda \geq 0$ then

$$\sum \frac{x^n}{\lambda + \sqrt{yz}} \geq \frac{1}{(3\lambda + 1) \cdot 3^{n-2}}.$$

Marin Chirciu

<u>Aplicația2.</u>

In ΔABC

$$\sum \frac{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^7}{\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^6 + \left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^6} \geq \frac{1}{2}.$$

Daniel Sitaru,RMM 8/2023

<u>Soluție.</u>

<u>Lema</u>

If $x, y, z > 0$, $xy + yz + zx = 1$ then

$$\sum \frac{x^7}{y^6 + z^6} \geq \frac{1}{2}$$

<u>Demonstrație.</u>

Folosim inegalitatea lui Chebyshev pentru tripletele la fel ordonate (x, y, z) și $\left(\frac{x^6}{y^6+z^6}, \frac{y^6}{z^6+x^6}, \frac{z^6}{x^6+y^6}\right)$.

$$\sum \frac{x^7}{y^6+z^6} = \sum x \frac{x^6}{y^6+z^6} \overset{Chebyshev}{\geq} \frac{1}{3} \sum x \sum \frac{x^6}{y^6+z^6} \overset{Nesbitt}{\geq} \frac{1}{3} \cdot 1 \cdot \frac{3}{2} = \frac{1}{2}.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$.

Folosind **Lema** pentru $(x, y, z) = \left(\tan\frac{A}{2}\tan\frac{B}{2}, \tan\frac{B}{2}\tan\frac{C}{2}, \tan\frac{C}{2}\tan\frac{A}{2}\right)$ obținem:

$$\sum \frac{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^7}{\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^6+\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^6} \geq \frac{1}{2}$$

Remarca.

In ΔABC

$$\sum \frac{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{n+1}}{\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^n+\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^n} \geq \frac{1}{2}, n \in \mathbf{N}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$, $xy + yz + zx = 1$ and $n \in \mathbf{N}$ then

$$\sum \frac{x^{n+1}}{y^n + z^n} \geq \frac{1}{2}.$$

Demonstrație.

Folosind **Lema** pentru $(x, y, z) = \left(\tan\frac{A}{2}\tan\frac{B}{2}, \tan\frac{B}{2}\tan\frac{C}{2}, \tan\frac{C}{2}\tan\frac{A}{2}\right)$ obținem:

$$\sum \frac{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{n+1}}{\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^n+\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^n} \geq \frac{1}{2}$$

Aplicația3.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum \sqrt{\frac{yz}{x + yz}} \leq \frac{3}{2}.$$

Kunihiko Chikaya, MathTime 8/2023

Remarca.

In ΔABC

$$\sum \frac{1}{\sqrt{1+\frac{p^2}{r_a^2}}} \le \frac{3}{2}.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $x+y+z=1$ then

$$\sum \sqrt{\frac{yz}{x+yz}} \le \frac{3}{2}.$$

If $x, y, z > 0$, $x+y+z=1$ then

$$\sqrt{\frac{yz}{x+yz}} \le \frac{1}{2}\left(\frac{y}{x+y}+\frac{z}{x+z}\right).$$

Demonstrație.

$$\sqrt{\frac{yz}{x+yz}} = \sqrt{\frac{yz}{x(x+y+z)+yz}} = \sqrt{\frac{yz}{(x+y)(x+z)}} = \sqrt{\frac{y}{x+y}\cdot\frac{z}{x+z}} \overset{AM-GM}{\le} \frac{1}{2}\left(\frac{y}{x+y}+\frac{z}{x+z}\right).$$

Obținem $\sum \sqrt{\frac{yz}{x+yz}} \le \sum \frac{1}{2}\left(\frac{y}{x+y}+\frac{z}{x+z}\right) = \frac{3}{2}$.

Se cunoaște identitatea în triunghi $\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{r}{r_a}, \frac{r}{r_b}, \frac{r}{r_c}\right)$ obținem:

$$\sum \sqrt{\frac{\frac{r}{r_b}\frac{r}{r_c}}{\frac{r}{r_a}+\frac{r}{r_b}\frac{r}{r_c}}} \le \frac{3}{2} \Leftrightarrow \sum \sqrt{\frac{rr_a}{rr_a+r_br_c}} \le \frac{3}{2} \Leftrightarrow \sum \frac{1}{\sqrt{1+\frac{r_br_c}{rr_a}}} \le \frac{3}{2} \Leftrightarrow \sum \frac{1}{\sqrt{1+\frac{r_ar_br_c}{rr_a^2}}} \le \frac{3}{2} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt{1+\frac{rp^2}{rr_a^2}}} \le \frac{3}{2} \Leftrightarrow \sum \frac{1}{\sqrt{1+\frac{p^2}{r_a^2}}} \le \frac{3}{2}.$$

Remarca.

In ΔABC

$$\sum \frac{1}{\sqrt{1+\frac{h_b h_c}{r h_a}}} \le \frac{3}{2}.$$

Marin Chirciu

Aplicația4.

If $a,b,c>0$, $a+b+c=3$ and $\lambda \ge 0$ then

$$\frac{(a+b)(b+c)(c+a)}{ab+bc+ca}+\frac{\lambda}{abc} \ge \lambda+\frac{8}{3}.$$

Marin Chirciu, IneMath 7/2023

Solution .

We use pqr -Method.

We note $p=a+b+c, q=ab+bc+ca, r=abc$.

We have $p=3$, $(a+b)(b+c)(c+a)=\prod(3-a)=3q-r$.

$3=a+b+c \ge 3\sqrt[3]{abc}=3\sqrt[3]{r} \Rightarrow r \le 1$.

$q^2=(ab+bc+ca)^2 \ge 3abc(a+b+c)=9r \Rightarrow q^2 \ge 9r$.

Inequality $\frac{(a+b)(b+c)(c+a)}{ab+bc+ca}+\frac{\lambda}{abc} \ge \lambda+\frac{8}{3}$ is written $\frac{3q-r}{q}+\frac{\lambda}{r} \ge \lambda+\frac{8}{3} \Leftrightarrow$

$\Leftrightarrow 3r(3q-r)+3\lambda q \ge (3\lambda+8)qr \Leftrightarrow$

$\Leftrightarrow q\left[3\lambda-(3\lambda-1)r\right] \ge 3r^2 \Leftrightarrow q \ge \frac{3r^2}{\left[3\lambda-(3\lambda-1)r\right]}$, (see: $r \le 1 < \frac{3\lambda}{3\lambda-1}$).

Using $q^2 \ge 9r$ is enough to show $3\sqrt{r} \ge \frac{3r^2}{\left[3\lambda-(3\lambda-1)r\right]} \Leftrightarrow 9r \ge \frac{9r^4}{\left[3\lambda-(3\lambda-1)r\right]^2} \Leftrightarrow$

$\Leftrightarrow \left[3\lambda-(3\lambda-1)r\right]^2 \ge r^3 \Leftrightarrow r^3+(1-3\lambda)r^3+6\lambda(3\lambda-1)r-9\lambda^2 \le 0 \Leftrightarrow$

$\Leftrightarrow (r-1)\left[r^2+\left(6\lambda-9\lambda^2\right)r+9\lambda^2\right] \le 0$, which result from : $(r-1) \le 0$ and

$\left[r^2+\left(6\lambda-9\lambda^2\right)r+9\lambda^2\right]>0$, for $0<r\le 1$ and $\lambda\ge 0$.

Equality occurs if and only if $a=b=c=1$.

Remark.

The problem can develop.

In ΔABC

$$\frac{\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{1}{r_c}+\frac{1}{r_a}\right)}{r_a+r_b+r_c}+\frac{1}{243r^4}\ge\frac{1}{F^2}.$$

Marin Chirciu

Lemma.

If $x,y,z>0$, $x+y+z=3$ then

$$\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}+\frac{1}{3xyz}\ge 3.$$

Solution .

We use pqr -Method.

We note $p=x+y+z, q=xy+yz+zx, r=xyz$.

We have $p=3$, $(x+y)(y+z)(z+x)=\prod(3-x)=3q-r$.

$3=x+y+z\ge 3\sqrt[3]{xyz}=3\sqrt[3]{r}\Rightarrow r\le 1$.

$q^2=(xy+yz+zx)^2\ge 3xyz(x+y+z)=9r\Rightarrow q^2\ge 9r$.

Inequality $\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}+\frac{1}{3xyz}\ge 3$ is written $\frac{3q-r}{q}+\frac{1}{3r}\ge 3\Leftrightarrow q\ge 3r^2$.

Using $q^2\ge 9r$ is enough to show $3\sqrt{r}\ge 3r^2\Leftrightarrow 9r\ge 9r^4\Leftrightarrow r\le 1$.

Equality occurs if and only if $x=y=z=1$.

The identity in the triangle is known $\sum\frac{r}{r_a}=1\Leftrightarrow\sum\frac{3r}{r_a}=3$.

Using **Lemma** for $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ we have:

$$\frac{\left(\frac{3r}{r_a}+\frac{3r}{r_b}\right)\left(\frac{3r}{r_b}+\frac{3r}{r_c}\right)\left(\frac{3r}{r_c}+\frac{3r}{r_a}\right)}{\frac{3r}{r_a}\cdot\frac{3r}{r_b}+\frac{3r}{r_b}\cdot\frac{3r}{r_c}+\frac{3r}{r_c}\cdot\frac{3r}{r_a}}+\frac{1}{3\cdot\frac{3r}{r_a}\cdot\frac{3r}{r_b}\cdot\frac{3r}{r_c}}\geq 3 \Leftrightarrow$$

$$\Leftrightarrow \frac{27r^3\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{1}{r_c}+\frac{1}{r_a}\right)}{9r^2\left(\frac{1}{r_a}\cdot\frac{1}{r_b}+\frac{1}{r_b}\cdot\frac{1}{r_c}+\frac{1}{r_c}\cdot\frac{1}{r_a}\right)}+\frac{1}{81r^3\cdot\frac{1}{r_a}\cdot\frac{1}{r_b}\cdot\frac{1}{r_c}}\geq 3 \Leftrightarrow$$

$$\Leftrightarrow \frac{\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{1}{r_c}+\frac{1}{r_a}\right)}{r_a+r_b+r_c}+\frac{1}{243r^4}\geq\frac{1}{r^2p^2} \Leftrightarrow$$

Equality occurs if and only if the triangle is equilateral.

Aplicația5.

If $x, y, z > 0$, $x+y+z=1$ then find min of

$$P=\frac{1}{xyz}+\frac{1}{x^2+y^2+z^2}.$$

Pham Van Tuyen, Vietnam, THCS7/2023

Soluție.

$$P=\frac{1}{xyz}+\frac{1}{x^2+y^2+z^2}=\frac{x+y+z}{xyz}+\frac{1}{x^2+y^2+z^2}=\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}+\frac{1}{x^2+y^2+z^2}.$$

Avem $\frac{1}{xy}+81xy\geq 2\sqrt{\frac{1}{xy}\cdot 81xy}=18$, cu egalitate pentru $\frac{1}{xy}=81xy \Leftrightarrow xy=\frac{1}{9}$.

Rezultă $\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}+18(xy+yz+zx)\geq 54 \Rightarrow \sum\frac{1}{xy}\geq 54-81\sum xy$, (1).

Avem $\frac{1}{x^2+y^2+z^2}+9\left(x^2+y^2+z^2\right)\overset{SOS}{\geq} 6 \Rightarrow \frac{1}{x^2+y^2+z^2}\geq 6-9\sum x^2$, (2).

Din (1) și (2) obținem:

$$P=\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}+\frac{1}{x^2+y^2+z^2}\geq 54-81\sum xy+6-9\sum x^2=60-9\sum x^2-81\sum xy=$$

$$=60-9\left(\sum x^2+2\sum xy\right)-63\sum xy=60-9\left(\sum x\right)^2-63\sum xy=60-9\cdot 1^2-63\sum xy=$$

$$=51-63\sum xy\overset{SOS}{\geq} 51-63\cdot\frac{1}{3}=51-21=30.$$

Egalitatea are loc dacă și numai dacă $x = y = z = \frac{1}{3}$.

Deducem că $\min P = 30$ pentru $x = y = z = \frac{1}{3}$.

Remarca.

Problema se poate dezvolta.

Let $0 \le \lambda \le \frac{9}{2}$ fixed. If $x, y, z > 0$, $x + y + z = 1$ then find min of

$$P = \frac{1}{xyz} + \frac{\lambda}{x^2 + y^2 + z^2}.$$

Marin Chirciu

Remarca.

Problema se poate dezvolta.

In ΔABC

$$p^2 + \frac{\lambda}{\frac{1}{r_a^2} + \frac{1}{r_b^2} + \frac{1}{r_c^2}} \ge (27 + 3\lambda) r^2, 0 \le \lambda \le \frac{9}{2}.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $x + y + z = 1$ and $0 \le \lambda \le \frac{9}{2}$ then

$$\frac{1}{xyz} + \frac{\lambda}{x^2 + y^2 + z^2} \ge 27 + 3\lambda.$$

Soluție.

Se cunoaște identitatea în triunghi $\frac{r}{r_a} + \frac{r}{r_b} + \frac{r}{r_c} = 1$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{r}{r_a}, \frac{r}{r_b}, \frac{r}{r_c}\right)$ obținem:

$$\frac{1}{\frac{r}{r_a}\frac{r}{r_b}\frac{r}{r_c}} + \frac{\lambda}{\left(\frac{r}{r_a}\right)^2 + \left(\frac{r}{r_b}\right)^2 + \left(\frac{r}{r_c}\right)^2} \ge 27 + 3\lambda \Leftrightarrow \frac{1}{\frac{r^3}{r_a r_b r_c}} + \frac{\lambda}{r^2\left(\frac{1}{r_a^2} + \frac{1}{r_b^2} + \frac{1}{r_c^2}\right)} \ge 27 + 3\lambda \Leftrightarrow$$

$$\Leftrightarrow \frac{\frac{1}{r^3}}{rp^2}+\frac{\lambda}{r^2\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)}\geq 27+3\lambda \Leftrightarrow \frac{\frac{1}{r^2}}{p^2}+\frac{\lambda}{r^2\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)}\geq 27+3\lambda \Leftrightarrow$$

$$\Leftrightarrow p^2+\frac{\lambda}{\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}}\geq (27+3\lambda)r^2.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\frac{2p^2}{Rr}+\frac{\lambda}{r^2\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)}\geq 27+3\lambda\ ,0\leq\lambda\leq\frac{9}{2}.$$

Marin Chirciu

Aplicația6.

If $a,b,c>0$, $a+b+c=1$ and $n>1$ then

$$\sum a\sqrt[n]{b}\geq\frac{1}{\sqrt[n]{3}}.$$

Marin Chirciu

Soluție.

Cu substituția $(x,y,z)=(3a,3b,3c)$ problema se reformulează:

If $x,y,z>0$, $x+y+z=3$ and $n>1$ then

$$\sum x\sqrt[n]{y}\leq 3.$$

Lema.

If $x,y,z>0$, $x+y+z=3$ and $n>1$ then

$$x\sqrt[n]{y}\leq\frac{x(y+n-1)}{n}.$$

Demonstrație

$$x\sqrt[n]{y}=x\sqrt[n]{y\cdot\underbrace{1\cdot1\ldots\cdot1}_{n-1}}\overset{AM-GM}{\leq}x\frac{y+\overbrace{1+1+\ldots+1}^{n-1}}{n}=\frac{x(y+n-1)}{n}$$,cu egalitate pentru $y=1$.

$$LHS=\sum x\sqrt[n]{y}\overset{Lema}{\leq}\sum\frac{x(y+n-1)}{n}=\frac{1}{n}\sum xy+\frac{n-1}{n}\sum x\leq\frac{1}{n}\cdot3+\frac{n-1}{n}\cdot3=3=RHS.$$

Remarca.

In ΔABC

$$\sum\frac{r}{r_a}\sqrt[n]{\frac{3r}{r_b}}\leq1, n>1.$$

Marin Chirciu

Soluție.

If $x,y,z>0$, $x+y+z=3$ and $n>1$ then

$$\sum x\sqrt[n]{y}\leq3.$$

Lema.

If $x,y,z>0$, $x+y+z=3$ and $n>1$ then

$$x\sqrt[n]{y}\leq\frac{x(y+n-1)}{n}.$$

Se cunoaște identitatea în triunghi $\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum\frac{3r}{r_a}\sqrt[n]{\frac{3r}{r_b}}\leq3\Leftrightarrow\sum\frac{r}{r_a}\sqrt[n]{\frac{3r}{r_b}}\leq1.$$

Remarca.

In ΔABC

$$\sum\frac{r}{h_a}\sqrt[n]{\frac{3r}{h_b}}\leq1, n>1.$$

Marin Chirciu

Aplicația7.

If $a,b,c>0$, $a+b+c=1$ then

$$\sum\sqrt[3]{a^4+2b^2c^2}\ge 1.$$

Nguyen Viet Hung, Vietnam, Mathematical Inequalities7/2023

Soluție.

$$LHS=\sum\sqrt[3]{a^4+2b^2c^2}=\sum\sqrt[3]{(a+b+c)(a+c+b)\left(a^4+b^2c^2+b^2c^2\right)}\overset{Holder}{\ge}\sum\sqrt[3]{\left(\sqrt[3]{a^6}+\sqrt[3]{b^3c^3}+\sqrt[3]{b^3c^3}\right)^3}=$$
$$\sum\sqrt[3]{\left(a^2+bc+bc\right)^3}=\sum\left(a^2+bc+bc\right)=\sum\left(a^2+2bc\right)=(a+b+c)^2=1=RHS.$$

Remarca.

In ΔABC

$$\sum\sqrt[3]{\frac{1}{r_a^4}+\frac{2}{r_b^2r_c^2}}\ge\sqrt[3]{\frac{1}{r^4}}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\sqrt[3]{x^4+2y^2z^2}\ge 1.$$

Se cunoaște identitatea în triunghi $\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\sqrt[3]{\left(\frac{r}{r_a}\right)^4+2\left(\frac{r}{r_b}\right)^2\left(\frac{r}{r_c}\right)^2}\ge 1\Leftrightarrow\sum\sqrt[3]{\frac{1}{r_a^4}+\frac{2}{r_b^2r_c^2}}\ge\sqrt[3]{\frac{1}{r^4}}.$$

Remarca.

In ΔABC

$$\sum\sqrt[3]{\frac{1}{h_a^4}+\frac{2}{h_b^2h_c^2}}\ge\sqrt[3]{\frac{1}{r^4}}.$$

Marin Chirciu

Aplicația8.

J636. If $a,b,c>0$, $ab+bc+ca=3$ then

$$\sum\frac{1}{a^2+1}\le\frac{a+b+c}{2}.$$

Marius Stănean, Zalău, Mathematical Reflections 4/2023,J636

Solution .

$$\sum\frac{1}{a^2+1}\le\frac{a+b+c}{2}\Leftrightarrow-\sum\frac{1}{a^2+1}\ge-\frac{a+b+c}{2}\Leftrightarrow 3-\sum\frac{1}{a^2+1}\ge 3-\frac{a+b+c}{2}\Leftrightarrow$$

$$\Leftrightarrow\sum\left(1-\frac{1}{a^2+1}\right)\ge 3-\frac{a+b+c}{2}\Leftrightarrow\sum\frac{a^2}{a^2+1}\ge 3-\frac{a+b+c}{2}\Leftrightarrow\sum\frac{a^2}{a^2+1}+\frac{a+b+c}{2}\ge 3,$$

which result from CS:

$$\sum\frac{a^2}{a^2+1}+\frac{a+b+c}{2}\overset{CS}{\ge}\frac{\left(\sum a\right)^2}{\sum\left(a^2+1\right)}+\frac{\sum a}{2}=\frac{\left(\sum a\right)^2}{\sum a^2+3}+\frac{\sum a}{2}=\frac{p^2}{p^2-3}+\frac{p}{2}\overset{(1)}{\ge}3,$$

$$(1)\Leftrightarrow\frac{p^2}{p^2-3}+\frac{p}{2}\ge 3\Leftrightarrow p^3-4p^2-3p+18\ge 0\Leftrightarrow (p-3)^2(p+2)\ge 0.$$

Equality occurs if and only if $a=b=c=1$.

Remark.

In ΔABC

$$\sum\frac{1}{1+3\tan^2\frac{A}{2}}\le 3\cdot\frac{R}{2r}$$

Marin Chirciu

Solution .

Lemma.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\sum\frac{1}{x^2+1}\le\frac{x+y+z}{2}.$$

Remark.

If $a,b,c>0$, $abc=1$ then

$$\sum\frac{1}{ab+1}\le\frac{ab+bc+ca}{2}.$$

Marin Chirciu

Solution .

We use pqr -Method.

We note $p=a+b+c, q=ab+bc+ca, r=abc$.

We have $r=1, \sum \frac{1}{ab+1}=\frac{\sum(bc+1)(ca+1)}{\prod(ab+1)}=\frac{\sum a+2\sum bc+3}{abc+\sum a+\sum bc+1}=\frac{p+2q+3}{p+q+2}$.

Inequality $\sum \frac{1}{ab+1} \le \frac{ab+bc+ca}{2}$ is written $\frac{p+2q+3}{p+q+2} \le \frac{q}{2} \Leftrightarrow q^2+pq \ge 2p+2q+6$.

Using $q^2=(ab+bc+ca)^2 \ge 3abc(a+b+c)=3rp=3p \Rightarrow q^2 \ge 3p$ is enough to show

$3p+pq \ge 2p+2q+6 \Leftrightarrow p+pq \ge 2q+6 \Leftrightarrow p(q+1) \ge 2q+6 \Leftrightarrow p \ge \frac{2q+6}{q+1}$.

Using $p^2 \ge 3q$ is enough to show that $3q \ge \left(\frac{2q+6}{q+1}\right)^2 \Leftrightarrow 3q^3+2q^2-21q-36 \ge 0 \Leftrightarrow$

$\Leftrightarrow (q-3)(3q^2+11q+12) \ge 0 \Leftrightarrow q \ge 3$, which is true from $q^2 \ge 3p$ and $p \ge 3$,

see $p=a+b+c \ge 3\sqrt[3]{abc}=3$.

Equality occurs if and only if $a=b=c=1$.

Aplicația9.

If $a,b,c>0$, $a+b+c=abc$ then

$$\frac{a}{b^3}+\frac{b}{c^3}+\frac{c}{a^3} \ge 1.$$

Sladjan Stankovik, Mathematical Inequalities,8/2014

Soluție.

$a+b+c=abc \Leftrightarrow \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=1$.

$$LHS=\frac{a}{b^3}+\frac{b}{c^3}+\frac{c}{a^3}=\left(\frac{a}{b^3}+\frac{b}{c^3}+\frac{c}{a^3}\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \overset{CBS}{\ge} \left(\sqrt{\frac{a}{b^3}\cdot\frac{1}{ab}}+\sqrt{\frac{b}{c^3}\cdot\frac{1}{bc}}+\sqrt{\frac{c}{a^3}\cdot\frac{1}{ca}}\right)^2=$$

$$=\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right)^2 \overset{SOS}{\ge} \left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2=1=RHS .$$

Remarca.

In ΔABC

$$\frac{\cot\frac{A}{2}}{\cot^3\frac{B}{2}}+\frac{\cot\frac{B}{2}}{\cot^3\frac{C}{2}}+\frac{\cot\frac{C}{2}}{\cot^3\frac{A}{2}}\geq 1.$$

Marin Chirciu

Soluție.

Lema

If $x,y,z>0$, $x+y+z=xyz$ then

$$\frac{x}{y^3}+\frac{y}{z^3}+\frac{z}{x^3}\geq 1.$$

Se cunoaște identitatea în triunghi $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{p}{r}$.

Folosind **Lema** pentru $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ obținem:

$$\frac{\cot\frac{A}{2}}{\cot^3\frac{B}{2}}+\frac{\cot\frac{B}{2}}{\cot^3\frac{C}{2}}+\frac{\cot\frac{C}{2}}{\cot^3\frac{A}{2}}\geq 1.$$

Aplicația10.

J628. If $a,b,c>0$, $ab+bc+ca=1$ then

$$\sum bc\sqrt{2a+b+c}\geq\frac{2}{\sqrt{a+b+c}}.$$

Nguyen Viet Hung, Vietnam, Mathematical Reflections3/2023,J628

Solution .

Using Holder-inequality we have:

$$\sum bc\sqrt{2a+b+c}\cdot\sum bc\sqrt{2a+b+c}\cdot\sum\frac{bc}{2a+b+c}\overset{Holder}{\geq}\left(\sum bc\right)^3=1.$$

It is enough to show that:

$\left(\sum bc\sqrt{2a+b+c}\right)^2\geq\frac{1}{\sum\frac{bc}{2a+b+c}}$,(1), which results from $\sum\frac{bc}{2a+b+c}\leq\frac{a+b+c}{4}$,(2) (see $\sum\frac{bc}{2a+b+c}=\sum\frac{bc}{(a+b)+(a+c)}\leq\frac{1}{4}\sum bc\left(\frac{1}{a+b}+\frac{1}{a+c}\right)=$

$=\frac{1}{4}\sum\left(\frac{bc}{a+b}+\frac{bc}{a+c}\right)=\frac{1}{4}\sum\left(\frac{ab}{b+c}+\frac{ac}{b+c}\right)=\frac{1}{4}\sum\frac{a(b+c)}{b+c}=\frac{1}{4}\sum a=\frac{a+b+c}{4}$).

From(1) and (2) we have:

$$\left(\sum bc\sqrt{2a+b+c}\right)^2\geq\frac{1}{\frac{a+b+c}{4}}\Leftrightarrow\left(\sum bc\sqrt{2a+b+c}\right)^2\geq\frac{4}{a+b+c}\Leftrightarrow$$

$$\Leftrightarrow\sum bc\sqrt{2a+b+c}\geq\frac{2}{\sqrt{a+b+c}}.$$

Equality occurs if and only if $a=b=c=\frac{1}{\sqrt{3}}$.

Remark.

The problem can develop.

If $a,b,c>0$, $a+b+c=1$ then

$$\sum a\sqrt{a+1}\geq\frac{2}{\sqrt{3}}.$$

Marin Chirciu

Solution .

Using Holder-inequality we have:

$$\sum a\sqrt{a+1}\cdot\sum a\sqrt{a+1}\cdot\sum\frac{a}{a+1}\overset{Holder}{\geq}\left(\sum a\right)^3=1.$$

It is enough to show that:

$\left(\sum a\sqrt{a+1}\right)^2\geq\frac{1}{\sum\frac{a}{a+1}}$,(1), which results from $\sum\frac{a}{a+1}\leq\frac{3}{4}$,(2)

(see $\sum\frac{a}{a+1}=\sum\frac{a}{2a+b+c}=\sum\frac{a}{(a+b)+(a+c)}\leq\frac{1}{4}a\left(\frac{1}{a+b}+\frac{1}{a+c}\right)=$

$=\frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{a}{a+c}\right)=\frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{b}{a+b}\right)=\frac{1}{4}\sum 1=\frac{1}{4}\cdot 3=\frac{3}{4}$).

From(1) and (2) we have:

$$\left(\sum a\sqrt{a+1}\right)^2 \geq \frac{1}{\frac{3}{4}} \Leftrightarrow \left(\sum a\sqrt{a+1}\right)^2 \geq \frac{4}{3} \Leftrightarrow \sum a\sqrt{a+1} \geq \frac{2}{\sqrt{3}}.$$

Equality occurs if and only if $a=b=c=\frac{1}{3}$.

Remark.

In ΔABC

$$\sum \frac{r}{r_a}\sqrt{\frac{3r}{r_a}+3} \geq 2.$$

Marin Chirciu

Solution .

Lemma.

If $x, y, z > 0$, $x+y+z=1$ then

$$\sum x\sqrt{x+1} \geq \frac{2}{\sqrt{3}}.$$

Using identity in triangle $\sum \frac{r}{r_a}=1$ and **Lemma** for $(x, y, z)=\left(\frac{r}{r_a}, \frac{r}{r_b}, \frac{r}{r_c}\right)$ we have>

$$\sum \frac{r}{r_a}\sqrt{\frac{r}{r_a}+1} \geq \frac{2}{\sqrt{3}} \Leftrightarrow \sum \frac{r}{r_a}\sqrt{\frac{3r}{r_a}+3} \geq 2.$$

Equality occurs if and only if the triangle is equilateral.

Remark.

In ΔABC

$$\sum \frac{r}{h_a}\sqrt{\frac{3r}{h_a}+3} \geq 2.$$

Marin Chirciu

Remark.

The problem can develop.

If $a,b,c>0$, $ab+bc+ca=1$ then

$$\sum bc\sqrt[3]{2a+b+c}\geq\frac{2}{\sqrt[3]{2(a+b+c)}}.$$

Marin Chirciu

Aplicația11.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$\sum\frac{x}{y+z}\leq\frac{3}{2}\left(x^2+y^2+z^2\right).$$

Elton Papanikolla,MathOlymp,7/2023

Soluţie.

$$\sum\frac{x}{y+z}\leq\frac{3}{2}\left(x^2+y^2+z^2\right)\Leftrightarrow 2\left(\sum x^3+\sum x\right)\leq 3\prod(y+z)\sum x^2$$

Folosim pqr -method.

Notăm $p=x+y+z, q=xy+yz+zx, r=xyz$.

Inegalitatea $2\left(\sum x^3+\sum x\right)\leq 3\prod(y+z)\sum x^2$ se scrie:

$$2\left(p^3-3pq-3r+p\right)\leq 3(pq-r)\left(p^2-2q\right)\overset{q=1}{\Leftrightarrow}2\left(p^3-3p-3r+p\right)\leq 3(p-r)\left(p^2-2\right)\Leftrightarrow$$

$\Leftrightarrow p^2-3pr-2\geq 0$, care rezultă din $p^2\geq 3q=3$ și $1=q^2\geq 3pr$.

Obținem: $p^2-3pr-2\geq 3-1-2=0$.

Remarca.

In ΔABC

$$\frac{3}{2}\leq\sum\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}\leq\frac{3}{2}\left(\frac{R}{r}-1\right).$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$\sum \frac{x}{y+z} \le \frac{3}{2}\left(x^2+y^2+z^2\right).$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$\sum \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \le \frac{3}{2}\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right).$$

Folosind: $\sum \tan^2\frac{A}{2}=\frac{(4R+r)^2}{p^2}-2 \overset{Gerretsen}{\le} \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}-2=\frac{R+r}{r}-2=\frac{R}{r}-1$, rezultă:

$$\sum \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \le \frac{3}{2}\left(\frac{R}{r}-1\right).$$

Aplicația12.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$\frac{x^2}{x+1}+\frac{y^2}{y+1}+\frac{z^2}{z+1} \ge \frac{\sqrt{3}}{\sqrt{3}+1}.$$

Matematics(College and High Scholl), Math for change 7/2023

Soluţie.

$$LHS=\sum\frac{x^2}{x+1} \overset{CS}{\ge} \frac{\left(\sum x\right)^2}{\sum(x+1)}=\frac{\left(\sum x\right)^2}{\sum x+3} \overset{(1)}{\ge} \frac{\sqrt{3}}{\sqrt{3}+1}=RHS,$$

unde (1) $\Leftrightarrow \frac{\left(\sum x\right)^2}{\sum x+3} \ge \frac{\sqrt{3}}{\sqrt{3}+1} \overset{\sqrt{x}=t}{\Leftrightarrow} \frac{t^2}{t+3} \ge \frac{\sqrt{3}}{\sqrt{3}+1} \Leftrightarrow \left(\sqrt{3}+1\right)t^2-\sqrt{3}t-3\sqrt{3}\ge 0 \Leftrightarrow$

$$\Leftrightarrow \left(t-\sqrt{3}\right)\left[\left(\sqrt{3}+1\right)t+3\right]\ge 0 \Leftrightarrow t\ge\sqrt{3}, \text{ vezi } 1=\sum xy \le \frac{1}{3}\left(\sum x\right)^2 \Rightarrow \sum x \ge \sqrt{3}.$$

Remarca.

If $x,y,z>0$, $xy+yz+zx=1$ and $\lambda \ge 0$ then

$$\frac{x^2}{x+\lambda}+\frac{y^2}{y+\lambda}+\frac{z^2}{z+\lambda}\geq\frac{\sqrt{3}}{\lambda\sqrt{3}+1}.$$

Marin Chirciu

Remarca.

In ΔABC

$$\sum\frac{\tan^2\frac{A}{2}}{\tan\frac{A}{2}+\lambda}\geq\frac{\sqrt{3}}{\lambda\sqrt{3}+1},\lambda\geq 0.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$, $xy+yz+zx=1$ and $\lambda\geq 0$ then

$$\frac{x^2}{x+\lambda}+\frac{y^2}{y+\lambda}+\frac{z^2}{z+\lambda}\geq\frac{\sqrt{3}}{\lambda\sqrt{3}+1}.$$

Se cunoaște identitatea în triunghi: $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$\sum\frac{\tan^2\frac{A}{2}}{\tan\frac{A}{2}+\lambda}\geq\frac{\sqrt{3}}{\lambda\sqrt{3}+1}.$$

Aplicația13.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$x\sqrt{y^2+1}+y\sqrt{z^2+1}+z\sqrt{x^2+1}\geq 2.$$

Vasile Cartoaje, Mathematical Inequalities7/2023

Soluţie.

Lema.

If $x>0$ then

$$\sqrt{x^2+1} \ge \frac{x+\sqrt{3}}{2}.$$

Demonstrație.

$\sqrt{x^2+1} \ge \frac{x+\sqrt{3}}{2} \Leftrightarrow \left(x\sqrt{3}-1\right)^2 \ge 0$, cu egalitate pentru $x = \frac{1}{\sqrt{3}}$.

$LHS = \sum x\sqrt{y^2+1} \overset{Lema}{\ge} \sum x \cdot \frac{y+\sqrt{3}}{2} = \frac{\sum xy + \sqrt{3}\sum x}{2} \overset{(1)}{\ge} \frac{1+\sqrt{3}\cdot\sqrt{3}}{2} = 2 = RHS$,

unde (1) $\Leftrightarrow \sum x \ge \sqrt{3}$, (vezi $\left(\sum x\right)^2 \ge 3\sum xy = 3\cdot 1 = 3 \Rightarrow \sum x \ge \sqrt{3}$).

Remarca.

If $x, y, z, \lambda > 0$, $xy + yz + zx = \lambda$ then

$$x\sqrt{y^2+\lambda} + y\sqrt{z^2+\lambda} + z\sqrt{x^2+\lambda} \ge 2\lambda.$$

Marin Chirciu

Remarca.

In ΔABC

$$\sum \tan\frac{A}{2}\sqrt{1+\tan^2\frac{B}{2}} \ge 2.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $xy + yz + zx = 1$ then

$$x\sqrt{y^2+1} + y\sqrt{z^2+1} + z\sqrt{x^2+1} \ge 2.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$.

Folosind **Lema** pentru $(x, y, z) = \left(\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right)$ obținem:

$\sum \tan\frac{A}{2}\sqrt{1+\tan^2\frac{B}{2}} \ge 2$.

Aplicația14.

If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\sum \frac{1}{x^3+1} \le \frac{3}{8xyz}.$$

AOPS, 7/2023

Soluţie.

Lema.

If $0 < x < \sqrt{3}$ then

$$\frac{1}{x^3+1} \le \frac{11-3x}{64}.$$

Demonstrație

Folosim Tangent Line Method pentru funcția $f:(0,\sqrt{3}) \to \mathbf{R}$ $f(x) = \frac{1}{x^3+1}$ în $x_0 = 1$.

Avem $f(1) = \frac{1}{8}$.

Ecuația tangentei în punctul $x_0 = 1$ este $y - f(x_0) = f'(x_0)(x - x_0)$.

Avem $f'(x) = \frac{-3x^2}{(x^3+7)^2}$, $f'(1) = \frac{-3}{64}$.

Ecuația tangentei în punctul $x_0 = 1$ este:

$$y - \frac{1}{8} = \frac{-3}{64}(x-1) \Leftrightarrow y = \frac{11-3x}{64}.$$

Arătăm că: $f(x) = \frac{1}{x^3+1} \le \frac{11-3x}{64} \Leftrightarrow 3x^4 - 11x^3 + 21x - 13 \le 0 \Leftrightarrow (x-1)^2(3x^2-5x-13) \le 0$,deoarece $(3x^2-5x-13) < 0$ pentru $0 < x < \sqrt{3}$ și $(x-1)^2$, cu egalitate pentru $x = 1$.

$$LHS = \sum \frac{1}{x^3+1} \overset{Lema}{\le} \sum \frac{11-3x}{64} = \frac{33-3\sum x}{64} \overset{(1)}{\le} \frac{33-3\cdot 3}{64} = \frac{3}{8} \overset{(2)}{\le} \frac{3}{8xyz} = RHS,$$

unde(1) $\Leftrightarrow \sum x \ge 3$, care rezultă din $\left(\sum x\right)^2 \ge 3\sum xy = 3\cdot 3 = 9 \Rightarrow \sum x \ge 3$.

(2) $\Leftrightarrow xyz \le 1$, vezi $3 = xy + yz + zx \ge 3\sqrt[3]{(xyz)^2} \Rightarrow xyz \le 1$.

Remarca.

In ΔABC

$$\sum \frac{1}{\left(\sqrt{3}\tan\frac{A}{2}\right)^3+7} \le \frac{3}{8}\cdot\frac{R}{2r}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\sum \frac{1}{x^3+1} \le \frac{3}{8xyz}.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\sqrt{3}\tan\frac{A}{2}, \sqrt{3}\tan\frac{B}{2}, \sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum \frac{1}{\left(\sqrt{3}\tan\frac{A}{2}\right)^3+7} \le \frac{3}{8\cdot\sqrt{3}\tan\frac{A}{2}\sqrt{3}\tan\frac{B}{2}\sqrt{3}\tan\frac{C}{2}} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\left(\sqrt{3}\tan\frac{A}{2}\right)^3+7} \le \frac{1}{8\sqrt{3}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}} \Leftrightarrow \sum \frac{1}{\left(\sqrt{3}\tan\frac{A}{2}\right)^3+7} \le \frac{1}{8\sqrt{3}\cdot\frac{r}{p}} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\left(\sqrt{3}\tan\frac{A}{2}\right)^3+7} \le \frac{p}{8\sqrt{3}\cdot r}.$$

Folosind inegalitatea lui Mitrinovic $p \le \frac{3R\sqrt{3}}{2}$ obținem:

$$\sum \frac{1}{\left(\sqrt{3}\tan\frac{A}{2}\right)^3+7} \le \frac{p}{8\sqrt{3}\cdot r} \le \frac{\frac{3R\sqrt{3}}{2}}{8\sqrt{3}\cdot r} = \frac{3}{8}\cdot\frac{R}{2r}$$

Remarca.

If $x, y, z > 0$, $xy + yz + zx = 3$ and $\lambda \ge 4$ then

$$\sum \frac{1}{x^3+\lambda} \le \frac{3}{(\lambda+1)xyz}.$$

Marin Chirciu

Soluţie.

Lema.

If $0<x<\sqrt{3}$ and $\lambda \geq 4$ then

$$\frac{1}{x^3+\lambda} \leq \frac{\lambda+4-3x}{(\lambda+1)^2}.$$

Demonstrație

Folosim Tangent Line Method pentru funcția $f:(0,\sqrt{3}) \to \mathbf{R}$ $f(x)=\frac{1}{x^3+\lambda}$ în $x_0=1$.

Aplicația15.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\sum \frac{x}{2y^3+1} \geq 1.$$

Pham Van Tuyen, Vietnam, THCS 6/2023

Soluţie.

Lema.

If $x,y,z>0$ then

$$\frac{x}{2y^3+1} \geq x-\frac{2}{3}xy.$$

Demonstrație

$\frac{x}{2y^3+1}=x\left(1-\frac{2y^3}{2y^3+1}\right) \overset{AM-GM}{\geq} x\left(1-\frac{2y^3}{3y^2}\right)=x-\frac{2}{3}xy$, cu egalitate pentru $y=1$.

$LHS=\sum \frac{x}{2y^3+1} \overset{Lema}{\geq} \sum\left(x-\frac{2}{3}xy\right)=\sum x-\frac{2}{3}\sum xy=\sum x-\frac{2}{3}\cdot 3=\sum x-2 \overset{(1)}{\geq} 3-2=1=RHS$,

unde(1) $\Leftrightarrow \sum x \geq 3$, care rezultă din $\left(\sum x\right)^2 \geq 3\sum xy=3\cdot 3=9 \Rightarrow \sum x \geq 3$.

Remarca.

In ΔABC

$$\sum \frac{\tan\frac{A}{2}}{6\tan^3\frac{A}{2}+\frac{1}{\sqrt{3}}} \geq 1.$$

Marin Chirciu

Soluţie.

Lema.

I If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\sum \frac{x}{2y^3+1} \geq 1.$$

Se cunoaşte identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2} \cdot \sqrt{3}\tan\frac{C}{2} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\sqrt{3}\tan\frac{A}{2}, \sqrt{3}\tan\frac{B}{2}, \sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum \frac{\sqrt{3}\tan\frac{A}{2}}{2\left(\sqrt{3}\tan\frac{A}{2}\right)^3+1} \geq 1 \Leftrightarrow \sum \frac{\tan\frac{A}{2}}{6\tan^3\frac{A}{2}+\frac{1}{\sqrt{3}}} \geq 1.$$

Remarca.

If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\sum \frac{x}{3y^4+1} \geq \frac{3}{4}.$$

Marin Chirciu

Remarca.

If $x, y, z > 0$, $xy + yz + zx = 3$ and $n \in \mathbf{N}$ then

$$\sum \frac{x}{ny^{n+1}+1} \geq \frac{3}{n+1}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$ then

$$\frac{x}{ny^{n+1}+1} \geq x - \frac{n}{n+1}xy.$$

Aplicația16.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum \frac{yz}{\sqrt{x+yz}} \le \frac{1}{2}.$$

Hoang Dot Toan, Vietnam,THCS 6/2023

Soluţie.

Lema.

If $x, y, z > 0$, $x+y+z=1$ then

$$\frac{yz}{\sqrt{x+yz}} \le \frac{yz}{2}\left(\frac{1}{x+y}+\frac{1}{x+z}\right).$$

Demonstrație

$$\frac{yz}{\sqrt{x+yz}} = \frac{yz}{\sqrt{x(x+y+z)+yz}} = \frac{yz}{\sqrt{x^2+xy+yz+zx}} = \frac{yz}{\sqrt{(x+y)(x+z)}} = yz\sqrt{\frac{1}{x+y}\cdot\frac{1}{x+z}} \overset{AM-GM}{\le}$$

$$\overset{AM-GM}{\le} \frac{yz}{2}\left(\frac{1}{x+y}+\frac{1}{x+z}\right).$$

$$LHS = \sum \frac{yz}{\sqrt{x+yz}} \le \sum \frac{yz}{2}\left(\frac{1}{x+y}+\frac{1}{x+z}\right) = \frac{1}{2}\sum \frac{xy+xz}{y+z} = \frac{1}{2}\sum x = \frac{1}{2} = RHS.$$

Remarca.

In ΔABC

$$\sum \frac{\sqrt{rr_a}}{\sqrt{1+\frac{p^2}{r_a^2}}} \le \frac{p}{2}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x+y+z=1$ then

$$\sum \frac{yz}{\sqrt{x+yz}} \le \frac{1}{2}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1.$

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{xyz}{\sqrt{x}\sqrt{x^2+xyz}}\le\frac{1}{2}\Leftrightarrow\sum\frac{\frac{r}{r_a}\frac{r}{r_b}\frac{r}{r_c}}{\sqrt{\frac{r}{r_a}}\sqrt{\left(\frac{r}{r_a}\right)^2+\frac{r}{r_a}\frac{r}{r_b}\frac{r}{r_c}}}\le\frac{1}{2}\Leftrightarrow\sum\frac{\sqrt{rr_a}}{\sqrt{1+\frac{p^2}{r_a^2}}}\le\frac{p}{2}.$$

Am folosit mai sus $r_ar_br_c=rp^2$.

Aplicația16.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{1}{2x^2+2x+yz}\ge\frac{1}{xy+yz+zx}.$$

Pham Van Tuyen, Vietnam

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$\frac{1}{2x^2+2x+yz}\ge\frac{yz}{(xy+yz+zx)^2}.$$

Demonstrație

$$2x^2+2x+yz=2x^2+2x(x+y+z)+yz=4x^2+2xy+2xz+yz=(2x+y)(2x+z).$$

$$\frac{1}{2x^2+2x+yz}=\frac{1}{(2x+y)(2x+z)}=\frac{yz}{(2xz+yz)(2xy+yz)}\overset{AM-GM}{\ge}\frac{yz}{\left[\frac{(2xz+yz)+(2xy+yz)}{2}\right]^2}=$$

$=\frac{yz}{(xy+yz+zx)^2}$, cu egalitate pentru $(2xz+yz)=(2xy+yz)\Leftrightarrow y=z$.

$$\sum\frac{1}{2x^2+2x+yz}\ge\sum\frac{yz}{(xy+yz+zx)^2}=\frac{1}{xy+yz+zx}.$$

Remarca.

In ΔABC

$$\sum\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\ge3r.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{1}{2x^2+2x+yz}\geq\frac{1}{xy+yz+zx}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{1}{2\left(\frac{r}{r_a}\right)^2+2\frac{r}{r_a}+\frac{r}{r_b}\frac{r}{r_c}}\geq\frac{1}{\frac{r}{r_a}\frac{r}{r_b}+\frac{r}{r_b}\frac{r}{r_c}+\frac{r}{r_c}\frac{r}{r_a}}\Leftrightarrow\sum\frac{1}{2\frac{r^2}{r_a^2}+2\frac{r}{r_a}+\frac{r^2}{r_br_c}}\geq\frac{1}{r^2\left(\frac{1}{r_ar_b}+\frac{1}{r_br_c}+\frac{1}{r_cr_a}\right)}\Leftrightarrow$$

$$\sum\frac{1}{2\frac{r}{r_a^2}+2\frac{1}{r_a}+\frac{r}{r_br_c}}\geq\frac{1}{r\frac{4R+r}{rp^2}}\Leftrightarrow\sum\frac{1}{2\frac{r}{r_a^2}+2\frac{1}{r_a}+\frac{r}{r_br_c}}\geq\frac{1}{\frac{4R+r}{p^2}}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\geq\frac{p^2}{4R+r}.$$

Folosind inegalitatea lui Doucet $p^2\geq3r(4R+r)$ obținem

$$\sum\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\geq\frac{3r(4R+r)}{4R+r}=3r.$$

Remarca.

In ΔABC

$$\sum\frac{1}{2\frac{r}{h_a^2}+2\frac{1}{h_a}+\frac{r}{h_bh_c}}\geq3r.$$

Aplicația17.

If $x,y,z>0$, $x+y+z=1$ then

$$33(xy+yz+zx)\leq54xyz+9.$$

Trinh Ha, Vietnam,THCS 6/2023

Soluţie.

Folosim pqr -Method.

Notăm $x+y+z=p, xy+yz+zx=q, xyz=r$.

Avem $p=1, q^2=(xy+yz+zx)^2 \geq 3xyz(x+y+z)=3rp=3r \Rightarrow q^2 \geq 3r$,

$1=x+y+z \geq 3\sqrt[3]{xyz}=3\sqrt[3]{r} \Rightarrow r \leq \frac{1}{27}$.

Omogenizând inegalitatea se scrie $33(xy+yz+zx)(x+y+z) \leq 54xyz+9(x+y+z)^3 \Leftrightarrow$

$\Leftrightarrow 11qp \leq 18r+3p^3$.

Folosind inegalitatea lui Schur $p^3+9r \geq 4pq$ este suficient să arătăm că:

$11pq \leq 18r+3(4pq-9r) \Leftrightarrow pq \geq 9r$, care rezultă din $p=3$, $q^2 \geq 3r$ și $r \leq \frac{1}{27}$.

Remarca.

If $x, y, z>0$, $x+y+z=1$ and $0 \leq \lambda \leq 4$ then

$$\lambda(xy+yz+zx) \leq 9(\lambda-3)xyz+1.$$

Marin Chirciu

Remarca.

In ΔABC

$$4r^2\left(\frac{1}{r_ar_b}+\frac{1}{r_br_c}+\frac{1}{r_cr_a}\right) \leq 9\frac{r^3}{r_ar_br_c}+1.$$

Marin Chirciu

Lema.

If $x, y, z>0$, $x+y+z=1$ then

$$4(xy+yz+zx) \leq 9xyz+1.$$

Soluţie.

Folosim pqr -Method.

Notăm $x+y+z=p, xy+yz+zx=q, xyz=r$.

Avem $p=1, q^2=(xy+yz+zx)^2 \geq 3xyz(x+y+z)=3rp=3r \Rightarrow q^2 \geq 3r$,

$1=x+y+z \geq 3\sqrt[3]{xyz}=3\sqrt[3]{r} \Rightarrow r \leq \frac{1}{27}$.

Omogenizând inegalitatea se scrie $4(xy+yz+zx)(x+y+z)\le 9xyz+(x+y+z)^3 \Leftrightarrow$

$\Leftrightarrow 4qp\le 9r+p^3$,(Inegalitatea lui Schur).

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{3}$.

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$4\left(\frac{r}{r_a}\frac{r}{r_b}+\frac{r}{r_b}\frac{r}{r_c}+\frac{r}{r_c}\frac{r}{r_a}\right)\le 9\frac{r}{r_a}\frac{r}{r_b}\frac{r}{r_c}+1\Leftrightarrow 4r^2\left(\frac{1}{r_ar_b}+\frac{1}{r_br_c}+\frac{1}{r_cr_a}\right)\le 9\frac{r^3}{r_ar_br_c}+1.$$

Aplicația18.

If $x,y,z>0$, $x+y+z=xyz$ then

$$\sum\frac{y}{x\sqrt{y^2+1}}\ge\frac{3}{2}.$$

Dung Nguyen Tan, Vietnam,THCS 5/2023

Soluție.

Folosim $x+y+z=xyz\Leftrightarrow\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=1$.

Cu substituția $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)=(a,b,c)$ problema se reformulează.

If $a,b,c>0$, $ab+bc+ca=1$ then

$$\sum\frac{a}{\sqrt{b^2+1}}\ge\frac{3}{2}.$$

Demonstrație.

$$\sum\frac{a}{\sqrt{b^2+1}}=\sum\frac{a}{\sqrt{b^2+ab+bc+ca}}=\sum\frac{a}{\sqrt{(b+a)(b+c)}}\overset{AM-GM}{\ge}\sum\frac{a}{\frac{(b+a)+(b+c)}{2}}=$$

$$=\sum\frac{2a}{a+2b+c}=\sum\frac{2a^2}{a^2+2ab+ac}\overset{CS}{\ge}\frac{2\left(\sum a\right)^2}{\sum a^2+3\sum ab}=\frac{2\sum a^2+4\sum ab}{\sum a^2+3\sum ab}\overset{(1)}{\ge}\frac{3}{2},$$

unde (1) $\Leftrightarrow \sum a^2 \geq \sum ab$, inegalitate cunoscută.

Remarca.

In ΔABC

$$\sum \frac{\cot\frac{B}{2}}{\cot\frac{A}{2}\sqrt{1+\cot^2\frac{B}{2}}} \geq \frac{3}{2}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x+y+z = xyz$ then

$$\sum \frac{y}{x\sqrt{y^2+1}} \geq \frac{3}{2}.$$

Se cunoaște identitatea în triunghi $\sum \cot\frac{A}{2} = \prod \cot\frac{A}{2} = \frac{p}{r}$.

Folosind **Lema** pentru $(x, y, z) = \left(\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}\right)$ obținem:

$$\sum \frac{\cot\frac{B}{2}}{\cot\frac{A}{2}\sqrt{1+\cot^2\frac{B}{2}}} \geq \frac{3}{2}.$$

Remarca.

In ΔABC

$$\sum \frac{\tan\frac{A}{2}}{\sqrt{1+\tan^2\frac{B}{2}}} \geq \frac{3}{2}.$$

Marin Chirciu

Soluţie.

Lema.

If $a, b, c > 0$, $ab+bc+ca = 1$ then

$$\sum \frac{a}{\sqrt{b^2+1}} \geq \frac{3}{2}.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2}=1$.

Folosind **Lema** pentru $(a,b,c)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$\sum\frac{\tan\frac{A}{2}}{\sqrt{1+\tan^2\frac{B}{2}}}\geq\frac{3}{2}.$$

Aplicația19.

If $x,y,z>0$, $x+y+z=1$ then find min of

$$P=\frac{1}{x^2+y^2+z^2}+\frac{2022}{xy+yz+zx}.$$

Tran Anh Tai, Vietnam,THCS 5/2023

Remarca.

Let $\lambda\geq 2$ fixed. If $x,y,z>0$, $x+y+z=1$ then find min of

$$P=\frac{1}{x^2+y^2+z^2}+\frac{\lambda}{xy+yz+zx}.$$

Marin Chirciu

Remarca.

In ΔABC

$$\frac{1}{\sum\frac{1}{r_a^2}}+\frac{\lambda}{\sum\frac{1}{r_br_c}}\geq 3(\lambda+1)r^2,\lambda\geq 2.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $x+y+z=1$ and $\lambda\geq 2$ then

$$\frac{1}{x^2+y^2+z^2}+\frac{\lambda}{xy+yz+zx}\geq 3(\lambda+1).$$

Demonstrație.

$$P=\frac{1}{\sum x^2}+\frac{\lambda}{\sum xy}=\left(\frac{1}{\sum x^2}+\frac{1}{\sum xy}+\frac{1}{\sum xy}\right)+\frac{\lambda-2}{\sum xy}\overset{CS}{\geq}\frac{(1+1+1)^2}{\sum x^2+2\sum xy}+\frac{\lambda-2}{\sum xy}=$$

$$=\frac{9}{\left(\sum x\right)^2}+\frac{\lambda-2}{\sum xy}\overset{CS}{\geq}\frac{9}{\left(\sum x\right)^2}+\frac{\lambda-2}{\frac{1}{3}\left(\sum x\right)^2}=\frac{9}{1^2}+\frac{\lambda-2}{\frac{1}{3}\cdot 1^2}=9+3(\lambda-2)=3(\lambda+1).$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\frac{1}{\sum\left(\frac{r}{r_a}\right)^2}+\frac{\lambda}{\sum\frac{r}{r_b}\frac{r}{r_c}}\geq 3(\lambda+1)\Leftrightarrow\frac{1}{\sum\frac{1}{r_a^2}}+\frac{\lambda}{\sum\frac{1}{r_b r_c}}\geq 3(\lambda+1)r^2.$$

Remarca.

In ΔABC

$$\frac{1}{\sum\frac{1}{h_a^2}}+\frac{\lambda}{\sum\frac{1}{h_b h_c}}\geq 3(\lambda+1)r^2,\lambda\geq 2.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $x+y+z=1$ and $\lambda\geq 2$ then

$$\frac{1}{x^2+y^2+z^2}+\frac{\lambda}{xy+yz+zx}\geq 3(\lambda+1).$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Leftrightarrow\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{h_a},\frac{r}{h_b},\frac{r}{h_c}\right)$ obținem:

$$\frac{1}{\sum\left(\frac{r}{h_a}\right)^2}+\frac{\lambda}{\sum\frac{r}{h_b}\frac{r}{h_c}}\geq 3(\lambda+1)\Leftrightarrow\frac{1}{\sum\frac{1}{h_a^2}}+\frac{\lambda}{\sum\frac{1}{h_b h_c}}\geq 3(\lambda+1)r^2.$$

Aplicaţia20.

If $x,y,z>0$ then

$$x^2+y^2+z^2+2xyz+\frac{18}{xy+yz+zx}\ge 11.$$

Dung Nguyen Tan, Vietnam, THCS 5/2023

Soluţie.

Lema.

If $x,y,z\ge 0$ then

$$x^2+y^2+z^2+2xyz+1\ge 2(xy+yz+zx).$$

Darij Grinberg

$$LHS=x^2+y^2+z^2+2xyz+\frac{18}{xy+yz+zx}\overset{Lema}{\ge}2(xy+yz+zx)-1+\frac{18}{xy+yz+zx}=$$

$$=2(xy+yz+zx)+\frac{18}{xy+yz+zx}-1\overset{AM-GM}{\ge}2\cdot 2\sqrt{(xy+yz+zx)\cdot\frac{9}{xy+yz+zx}}-1=4\cdot 3-1=11.$$

Remarca.

In ΔABC

$$\frac{\sum\tan\frac{B}{2}\tan\frac{C}{2}}{\sum\tan^2\frac{A}{2}}+\frac{\left(\sum\tan\frac{A}{2}\right)^3}{9\prod\tan\frac{A}{2}}\ge 4.$$

Marin Chirciu

Soluţie.

If $x,y,z>0$ then

$$x^2+y^2+z^2+2xyz+\frac{18}{xy+yz+zx}\ge 11.$$

Soluţie.

Lema.

If $x,y,z\ge 0$ then

$$x^2+y^2+z^2+2xyz+1\ge 2(xy+yz+zx).$$

Darij Grinberg

Folosind **Lema** pentru $(x,y,z)=\left(2\sin\frac{A}{2},2\sin\frac{B}{2},2\sin\frac{C}{2}\right)$ obținem:

$$\sum\left(2\sin\frac{A}{2}\right)^2+2\prod\left(2\sin\frac{A}{2}\right)+\frac{18}{\sum\left(2\sin\frac{A}{2}\right)\left(2\sin\frac{A}{2}\right)}\ge 11 \Leftrightarrow$$

$$\Leftrightarrow 4\sum\sin^2\frac{A}{2}+16\prod\sin\frac{A}{2}+\frac{18}{4\sum\sin\frac{B}{2}\sin\frac{C}{2}}\ge 11 \Leftrightarrow$$

$$\Leftrightarrow 8\sum\sin^2\frac{A}{2}+32\prod\sin\frac{A}{2}+\frac{9}{\sum\sin\frac{B}{2}\sin\frac{C}{2}}\ge 22$$

Aplicația21.

If $x,y,z>0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{xyz}\ge 28.$$

An Khanh Le, Vietnam, THCS 5/2023

Remarca.

If $x,y,z>0$ and $\lambda\ge\frac{1}{9}$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\lambda\frac{(x+y+z)^3}{xyz}\ge 27\lambda+1.$$

Marin Chirciu

Soluție.

Lema

If $x,y,z>0$ then

$$\frac{(x+y+z)^3}{xyz}\ge\frac{9\sum x^2}{\sum yz}+18.$$

Demonstrație.

Avem $\sum x\sum yz\ge 9xyz \Rightarrow \frac{(x+y+z)^3}{xyz}\ge\frac{9(x+y+z)^2}{xy+yz+zx}=\frac{9\sum x^2+18\sum yz}{\sum yz}=\frac{9\sum x^2}{\sum yz}+18.$

$$LHS=\frac{xy+yz+zx}{x^2+y^2+z^2}+\lambda\frac{(x+y+z)^3}{xyz}\overset{Lema}{\ge}\frac{\sum yz}{\sum x^2}+\frac{9\lambda\sum x^2}{\sum yz}+18\lambda=\left(\frac{\sum yz}{\sum x^2}+\frac{\sum x^2}{\sum yz}\right)+$$

$$\frac{(9\lambda-1)\sum x^2}{\sum yz}+18\lambda \overset{SOS}{\geq} 2+(9\lambda-1)+18\lambda=27\lambda+1=RHS.$$

Remarca.

Cazul $\lambda=\frac{1}{9}$

If $x,y,z>0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\geq 4.$$

Marin Chirciu

Remarca.

In ΔABC

$$\frac{\sum \tan\frac{B}{2}\tan\frac{C}{2}}{\sum \tan^2\frac{A}{2}}+\frac{\left(\sum \tan\frac{A}{2}\right)^3}{9\prod \tan\frac{A}{2}}\geq 4.$$

Marin Chirciu

Soluţie.

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\geq \frac{\sum yz}{\sum x^2}+\frac{\sum x^2}{\sum yz}+2 \overset{AM-GM}{\geq} 2+2=4.$$

Folosind Lema pentru $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$\frac{\sum \tan\frac{B}{2}\tan\frac{C}{2}}{\sum \tan^2\frac{A}{2}}+\frac{\left(\sum \tan\frac{A}{2}\right)^3}{9\prod \tan\frac{A}{2}}\geq 4.$$

Remarca.

In ΔABC

$$\frac{\sum \cot\frac{B}{2}\cot\frac{C}{2}}{\sum \cot^2\frac{A}{2}}+\frac{\left(\sum \cot\frac{A}{2}\right)^3}{9\prod \cot\frac{A}{2}}\geq 4.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\geq 4.$$

Folosind Lema pentru $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ obținem:

$$\frac{\sum\cot\frac{B}{2}\cot\frac{C}{2}}{\sum\cot^2\frac{A}{2}}+\frac{\left(\sum\cot\frac{A}{2}\right)^3}{9\prod\cot\frac{A}{2}}\geq 4.$$

Remarca.

In ΔABC

$$\frac{\sum\sin\frac{B}{2}\sin\frac{C}{2}}{\sum\sin^2\frac{A}{2}}+\frac{\left(\sum\sin\frac{A}{2}\right)^3}{9\prod\sin\frac{A}{2}}\geq 4.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\geq 4.$$

Folosind Lema pentru $(x,y,z)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ obținem:

$$\frac{\sum\sin\frac{B}{2}\sin\frac{C}{2}}{\sum\sin^2\frac{A}{2}}+\frac{\left(\sum\sin\frac{A}{2}\right)^3}{9\prod\sin\frac{A}{2}}\geq 4.$$

Remarca.

In ΔABC

$$\frac{\sum \cos\frac{B}{2}\cos\frac{C}{2}}{\sum \cos^2\frac{A}{2}}+\frac{\left(\sum \cos\frac{A}{2}\right)^3}{9\prod \cos\frac{A}{2}}\ge 4.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\ge 4.$$

Folosind **Lema** pentru $(x,y,z)=\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right)$ obținem:

$$\frac{\sum \cos\frac{B}{2}\cos\frac{C}{2}}{\sum \cos^2\frac{A}{2}}+\frac{\left(\sum \cos\frac{A}{2}\right)^3}{9\prod \cos\frac{A}{2}}\ge 4.$$

Remarca.

In ΔABC

$$\frac{\sum \sec\frac{B}{2}\sec\frac{C}{2}}{\sum \sec^2\frac{A}{2}}+\frac{\left(\sum \sec\frac{A}{2}\right)^3}{9\prod \sec\frac{A}{2}}\ge 4.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\ge 4.$$

Folosind Lema pentru $(x,y,z)=\left(\sec\frac{A}{2},\sec\frac{B}{2},\sec\frac{C}{2}\right)$ obținem:

$$\frac{\sum \sec\frac{B}{2}\sec\frac{C}{2}}{\sum \sec^2\frac{A}{2}}+\frac{\left(\sum \sec\frac{A}{2}\right)^3}{9\prod \sec\frac{A}{2}}\geq 4.$$

Remarca.

In ΔABC

$$\frac{\sum \csc\frac{B}{2}\csc\frac{C}{2}}{\sum \csc^2\frac{A}{2}}+\frac{\left(\sum \csc\frac{A}{2}\right)^3}{9\prod \csc\frac{A}{2}}\geq 4.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$ then

$$\frac{xy+yz+zx}{x^2+y^2+z^2}+\frac{(x+y+z)^3}{9xyz}\geq 4.$$

Folosind Lema pentru $(x,y,z)=\left(\sec\frac{A}{2},\sec\frac{B}{2},\sec\frac{C}{2}\right)$ obținem:

$$\frac{\sum \csc\frac{B}{2}\csc\frac{C}{2}}{\sum \csc^2\frac{A}{2}}+\frac{\left(\sum \csc\frac{A}{2}\right)^3}{9\prod \csc\frac{A}{2}}\geq 4.$$

Aplicația22.

If $x, y, z > 0$, $x+y+z=xyz$ then

$$\sum \frac{1}{1+yz}\leq \frac{3}{4}.$$

Boris Colakovic, MathAtelier 5/2023

Remarca.

In ΔABC

$$\sum \frac{1}{1+\cot\frac{B}{2}\cot\frac{C}{2}}\leq \frac{3}{4}.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$, $x+y+z=xyz$ then

$$\sum\frac{1}{1+yz}\leq\frac{3}{4}.$$

Avem $x+y+z=xyz\Leftrightarrow\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}=1$.

Cu substituția $\left(\frac{1}{yz},\frac{1}{zx},\frac{1}{xy}\right)=(a,b,c)$ problema se reformulează.

If $a,b,c>0$, $a+b+c=1$ then

$$\sum\frac{a}{1+a}\leq\frac{3}{4}.$$

Demonstrație

$\sum\frac{a}{1+a}\leq\frac{3}{4}\Leftrightarrow\sum\frac{1}{1+a}\geq\frac{9}{4}$, care rezultă din :

$\sum\frac{1}{1+a}\overset{CS}{\geq}\frac{9}{\sum(1+a)}=\frac{9}{3+\sum a}=\frac{9}{3+1}=\frac{9}{4}$.

Se cunoaște identitatea în triunghi $\sum\cot\frac{A}{2}=\prod\cot\frac{A}{2}=\frac{p}{r}$.

Folosind **Lema** pentru $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ obținem:

$\sum\frac{1}{1+\cot\frac{B}{2}\cot\frac{C}{2}}\leq\frac{3}{4}$.

Aplicația23.

If $x,y,z>0$, $x+y+z=3$ then find max of

$$P=\sum\sqrt{x+yz}.$$

Anh Duc, Vietnam,THCS 5/2023

Remarca.

Let $\lambda\geq 0$ fixed. If $x,y,z>0$, $x+y+z=3$ then find max of

$$P=\sum\sqrt{x+\lambda yz}\,.$$

Marin Chirciu

Soluţie.

$$P=\frac{1}{\sqrt{\lambda+1}}\sum\sqrt{(\lambda+1)(x+\lambda yz)}\overset{AM-GM}{\le}\frac{1}{\sqrt{\lambda+1}}\sum\frac{\lambda+1+(x+\lambda yz)}{2}=\frac{1}{2\sqrt{\lambda+1}}\sum(\lambda+1+x+yz)=$$

$$=\frac{1}{2\sqrt{\lambda+1}}\left(3\lambda+3+\sum x+\lambda\sum yz\right)\overset{SOS}{\le}\frac{1}{2\sqrt{2}}\left(3\lambda+3+\sum x+\frac{\lambda}{3}\left(\sum x\right)^2\right)\overset{SOS}{\le}\frac{1}{2\sqrt{\lambda+1}}\left(3\lambda+6+\frac{\lambda}{3}3^2\right)=$$

$$=\frac{1}{2\sqrt{\lambda+1}}\cdot 6(\lambda+1)=3\sqrt{\lambda+1}\,.$$

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Deducem că $\max P=3\sqrt{\lambda+1}$ pentru $x=y=z=1$.

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\sum\sqrt{\frac{1}{r_a}+\frac{3\lambda r}{r_b r_c}}\le\sqrt{\frac{3(\lambda+1)}{r}}\,.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$, $x+y+z=3$ and $\lambda\ge 0$ then

$$\sum\sqrt{x+\lambda yz}\le 3\sqrt{\lambda+1}\,.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum\sqrt{\frac{3r}{r_a}+\lambda\frac{3r}{r_b}\frac{3r}{r_c}}\le 3\sqrt{\lambda+1}\Leftrightarrow=\sum\sqrt{\frac{1}{r_a}+\frac{3\lambda r}{r_b r_c}}\le\sqrt{\frac{3(\lambda+1)}{r}}\quad\Leftrightarrow$$

Remarca.

In ΔABC

$$\sum\sqrt{\frac{1}{h_a}+\frac{3\lambda r}{h_b h_c}}\le\sqrt{\frac{3(\lambda+1)}{r}}.$$

Marin Chirciu

Aplicația24.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{yz}{x^2+xyz}\ge\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}.$$

Quan Le, Vietnam,THCS 5/2023

Remarca.

In ΔABC

$$\sum\frac{\frac{1}{r_b r_c}}{\frac{1}{r_a^2}+\frac{1}{p^2}}\ge\frac{R}{r}+\frac{1}{4}.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{yz}{x^2+xyz}\ge\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}.$$

If $x,y,z>0$, $x+y+z=1$ then

$$\frac{yz}{x^2+xyz}=\frac{1}{x}-\frac{1}{(x+y)(x+z)}.$$

Demonstrație.

$$\frac{yz}{x^2+xyz}=\frac{1}{x}-\frac{1}{x+yz}=\frac{1}{x}-\frac{1}{x(x+y+z)+yz}=\frac{1}{x}-\frac{1}{(x+y)(x+z)}.$$

$$LHS=\sum\frac{yz}{x^2+xyz}=\sum\left(\frac{1}{x}-\frac{1}{(x+y)(x+z)}\right)=\sum\frac{1}{x}-\sum\frac{1}{(x+y)(x+z)}\overset{Lema8/9}{\ge}\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}=RHS$$

$$\sum\frac{1}{x}-\sum\frac{1}{(x+y)(x+z)}\overset{Lema8/9}{\ge}\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}\Leftrightarrow\frac{3}{4}\sum\frac{1}{x}\ge\sum\frac{1}{(x+y)(x+z)}\Leftrightarrow$$

$$\Leftrightarrow \frac{\sum(y+z)}{(x+y)(y+z)(z+x)} \leq \frac{3}{4}\sum\frac{1}{x} \Leftrightarrow \frac{2}{(x+y)(y+z)(z+x)} \leq \frac{3}{4}\sum\frac{1}{x} \Leftrightarrow$$

$$\Leftrightarrow (x+y)(y+z)(z+x) \geq \frac{8}{3}\cdot\frac{1}{\sum\frac{1}{x}}.$$

Lema8/9 .

If $x, y, z > 0$ then

$$(x+y)(y+z)(z+x) \geq \frac{8}{9}(x+y+z)(xy+yz+zx).$$

Lema8/9

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x, y, z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{\frac{r}{r_b}\frac{r}{r_c}}{\left(\frac{r}{r_a}\right)^2+\frac{r}{r_a}\frac{r}{r_b}\frac{r}{r_c}} \geq \frac{1}{4\frac{r}{r_a}}+\frac{1}{4\frac{r}{r_b}}+\frac{1}{4\frac{r}{r_c}} \Leftrightarrow \sum\frac{\frac{r^2}{r_br_c}}{\frac{r^2}{r_a^2}+\frac{r^3}{r_ar_br_c}} \geq \frac{r_a+r_b+r_c}{4r} \Leftrightarrow$$

$$\Leftrightarrow \sum\frac{\frac{r^2}{r_br_c}}{\frac{r^2}{r_a^2}+\frac{r^3}{rp^2}} \geq \frac{4R+r}{4r} \Leftrightarrow \sum\frac{\frac{1}{r_br_c}}{\frac{1}{r_a^2}+\frac{1}{p^2}} \geq \frac{R}{r}+\frac{1}{4}.$$

Remarca.

In ΔABC

$$\sum\frac{\frac{1}{h_bh_c}}{\frac{1}{h_a^2}+\frac{R}{p^2r}} \geq \frac{5}{2}-\frac{r}{2R}.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$, $x+y+z=1$ then

$$\sum\frac{yz}{x^2+xyz}\ge\frac{1}{4x}+\frac{1}{4y}+\frac{1}{4z}.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Leftrightarrow\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{h_a},\frac{r}{h_b},\frac{r}{h_c}\right)$ obținem:

$$\sum\frac{\frac{r}{h_b}\frac{r}{h_c}}{\left(\frac{r}{h_a}\right)^2+\frac{r}{h_a}\frac{r}{h_b}\frac{r}{h_c}}\ge\frac{1}{4\frac{r}{h_a}}+\frac{1}{4\frac{r}{h_b}}+\frac{1}{4\frac{r}{h_c}}\Leftrightarrow\sum\frac{\frac{r^2}{h_bh_c}}{\frac{r^2}{h_a^2}+\frac{r^3}{h_ah_bh_c}}\ge\frac{h_a+h_b+h_c}{4r}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\frac{r^2}{h_bh_c}}{\frac{r^2}{h_a^2}+\frac{r^3}{\frac{2p^2r^2}{2R}}}\ge\frac{\frac{p^2+r^2+4Rr}{2R}}{4r}\Leftrightarrow\sum\frac{\frac{1}{h_bh_c}}{\frac{1}{h_a^2}+\frac{R}{p^2r}}\ge\frac{p^2+r^2+4Rr}{8Rr}.$$

$$\sum\frac{\frac{1}{h_bh_c}}{\frac{1}{h_a^2}+\frac{R}{p^2r}}\ge\frac{p^2+r^2+4Rr}{8Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2+r^2+4Rr}{8Rr}=\frac{20Rr-4r^2}{8Rr}=$$

$$=\frac{4r(5R-r)}{8Rr}=\frac{(5R-r)}{2R}=\frac{5}{2}-\frac{r}{2R}$$

<u>Aplicația24.</u>

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}\ge\frac{8}{x^2+7}+\frac{8}{y^2+7}+\frac{8}{z^2+7}.$$

Nguyen Huy Dang, Vietnam,THCS 5/2023

<u>Remarca.</u>

If $x,y,z>0$, $xy+yz+zx=3$ and $n\in\mathbf{N}^*$ then

$$\frac{1}{\sqrt[n]{xy}}+\frac{1}{\sqrt[n]{yz}}+\frac{1}{\sqrt[n]{zx}}\ge\frac{4n}{x^2+4n-1}+\frac{4n}{y^2+4n-1}+\frac{4n}{z^2+4n-1}.$$

Marin Chirciu

<u>Soluţie.</u>

<u>Lema.</u>

If $x>0$ and $n\in\mathbf{N}^*$ then

$$\frac{4n}{x^2+4n-1}\le\frac{1}{\sqrt[2n]{x}}.$$

Demonstrație.

$x^2+4n-1\overset{AM-GM}{\ge}4n\sqrt[4n]{x^2\cdot\underbrace{1\cdot1\ldots\cdot1}_{n}}=4n\sqrt[4n]{x^2}=4n\sqrt[2n]{x}$, cu egalitate pentru $x=1$.

Din $x^2+4n-1\ge4n\sqrt[2n]{x}\Rightarrow\frac{4n}{x^2+4n-1}\le\frac{1}{\sqrt[2n]{x}}$.

Remarca.

In ΔABC

$$\sum\frac{1}{\tan\frac{B}{2}\tan\frac{C}{2}}\ge\sum\frac{4}{\tan^2\frac{A}{2}+1}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\ge\frac{4}{x^2+3}+\frac{4}{y^2+3}+\frac{4}{z^2+3}.$$

Demonstrație.

If $x>0$ then

$$\frac{4}{x^2+3}\le\frac{1}{\sqrt{x}}.$$

$x^2+3\overset{AM-GM}{\ge}4\sqrt[4]{x^2\cdot1\cdot1\cdot1}=4\sqrt[4]{x^2}=4\sqrt{x}$, cu egalitate pentru $x=1$.

Din $x^2+3\ge4\sqrt{x}\Rightarrow\frac{4}{x^2+3}\le\frac{1}{\sqrt{x}}$.

Folosind $\sum a^2\ge\sum bc$ pentru $(a,b,c)=\left(\frac{1}{\sqrt{yz}},\frac{1}{\sqrt{zx}},\frac{1}{\sqrt{xy}}\right)$ rezultă

$$\sum\frac{1}{yz}\ge\sum\frac{1}{\sqrt{zx}}\frac{1}{\sqrt{xy}}=\frac{1}{\sqrt{xyz}}\sum\frac{1}{\sqrt{x}}\overset{xyz\le1}{\ge}\sum\frac{1}{\sqrt{x}}\Rightarrow\sum\frac{1}{yz}\ge\sum\frac{1}{\sqrt{x}}, (1).$$

Am folosit mai sus $xyz\le1$, vezi $3=xy+yz+zx\ge3\sqrt[3]{x^2y^2z^2}\Rightarrow xyz\le1$.

Obținem $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\overset{(1)}{\geq}\sum\frac{1}{\sqrt{x}}\overset{Lema}{\geq}\frac{4}{x^2+3}+\frac{4}{y^2+3}+\frac{4}{z^2+3}$.

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\Leftrightarrow\sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum\frac{1}{\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}}\geq\sum\frac{4}{3\tan^2\frac{A}{2}+3}\Leftrightarrow\sum\frac{1}{3\tan\frac{B}{2}\tan\frac{C}{2}}\geq\sum\frac{4}{3\tan^2\frac{A}{2}+3}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{\tan\frac{B}{2}\tan\frac{C}{2}}\geq\sum\frac{4}{\tan^2\frac{A}{2}+1}.$$

Aplicația25.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\sum\frac{1}{y+z}+\frac{x^2+y^2+z^2}{2}\geq 3.$$

Trinh Ha, Vietnam,THCS 5/2023

Remarca.

In ΔABC

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\cdot\frac{(4R+r)^2}{p^2}\geq 6.$$

Marin Chirciu

Soluţie.

Lema

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\sum\frac{1}{y+z}+\frac{x^2+y^2+z^2}{2}\geq 3.$$

Soluţie.

$$LHS=\sum\frac{1}{y+z}+\frac{x^2+y^2+z^2}{2}\overset{CS}{\geq}\frac{9}{2\sum x}+\frac{\left(\sum x\right)^2-2\sum yz}{2}=\frac{9}{2p}+\frac{p^2-2\cdot 3}{2}\overset{(1)}{\geq}3=RHS,$$

unde (1) $\Leftrightarrow\ \frac{9}{2p}+\frac{p^2-2\cdot 3}{2}\geq 3\Leftrightarrow p^3-12p+9\geq 0\Leftrightarrow (p-3)\left(p^2+3p-3\right)\geq 0$,

care rezultă din $p\geq 3$, vezi $p^2=(x+y+z)^2\geq 3(xy+yz+zx)=3\cdot 3=9$.

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\Leftrightarrow\sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\sum\tan^2\frac{A}{2}\geq 3\Leftrightarrow\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\cdot\frac{(4R+r)^2-2p^2}{p^2}\geq 3\Leftrightarrow$$

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\cdot\frac{(4R+r)^2}{p^2}\geq 6.$$

Am folosit mai sus:

$$\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2-2p^2}{p^2}.$$

<u>**Remarca.**</u>

In ΔABC

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\geq\frac{3}{2}\left(3-\frac{R}{r}\right).$$

Marin Chirciu

<u>**Soluţie.**</u>

<u>**Lema**</u>

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\sum\frac{1}{y+z}+\frac{x^2+y^2+z^2}{2}\geq 3.$$

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\Leftrightarrow\sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\sum\tan^2\frac{A}{2}\geq 3\Leftrightarrow\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\cdot\frac{(4R+r)^2-2p^2}{p^2}\geq 3\Leftrightarrow$$

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}+\frac{3}{2}\cdot\frac{(4R+r)^2}{p^2}\geq 6\Leftrightarrow\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\geq 6-\frac{3}{2}\cdot\frac{(4R+r)^2}{p^2}\Leftrightarrow$$

$$\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\geq 6-\frac{3}{2}\cdot\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\Leftrightarrow\sum\frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\geq\frac{3}{2}\left(3-\frac{R}{r}\right).$$

Am folosit mai sus:

$$\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2-2p^2}{p^2}.$$

Remarca.

If $x,y,z>0$, $xy+yz+zx=3$ and $\lambda\geq 0$ then

$$\sum\frac{1}{y+\lambda z}+\frac{x^2+y^2+z^2}{2}\geq\frac{3}{2}\cdot\frac{\lambda+3}{\lambda+1}.$$

Marin Chirciu

Soluţie.

$$LHS=\sum\frac{1}{y+\lambda z}+\frac{x^2+y^2+z^2}{2}\overset{CS}{\geq}\frac{9}{(\lambda+1)\sum x}+\frac{\left(\sum x\right)^2-2\sum yz}{2}=$$

$$=\frac{9}{(\lambda+1)p}+\frac{p^2-2\cdot 3}{2}\overset{(1)}{\geq}\frac{3}{2}\cdot\frac{\lambda+3}{\lambda+1}=RHS$$, unde (1) $\Leftrightarrow$

$$\Leftrightarrow\frac{9}{(\lambda+1)p}+\frac{p^2-2\cdot 3}{2}\geq\frac{3}{2}\cdot\frac{\lambda+3}{\lambda+1}\Leftrightarrow(\lambda+1)p^3-(9\lambda+15)p+18\geq 0\Leftrightarrow$$

$(p-3)\left((\lambda+1)p^2+3(\lambda+1)p-6\right)\geq 0$, care rezultă din $\lambda\geq 0$ și $p\geq 3$, vezi $p^2=(x+y+z)^2\geq 3(xy+yz+zx)=3\cdot 3=9$.

Remarca.

In ΔABC

$$\sum \frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\lambda\tan\frac{C}{2}\right)} \geq \frac{3}{2}\cdot\left(\frac{2\lambda+4}{\lambda+1}-\frac{R}{r}\right), \lambda \geq 0.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$, $xy + yz + zx = 3$ and $\lambda \geq 0$ then

$$\sum \frac{1}{y+\lambda z} + \frac{x^2+y^2+z^2}{2} \geq \frac{3}{2}\cdot\frac{\lambda+3}{\lambda+1}.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\sqrt{3}\tan\frac{A}{2}, \sqrt{3}\tan\frac{B}{2}, \sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum \frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\lambda\tan\frac{C}{2}\right)} + \frac{3}{2}\sum \tan^2\frac{A}{2} \geq \frac{3}{2}\cdot\frac{\lambda+3}{\lambda+1} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\lambda\tan\frac{C}{2}\right)} + \frac{3}{2}\cdot\frac{(4R+r)^2-2p^2}{p^2} \geq \frac{3}{2}\cdot\frac{\lambda+3}{\lambda+1} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt{3}\left(\tan\frac{B}{2}+\lambda\tan\frac{C}{2}\right)} \geq \frac{3}{2}\cdot\left(\frac{2\lambda+4}{\lambda+1}-\frac{R}{r}\right).$$

Am folosit mai sus:

$$\sum \tan^2\frac{A}{2} = \frac{(4R+r)^2-2p^2}{p^2}.$$

Aplicația26.

If $x, y, z > 0$ then

$$\sum \frac{x}{9yz+1} \geq \frac{x+y+z}{1+(x+y+z)^2}.$$

Amir Sofi, Kosovo, Mathematics(College and High Schoool)5/2023

Remarca.

If $x, y, z > 0$ and $\lambda \geq 0$ then

$$\sum \frac{x}{9yz+\lambda} \geq \frac{x+y+z}{\lambda+(x+y+z)^2}.$$

Marin Chirciu

Soluţie.

$$LHS = \sum \frac{x}{9yz+\lambda} = \sum \frac{x^2}{9xyz+\lambda x} \overset{CS}{\geq} \frac{\left(\sum x\right)^2}{27xyz+\lambda\sum x} \overset{(1)}{\geq} \frac{\sum x}{\lambda+\left(\sum x\right)^2} = RHS,$$

unde (1) $\Leftrightarrow \dfrac{\left(\sum x\right)^2}{27xyz+\lambda\sum x} \geq \dfrac{\sum x}{\lambda+\left(\sum x\right)^2} \Leftrightarrow \left(\sum x\right)^3 \geq 3xyz$, vezi AM-GM.

Remarca.

In ΔABC

$$\sum \frac{\frac{r}{r_a}}{\frac{9r^2}{r_b r_c}+\lambda} \geq \frac{1}{\lambda+1}, \lambda \geq 0.$$

Soluţie.

Lema

If $x, y, z > 0$ and $\lambda \geq 0$ then

$$\sum \frac{x}{9yz+\lambda} \geq \frac{x+y+z}{\lambda+(x+y+z)^2}.$$

Soluţie.

$$LHS = \sum \frac{x}{9yz+\lambda} = \sum \frac{x^2}{9xyz+\lambda x} \overset{CS}{\geq} \frac{\left(\sum x\right)^2}{27xyz+\lambda\sum x} \overset{(1)}{\geq} \frac{\sum x}{\lambda+\left(\sum x\right)^2} = RHS,$$

unde (1) $\Leftrightarrow \dfrac{\left(\sum x\right)^2}{27xyz+\lambda\sum x} \geq \dfrac{\sum x}{\lambda+\left(\sum x\right)^2} \Leftrightarrow \left(\sum x\right)^3 \geq 3xyz$, vezi AM-GM.

Se cunoaște identitatea în triunghi $\dfrac{1}{r_a}+\dfrac{1}{r_b}+\dfrac{1}{r_c}=\dfrac{1}{r} \Leftrightarrow \dfrac{r}{r_a}+\dfrac{r}{r_b}+\dfrac{r}{r_c}=1$.

Folosind **Lema** pentru $(x, y, z) = \left(\dfrac{r}{r_a}, \dfrac{r}{r_b}, \dfrac{r}{r_c}\right)$ obținem:

$$\sum \frac{\frac{r}{r_a}}{9\frac{r}{r_b}\frac{r}{r_c}+\lambda} \geq \frac{1}{\lambda+1} \Leftrightarrow \sum \frac{\frac{r}{r_a}}{\frac{9r^2}{r_b r_c}+\lambda} \geq \frac{1}{\lambda+1}.$$

Remarca.

In ΔABC

$$\sum \frac{\frac{r}{h_a}}{\frac{9r^2}{h_b h_c}+\lambda} \geq \frac{1}{\lambda+1}, \lambda \geq 0.$$

Marin Chirciu

Aplicația27.

If $x, y, z > 0$, $x+y+z=\frac{3}{4}$ then find min of

$$P=\sum \sqrt{x^2+\frac{1}{y^2}}.$$

THCS5/2023, Vietnam

Remarca.

Let $0 \leq \lambda \leq 256$ fixed. If $x, y, z > 0$, $x+y+z=\frac{3}{4}$ then find min of

$$P=\sum \sqrt{x^2+\frac{\lambda}{y^2}}.$$

Marin Chirciu

Remarca.

Let $0 \leq \lambda \leq 81$ fixed. If $x, y, z > 0$, $x+y+z=1$ then find min of

$$P=\sum \sqrt{x^2+\frac{\lambda}{y^2}}.$$

Marin Chirciu

Remarca.

In ΔABC

$$\sum\sqrt{\frac{r^2}{r_a^2}+\frac{\lambda r_b^2}{r^2}}\geq\sqrt{81+\lambda},\,0\leq\lambda\leq 81.$$

Marin Chirciu

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\sqrt{x^2+\frac{\lambda}{y^2}}\geq\sqrt{81+\lambda}.$$

Soluţie.

$$P=\sum\sqrt{x^2+\frac{\lambda}{y^2}}\overset{CBS}{\geq}\sqrt{\left(\sum x\right)^2+\lambda\left(\sum\frac{1}{y}\right)^2}=\sqrt{\lambda\left[\left(\sum x\right)^2+\frac{1}{81}\left(\sum\frac{1}{x}\right)^2\right]+\frac{81-\lambda}{81}\left(\sum\frac{1}{x}\right)^2}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\sqrt{2\lambda\sqrt{\left(\sum x\right)^2\cdot\frac{1}{81}\left(\sum\frac{1}{x}\right)^2}+\frac{81-\lambda}{81}\left(\sum\frac{1}{x}\right)^2}=\sqrt{2\lambda\cdot\frac{1}{9}\sum x\sum\frac{1}{x}+\frac{81-\lambda}{81}\left(\sum\frac{1}{x}\right)^2}\overset{CS}{\geq}$$

$$\overset{CS}{\geq}\sqrt{2\lambda\cdot\frac{1}{9}\cdot 9+\frac{81-\lambda}{81}\left(\sum\frac{1}{x}\right)^2}\overset{CS}{\geq}\sqrt{2\lambda+\frac{81-\lambda}{81}\left(\frac{9}{\sum x}\right)^2}=\sqrt{2\lambda+\frac{81-\lambda}{81}\left(\frac{9}{1}\right)^2}=$$

$$=\sqrt{2\lambda+\frac{81-\lambda}{81}\cdot 9^2}=\sqrt{2\lambda+(81-\lambda)}=\sqrt{81+\lambda}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\sqrt{\left(\frac{r}{r_a}\right)^2+\frac{\lambda}{\left(\frac{r}{r_b}\right)^2}}\geq\sqrt{81+\lambda}\Leftrightarrow\sum\sqrt{\frac{r^2}{r_a^2}+\frac{\lambda r_b^2}{r^2}}\geq\sqrt{81+\lambda}.$$

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\sum\sqrt{\frac{r^2}{h_a^2}+\frac{\lambda h_b^2}{r^2}}\geq\sqrt{81+\lambda},\,0\leq\lambda\leq 81.$$

Marin Chirciu

Lema.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum \sqrt{x^2 + \frac{\lambda}{y^2}} \geq \sqrt{81 + \lambda}.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \Leftrightarrow \frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c} = 1$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{r}{h_a}, \frac{r}{h_b}, \frac{r}{h_c}\right)$ obținem:

$$\sum \sqrt{\left(\frac{r}{h_a}\right)^2 + \frac{\lambda}{\left(\frac{r}{h_b}\right)^2}} \geq \sqrt{81 + \lambda} \Leftrightarrow \sum \sqrt{\frac{r^2}{h_a^2} + \frac{\lambda h_b^2}{r^2}} \geq \sqrt{81 + \lambda}.$$

Aplicația28.

If $x, y, z > 0$, $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 1$ then find min of

$$P = \sum \frac{y^2 z^2}{x\left(y^2 + z^2\right)}.$$

Le Tran, Vietnam,THCS 5/2023

Remarca.

In ΔABC

$$\sum \frac{r_b r_c}{\sqrt{r_a}\left(r_b + r_c\right)} \geq \frac{3\sqrt{3r}}{2}.$$

Marin Chirciu

Lema

If $x, y, z > 0$, $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 1$ then

$$\sum \frac{y^2 z^2}{x\left(y^2 + z^2\right)} \geq \frac{3\sqrt{3}}{2}.$$

Soluţie.

Cu substituția $(a,b,c)=\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ problema se reformulează:

If $a,b,c>0$, $a^2+b^2+c^2=1$ then

$$\sum\frac{a}{b^2+c^2}\ge\frac{3\sqrt{3}}{2}.$$

Demonstrație.

Lema

$$\frac{a}{b^2+c^2}\ge\frac{3\sqrt{3}}{2}a^2.$$

$\frac{a}{b^2+c^2}=\frac{a}{1-a^2}=\frac{a^2}{a\left(1-a^2\right)}\ge\frac{3\sqrt{3}}{2}a^2$, care rezultă din:

$$a^2\left(1-a^2\right)^2=\frac{1}{2}\cdot 2a^2\left(1-a^2\right)\left(1-a^2\right)\overset{AM-GM}{\le}\frac{1}{2}\left[\frac{2a^2+\left(1-a^2\right)+\left(1-a^2\right)}{3}\right]^3=\frac{1}{2}\left(\frac{2}{3}\right)^3=\frac{4}{27},$$

cu egalitate pentru $2a^2=\left(1-a^2\right)\Leftrightarrow a=\frac{1}{\sqrt{3}}$.

$$a^2\left(1-a^2\right)^2\le\frac{4}{27}\Rightarrow a\left(1-a^2\right)\le\frac{2}{3\sqrt{3}}\Rightarrow\frac{1}{a\left(1-a^2\right)}\ge\frac{3\sqrt{3}}{2}\Rightarrow\frac{a^2}{a\left(1-a^2\right)}\ge\frac{3\sqrt{3}}{2}a^2.$$

Obținem: $\sum\frac{a}{b^2+c^2}\ge\sum\frac{3\sqrt{3}}{2}a^2=\frac{3\sqrt{3}}{2}\sum a^2=\frac{3\sqrt{3}}{2}$.

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{\frac{r_a}{r}},\sqrt{\frac{r_b}{r}},\sqrt{\frac{r_c}{r}}\right)$ obținem:

$$\sum\frac{\left(\frac{r_b}{r}\right)\left(\frac{r_c}{r}\right)}{\sqrt{\frac{r_a}{r}}\left(\left(\frac{r_b}{r}\right)+\left(\frac{r_c}{r}\right)\right)}\ge\frac{3\sqrt{3}}{2}\Leftrightarrow\sum\frac{r_br_c}{\sqrt{r_a}\left(r_b+r_c\right)}\ge\frac{3\sqrt{3r}}{2}.$$

Remarca.

In ΔABC

$$\sum\frac{h_bh_c}{\sqrt{h_a}\left(h_b+h_c\right)}\ge\frac{3\sqrt{3r}}{2}.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$, $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=1$ then

$$\sum \frac{y^2z^2}{x\left(y^2+z^2\right)} \geq \frac{3\sqrt{3}}{2}.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r} \Leftrightarrow \frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1$.

Folosind **Lema** pentru $(x, y, z)=\left(\sqrt{\frac{h_a}{r}}, \sqrt{\frac{h_b}{r}}, \sqrt{\frac{h_c}{r}}\right)$ obținem:

$$\sum \frac{\left(\frac{h_b}{r}\right)\left(\frac{h_c}{r}\right)}{\sqrt{\frac{h_a}{r}}\left(\left(\frac{h_b}{r}\right)+\left(\frac{h_c}{r}\right)\right)} \geq \frac{3\sqrt{3}}{2} \Leftrightarrow \sum \frac{h_b h_c}{\sqrt{h_a}\left(h_b+h_c\right)} \geq \frac{3\sqrt{3r}}{2}.$$

Aplicația29.

If $x, y, z > 0$, $xy+yz+zx=3$ then

$$\sum \frac{1}{2x+y+z} \leq \frac{x^2+y^2+z^2+9}{16}.$$

Ng Bao Ngoc, Vietnam,THCS5/2023

Remarca.

In ΔABC

$$\sum \frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}} \leq \frac{3\sqrt{3}}{16}\left(2+\frac{R}{r}\right).$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $xy+yz+zx=3$ then

$$\sum \frac{1}{2x+y+z} \leq \frac{x^2+y^2+z^2+9}{16}.$$

Soluţie.

$$LHS=\sum\frac{1}{2x+y+z}\le\frac{1}{4}\sum\left(\frac{1}{x+y}+\frac{1}{x+z}\right)=\frac{1}{2}\sum\frac{1}{y+z}=\frac{1}{2}\frac{\sum(x+y)(x+z)}{\prod(y+z)}=$$

$$=\frac{1}{2}\frac{\sum\left(x^2+xy+yz+zx\right)}{\prod(y+z)}=\frac{1}{2}\frac{\sum\left(x^2+3\right)}{\prod(y+z)}=\frac{1}{2}\frac{\sum x^2+9}{\prod(y+z)}\overset{(1)}{\le}\frac{\sum x^2+9}{16}=RHS,$$

unde (1) $\Leftrightarrow\ \prod(y+z)\ge 8$, vezi **Lema8/9.**

Lema 8/9.

If $x,y,z>0$ then

$$(x+y)(y+z)(z+x)\ge\frac{8}{9}(x+y+z)(xy+yz+zx).$$

Lema8/9

Demonstrație.

Avem:

$$(x+y)(y+z)(z+x)\ge\frac{8}{9}(x+y+z)(xy+yz+zx)\Leftrightarrow$$

$$\Leftrightarrow 9\left(\sum yz(y+z)+2xyz\right)\ge\left(\sum yz(y+z)+3xyz\right)\Leftrightarrow\sum yz(y+z)\ge 6xyz\Leftrightarrow$$

$\Leftrightarrow\sum x(y-z)^2\ge 0$, evident cu egalitate pentru $x=y=z$.

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\Leftrightarrow\sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum\frac{1}{2\sqrt{3}\tan\frac{A}{2}+\sqrt{3}\tan\frac{B}{2}+\sqrt{3}\tan\frac{C}{2}}\le\frac{3\tan^2\frac{A}{2}+3\tan^2\frac{B}{2}+3\tan^2\frac{C}{2}+9}{16}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\le\frac{3\sqrt{3}\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+3\right)}{16}\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\le\frac{3\sqrt{3}\left(\frac{(4R+r)^2-2p^2}{p^2}+3\right)}{16}\Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}} \le \frac{3\sqrt{3}}{16}\left(1+\frac{(4R+r)^2}{p^2}\right), (1).$$

Folosind inegalitatea lui Gerretsen $p^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$ rezultă

$$\frac{(4R+r)^2}{p^2} \le \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} = \frac{R+r}{r} = 1+\frac{R}{r}, (2).$$

Din (1) și (2) rezultă:

$$\sum \frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}} \le \frac{3\sqrt{3}}{16}\left(2+\frac{R}{r}\right).$$

Am folosit mai sus $\sum \tan\frac{A}{2} = \frac{(4R+r)^2-2p^2}{p^2}$.

Aplicația30.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum \sqrt{2x(y^2+z^2)} + xy+yz+zx \le 9.$$

Nhat Viet, Vietnam,THCS 5/2023

Remarca.

In ΔABC

$$\sum \sqrt{\frac{6r}{r_a}\left(\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)} + \frac{3(4R+r)}{p^2} \le 9.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum \sqrt{2x(y^2+z^2)} + xy+yz+zx \le 9.$$

If $x, y, z > 0$, $x+y+z=3$ then

$$1).\ \sqrt{2x(y^2+z^2)} \le \frac{x^2+y^2+z^2+1}{2};$$

$$2).\ \sqrt{2y\left(x^2+z^2\right)}+\sqrt{2z\left(x^2+y^2\right)}\le 4.$$

Demonstrație.

1).

$$\sqrt{2x\left(y^2+z^2\right)}\overset{SOS}{\le}\sqrt{\left(x^2+1\right)\left(y^2+z^2\right)}\overset{AM-GM}{\le}\frac{\left(x^2+1\right)+\left(y^2+z^2\right)}{2}=\frac{x^2+y^2+z^2+1}{2}$$, cu egalitate pentru $x=1$ și $y^2+z^2=x^2+1$.

2).

Principiul lui Dirichlet:

Din trei numere pozitive a,b,c există cel puțin două numere care sunt situate de aceeași parte a lui 1. Fie b,c numerele, deci $(1-b)(1-c)\le 0$.

Folosind principiul lui Dirichlet pentru $(b,c)=\left(\frac{y}{x},\frac{z}{x}\right)$ obținem $\left(1-\frac{y}{x}\right)\left(1-\frac{z}{x}\right)\le 0\Leftrightarrow$

$\Leftrightarrow (x-y)(x-z)\le 0\Leftrightarrow x^2+yz\le x(y+z)$,(1).

Obținem:

$$\sqrt{2y\left(x^2+z^2\right)}+\sqrt{2z\left(x^2+y^2\right)}\overset{CBS}{\le}\sqrt{2}\sqrt{2y\left(x^2+z^2\right)+2z\left(x^2+y^2\right)}=2\sqrt{y\left(x^2+z^2\right)+z\left(x^2+y^2\right)}=$$

$$=2\sqrt{(y+z)\left(x^2+yz\right)}\overset{(1)}{\le}2\sqrt{x(y+z)^2}\overset{(2)}{\le}2\cdot\sqrt{4}=4$$, unde (2) rezultă din:

$$x(y+z)^2=4x\cdot\frac{y+z}{2}\cdot\frac{y+z}{2}\overset{AM-GM}{\le}4\left(\frac{x+\frac{y+z}{2}+\frac{y+z}{2}}{3}\right)^3=4\left(\frac{x+y+z}{3}\right)^3=4\left(\frac{3}{3}\right)^3=4.$$

$$LHS=\sum\sqrt{2x\left(y^2+z^2\right)}+(xy+yz+zx)\le\frac{x^2+y^2+z^2+1}{2}+4+(xy+yz+zx)=$$

$$=\frac{x^2+y^2+z^2+2(xy+yz+zx)+9}{2}=\frac{(x+y+z)^2+9}{2}=\frac{3^2+9}{2}=9.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum\sqrt{2\frac{3r}{r_a}\left(\left(\frac{3r}{r_b}\right)^2+\left(\frac{3r}{r_c}\right)^2\right)}+\frac{3r}{r_a}\frac{3r}{r_b}+\frac{3r}{r_b}\frac{3r}{r_c}+\frac{3r}{r_c}\frac{3r}{r_a}\le 9\Leftrightarrow$$

$$\Leftrightarrow 3r\sum\sqrt{2\frac{3r}{r_a}\left(\left(\frac{1}{r_b}\right)^2+\left(\frac{1}{r_c}\right)^2\right)}+9r^2\frac{4R+r}{rp^2}\le 9 \Leftrightarrow \sum\sqrt{\frac{6r}{r_a}\left(\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)}+\frac{3(4R+r)}{p^2}\le 9$$

Aplicația31.

If $x, y, z > 0$, $x+y+z=3$ then find max of

$$P=\sum\frac{yz}{3+x^2}.$$

Duc Tran Truong, Vietnam,THCS 5/2023

Remarca.

In ΔABC

$$\sum\frac{\frac{1}{r_br_c}}{1+\frac{3r^2}{r_a^2}}\le\frac{1}{4r^2}.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum\frac{yz}{3+x^2}\le\frac{3}{4}.$$

Demonstrație.

$x+y+z=3 \Rightarrow x^2+y^2+z^2\ge 3.$

$$P=\sum\frac{yz}{3+x^2}\le\frac{1}{4}\sum\frac{(y+z)^2}{(x^2+y^2+z^2)+x^2}=\frac{1}{4}\sum\frac{(y+z)^2}{(x^2+y^2)+(x^2+z^2)}\overset{CS}{\le}$$

$$\overset{CS}{\le}\frac{1}{4}\sum\left(\frac{y^2}{x^2+y^2}+\frac{z^2}{x^2+z^2}\right)=\frac{1}{4}\cdot 3=\frac{3}{4}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$

Folosind **Lema** pentru $(x, y, z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum \frac{\frac{3r}{r_b} \cdot \frac{3r}{r_c}}{3+\left(\frac{3r}{r_a}\right)^2} \le \frac{3}{4} \Leftrightarrow \sum \frac{\frac{1}{r_b r_c}}{1+\frac{3r^2}{r_a^2}} \le \frac{1}{4r^2}.$$

Remarca.

In ΔABC

$$\sum \frac{\frac{1}{h_b h_c}}{1+\frac{3r^2}{h_a^2}} \le \frac{1}{4r^2}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum \frac{yz}{3+x^2} \le \frac{3}{4}.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r} \Leftrightarrow \frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1 \Leftrightarrow \frac{3r}{h_a}+\frac{3r}{h_b}+\frac{3r}{h_c}=3$

Folosind **Lema** pentru $(x, y, z)=\left(\frac{3r}{h_a}, \frac{3r}{h_b}, \frac{3r}{h_c}\right)$ obținem:

$$\sum \frac{\frac{3r}{h_b} \cdot \frac{3r}{h_c}}{3+\left(\frac{3r}{h_a}\right)^2} \le \frac{3}{4} \Leftrightarrow \sum \frac{\frac{1}{h_b h_c}}{1+\frac{3r^2}{h_a^2}} \le \frac{1}{4r^2}.$$

Aplicația32.

If $x, y, z > 0$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ then

$$\sum \sqrt{\frac{3}{x^2+xy+y^2}} \le 1.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$ then

$$\sqrt{\frac{3}{x^2+xy+y^2}}\le\frac{2}{x+y}.$$

Demonstrație.

$\sqrt{\frac{3}{x^2+xy+y^2}}\le\frac{2}{x+y}\Leftrightarrow(x-y)^2\ge 0$, cu egalitate pentru $x=y$.

$LHS=\sum\sqrt{\frac{3}{x^2+xy+y^2}}\le\sum\frac{2}{x+y}\le 2\cdot\frac{1}{4}\sum\left(\frac{1}{x}+\frac{1}{y}\right)=\sum\frac{1}{x}=1=RHS$.

Remarca.

In ΔABC

$$\sum\sqrt{\frac{3}{r_a^2+r_ar_b+r_b^2}}\le\frac{1}{r}.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$then

$$\sum\sqrt{\frac{3}{x^2+xy+y^2}}\le 1.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r_a}{r},\frac{r_b}{r},\frac{r_c}{r}\right)$obținem:

$$\sum\sqrt{\frac{3}{\left(\frac{r_a}{r}\right)^2+\frac{r_a}{r}\frac{r_b}{r}+\left(\frac{r_b}{r}\right)^2}}\le 1\Leftrightarrow\sum\sqrt{\frac{3}{r_a^2+r_ar_b+r_b^2}}\le\frac{1}{r}.$$

Remarca.

In ΔABC

$$\sum\sqrt{\frac{3}{h_a^2+h_ah_b+h_b^2}}\le\frac{1}{r}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ then

$$\sum\sqrt{\frac{3}{x^2+xy+y^2}}\le 1.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Leftrightarrow\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{h_a}{r},\frac{h_b}{r},\frac{h_c}{r}\right)$ obținem:

$$\sum\sqrt{\frac{3}{\left(\frac{h_a}{r}\right)^2+\frac{h_a}{r}\frac{h_b}{r}+\left(\frac{h_b}{r}\right)^2}}\le 1\Leftrightarrow\sum\sqrt{\frac{3}{h_a^2+h_ah_b+h_b^2}}\le\frac{1}{r}.$$

Aplicația33.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\sqrt{3\left(x^2+2\right)}\ge 9.$$

Marin Chirciu

Soluție.

Lema.

If $x>0$ then

$$\sqrt{3\left(x^2+2\right)}\ge 2+x.$$

Demonstrație.

$\sqrt{3\left(x^2+2\right)}\ge 2+x\Leftrightarrow(x-1)^2\ge 0$, cu egalitate pentru $x=1$.

$\sum\sqrt{3\left(x^2+2\right)}\ge\sum(2+x)=6+\sum x=6+3=9$.

Remarca.

In ΔABC

$$\sum\sqrt{2+\left(\frac{3r}{r_a}\right)^2}\ge 3\sqrt{3}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum \sqrt{3\left(x^2+2\right)} \geq 9.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1 \Leftrightarrow \frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x, y, z)=\left(\frac{3r}{r_a}, \frac{3r}{r_b}, \frac{3r}{r_c}\right)$ obținem:

$$\sum \sqrt{3\left(\left(\frac{3r}{r_a}\right)^2+2\right)} \geq 9 \Leftrightarrow \sum \sqrt{2+\left(\frac{3r}{r_a}\right)^2} \geq 3\sqrt{3}.$$

Remarca.

In ΔABC

$$\sum \sqrt{2+\left(\frac{3r}{h_a}\right)^2} \geq 3\sqrt{3}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum \sqrt{3\left(x^2+2\right)} \geq 9.$$

$$\sum \sqrt{3\left(x^2+2\right)} \geq \sum (2+x)=6+\sum x=6+3=9.$$

Se cunoaște identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r} \Leftrightarrow \frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1 \Leftrightarrow \frac{3r}{h_a}+\frac{3r}{h_b}+\frac{3r}{h_c}=3$.

Folosind **Lema** pentru $(x, y, z)=\left(\frac{3r}{h_a}, \frac{3r}{h_b}, \frac{3r}{h_c}\right)$ obținem:

$$\sum \sqrt{3\left(\left(\frac{3r}{h_a}\right)^2+2\right)} \geq 9 \Leftrightarrow \sum \sqrt{2+\left(\frac{3r}{h_a}\right)^2} \geq 3\sqrt{3}.$$

Aplicația34.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum \frac{x^2 + 3y^2}{x + 3y} \geq 1.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y > 0$ then

$$\frac{x^2 + 3y^2}{x + 3y} \geq \frac{x + 3y}{4}.$$

Demonstrație.

$\frac{x^2 + 3y^2}{x + 3y} \geq \frac{x + 3y}{4} \Leftrightarrow 3(x - y)^2 \geq 0$, cu egalitate pentru $x = y$.

$$LHS = \sum \frac{x^2 + 3y^2}{x + 3y} \overset{Lema}{\geq} \sum \frac{x + 3y}{4} = \frac{4\sum x}{4} = \sum x = 1 = RHS.$$

Remarca.

In ΔABC

$$\sum \frac{\frac{1}{r_a^2} + \frac{3}{r_b^2}}{\frac{1}{r_a} + \frac{3}{r_b}} \geq \frac{1}{r}.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum \frac{x^2 + 3y^2}{x + 3y} \geq 1.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \Leftrightarrow \frac{r}{r_a} + \frac{r}{r_b} + \frac{r}{r_c} = 1$

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{\left(\frac{r}{r_a}\right)^2+3\left(\frac{r}{r_b}\right)^2}{\frac{r}{r_a}+3\frac{r}{r_b}}\ge 1\Leftrightarrow\sum\frac{\frac{1}{r_a^2}+\frac{3}{r_b^2}}{\frac{1}{r_a}+\frac{3}{r_b}}\ge\frac{1}{r}.$$

Remarca.

In ΔABC

$$\sum\frac{\frac{1}{h_a^2}+\frac{3}{h_b^2}}{\frac{1}{h_a}+\frac{3}{h_b}}\ge\frac{1}{r}.$$

Marin Chirciu

Remarca.

If $x,y,z>0$, $x+y+z=1$ and $\lambda\ge 0$ then

$$\sum\frac{x^2+\lambda y^2}{x+\lambda y}\ge 1.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y>0$ and $\lambda\ge 0$ then

$$\frac{x^2+\lambda y^2}{x+\lambda y}\ge\frac{x+\lambda y}{\lambda+1}.$$

Demonstrație.

$\frac{x^2+\lambda y^2}{x+\lambda y}\ge\frac{x+\lambda y}{\lambda+1}\Leftrightarrow\lambda(x-y)^2\ge 0$, cu egalitate pentru $x=y$.

$$LHS=\sum\frac{x^2+\lambda y^2}{x+\lambda y}\overset{Lema}{\ge}\sum\frac{x+\lambda y}{\lambda+1}=\frac{(\lambda+1)\sum x}{\lambda+1}=\sum x=1=RHS.$$

Remarca.

In ΔABC

$$\sum\frac{\frac{1}{r_a^2}+\frac{\lambda}{r_b^2}}{\frac{1}{r_a}+\frac{\lambda}{r_b}}\geq\frac{1}{r},\lambda\geq 0.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $x+y+z=1$ and $\lambda\geq 0$ then

$$\sum\frac{x^2+\lambda y^2}{x+\lambda y}\geq 1.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$

Să trecem la rezolvarea problemei din enunț.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{\left(\frac{r}{r_a}\right)^2+\lambda\left(\frac{r}{r_b}\right)^2}{\frac{r}{r_a}+\lambda\frac{r}{r_b}}\geq 1\Leftrightarrow\sum\frac{\frac{1}{r_a^2}+\frac{\lambda}{r_b^2}}{\frac{1}{r_a}+\frac{\lambda}{r_b}}\geq\frac{1}{r}.$$

Remarca.

In ΔABC

$$\sum\frac{\frac{1}{h_a^2}+\frac{\lambda}{h_b^2}}{\frac{1}{h_a}+\frac{\lambda}{h_b}}\geq\frac{1}{r},\lambda\geq 0.$$

Marin Chirciu

Aplicaţia35.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{x}{1+y^2}\geq\frac{3}{2}.$$

Le Hai Trung, Vietnam,THCS 5/2023

Soluţie.

Lema

$x, y > 0$

$$\frac{x}{1+y^2} \geq x - \frac{xy}{2}.$$

Demonstraţie.

$\frac{x}{1+y^2} = x\left(1-\frac{y^2}{1+y^2}\right) \overset{AM-GM}{\geq} x\left(1-\frac{y^2}{2y}\right) = x - \frac{xy}{2}$, cu egalitate pentru $y=1$.

$LHS = \sum\frac{x}{1+y^2} \overset{Lema}{\geq} \sum\left(x-\frac{xy}{2}\right) = \sum x - \frac{1}{2}\sum xy \overset{(1)}{\geq} 3 - \frac{1}{2}\cdot 3 = \frac{3}{2} = RHS$, unde (1) $\Leftrightarrow$

$\Leftrightarrow \sum xy \leq 3$, vezi $\sum xy \leq \frac{1}{3}\left(\sum x\right)^2 = \frac{1}{3}\cdot 3^2 = 3$.

Remarca.

In ΔABC

$$\sum\frac{1}{r_a\left(1+\frac{9r^2}{r_b^2}\right)} \geq \frac{1}{2r}.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$, $x+y+z=3$ then

$$\sum\frac{x}{1+y^2} \geq \frac{3}{2}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1 \Leftrightarrow \frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\sum\frac{\frac{3r}{r_a}}{1+\left(\frac{3r}{r_b}\right)^2} \geq \frac{3}{2} \Leftrightarrow \sum\frac{1}{r_a\left(1+\frac{9r^2}{r_b^2}\right)} \geq \frac{1}{2r}.$$

Remarca.

In ΔABC

$$\sum \frac{1}{h_a\left(1+\frac{9r^2}{h_b^2}\right)} \geq \frac{1}{2r}.$$

Marin Chirciu

Aplicația36.

If $a,b,c \geq 0$, $a+b+c=2$ then

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} \geq 2.$$

Mehmet Şahin, Turkey, Matematik Olympiyat Okulu 4/2023

Remarca.

If $a,b,c>0$, $a+b+c=3$ then

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} \geq \frac{3}{2}.$$

Marin Chirciu

Soluţie.

Lema.

If $a>0$ then

$$\frac{1}{1+a^2} \geq \frac{2-a}{2}.$$

Demonstrație.

Avem $\frac{1}{1+a^2} \geq \frac{2-a}{2} \Leftrightarrow a(a-1)^2 \geq 0$, cu egalitate pentru $a=1$.

$$LHS=\sum \frac{1}{1+a^2} \overset{Lema}{\geq} \sum \frac{2-a}{2}=\frac{6-\sum a}{2}=\frac{6-3}{2}=\frac{3}{2}=RHS.$$

Remarca.

If $a,b,c>0$, $a+b+c=3$ and $0 \leq \lambda \leq 1$ then

$$\frac{1}{\lambda+a^2}+\frac{1}{\lambda+b^2}+\frac{1}{\lambda+c^2} \geq \frac{3}{\lambda+1}.$$

Marin Chirciu

Soluţie.

Lema.

If $a > 0$ and $\lambda \geq 0$ then

$$\frac{1}{\lambda + a^2} \geq \frac{\lambda + 3 - 2a}{(\lambda + 1)^2}.$$

Demonstrație.

Avem $\frac{1}{\lambda + a^2} \geq \frac{\lambda + 3 - 2a}{(\lambda + 1)^2} \Leftrightarrow 2a^3 - (\lambda + 3)a^2 + 2a\lambda + 1 - \lambda \geq 0 \Leftrightarrow (a-1)^2(2a + 1 - \lambda) \geq 0$, care rezultă din $0 \leq \lambda \leq 1$, care asigură $(2a + 1 - \lambda) > 0$ și $(a-1)^2 \geq 0$, cu egalitate pentru $a = 1$.

$$LHS = \sum \frac{1}{\lambda + a^2} \overset{Lema}{\geq} \sum \frac{\lambda + 3 - 2a}{(\lambda + 1)^2} = \frac{3(\lambda + 3) - 2\sum a}{(\lambda + 1)^2} = \frac{3\lambda + 9 - 2 \cdot 3}{(\lambda + 1)^2} = \frac{3(\lambda + 1)}{(\lambda + 1)^2} =$$

$$= \frac{3}{\lambda + 1} = RHS$$

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\sum \frac{1}{\lambda + \left(\frac{3r}{r_a}\right)^2} \geq \frac{3}{\lambda + 1}, 0 \leq \lambda \leq 1.$$

Marin Chirciu

Soluţie.

Lema.

If $x, y, z > 0$, $x + y + z = 3$ and $0 \leq \lambda \leq 1$ then

$$\frac{1}{\lambda + x^2} + \frac{1}{\lambda + y^2} + \frac{1}{\lambda + z^2} \geq \frac{3}{\lambda + 1}.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \Leftrightarrow \frac{r}{r_a} + \frac{r}{r_b} + \frac{r}{r_c} = 1 \Leftrightarrow \frac{3r}{r_a} + \frac{3r}{r_b} + \frac{3r}{r_c} = 3$

Folosind **Lema** pentru $(x, y, z) = \left(\frac{3r}{r_a}, \frac{3r}{r_b}, \frac{3r}{r_c}\right)$ obținem:

$$\sum \frac{1}{\lambda + \left(\frac{3r}{r_a}\right)^2} \geq \frac{3}{\lambda + 1}.$$

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\sum \frac{1}{\lambda + \left(\frac{3r}{h_a}\right)^2} \geq \frac{3}{\lambda + 1}, 0 \leq \lambda \leq 1.$$

Marin Chirciu

Aplicația37.

If $a, b, c > 0$, $a + b + c = 1$ then

$$\sum \frac{a}{a + b^2} \leq \frac{1}{4}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Nguyen Huy Gia Bao, Vietnam,THCS 4/2023

Remarca.

In ΔABC

$$\sum \frac{r_b^2}{rr_a + r_b^2} \leq \frac{4R + r}{4r}.$$

Marin Chirciu

Soluţie.

Lema

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum \frac{x}{x + y^2} \leq \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

If $x, y, z > 0$, $x + y + z = 1$ then

$$\frac{x}{x + y^2} \leq \frac{1}{4}\left[\frac{1}{x + z} + \frac{x}{y(x + y)}\right].$$

Demonstrație

$$\frac{x}{x + y^2} = \frac{x}{x(x + y + z) + y^2} = \frac{x}{x^2 + y^2 + x(y + z)} = \frac{x}{x(x + z) + y(x + y)} \leq$$

$$\le \frac{x}{4}\left[\frac{1}{x(x+z)}+\frac{1}{y(x+y)}\right]=\frac{1}{4}\left[\frac{1}{x+z}+\frac{x}{y(x+y)}\right].$$

Obținem:

$$LHS=\sum\frac{x}{x+y^2}\overset{Lema}{\le}\sum\frac{1}{4}\left[\frac{1}{x+z}+\frac{x}{y(x+y)}\right]=\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=RHS$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$

Folosind Lema pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum\frac{\frac{r}{r_a}}{\frac{r}{r_a}+\left(\frac{r}{r_a}\right)^2}\le\frac{1}{4}\left(\frac{r_a}{r}+\frac{r_b}{r}+\frac{r_c}{r}\right)\Leftrightarrow\sum\frac{\frac{1}{r_a}}{\frac{1}{r_a}+\frac{r}{r_b^2}}\le\frac{1}{4}\cdot\frac{4R+r}{r}\Leftrightarrow\sum\frac{r_b^2}{rr_a+r_b^2}\le\frac{4R+r}{4r}.$$

Remarca.

In ΔABC

$$\sum\frac{h_b^2}{rh_a+h_b^2}\le\frac{(R+r)^2}{2Rr}.$$

Marin Chirciu

Aplicația38.

If $a,b,c>0$

$$\sum\frac{1}{4a+b+c}\le\frac{a+b+c}{2(ab+bc+ca)}.$$

Remarca.

If $a,b,c>0$, $ab+bc+ca=1$

$$\sum\frac{1}{4a+b+c}\le\frac{a+b+c}{2}.$$

Marin Chirciu

Soluţie.

Lema

If $a,b,c>0$,then

$$\frac{1}{4a+b+c} \le \frac{b+c}{4(ab+bc+ca)}.$$

Demonstrație.

$$\frac{1}{4a+b+c} \le \frac{b+c}{4(ab+bc+ca)} \Leftrightarrow 4(ab+bc+ca) \le (4a+b+c)(b+c) \Leftrightarrow (b-c)^2 \ge 0.$$

Folosind **Lema** obținem:

$$LHS = \sum \frac{1}{4a+b+c} \overset{Lema}{\le} \sum \frac{b+c}{4(ab+bc+ca)} = \frac{\sum(b+c)}{4} = \frac{2\sum a}{4} = \frac{\sum a}{2} = RHS.$$

Remarca.

In ΔABC

$$\sum \frac{1}{4\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}} \le \frac{4R+r}{2p}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$, $xy + yz + zx = 1$

$$\sum \frac{1}{4x+y+z} \le \frac{x+y+z}{2}.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$.

Folosind **Lema** pentru $(x, y, z) = \left(\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right)$ obținem:

$$\sum \frac{1}{4\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}} \le \frac{\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}}{2}.$$

Folosind $\sum \tan\frac{A}{2} = \frac{4R+r}{p}$ rezultă $\sum \frac{1}{4\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}} \le \frac{\frac{4R+r}{p}}{2} = \frac{4R+r}{2p}$.

Aplicația39.

If $x, y, z > 0$, $x + y + z = xyz$ then

$$xy+yz+zx\geq 3+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}.$$

Florina și Marian Tetiva, Pure Inequalities 4/2017

Remarca.

In acute ΔABC

$$\sum \tan A\tan B\geq 3+\sum\sqrt{1+\tan^2 A}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=xyz$ then

$$xy+yz+zx\geq 3+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}.$$

Demonstrație.

$\sum(xy)^2\geq xyz(x+y+z)=(x+y+z)^2$,(1).

$\left(\sum xy\right)^2=\sum(xy)^2+2xyz(x+y+z)\overset{(1)}{\geq}(x+y+z)^2+2(x+y+z)^2=3(x+y+z)^2$,(2).

$\left(\sum xy-3\right)^2=\left(\sum xy\right)^2-6\sum xy+9\overset{(2)}{\geq}3(x+y+z)^2-6\sum xy+9=3\sum x^2+9=3\left(\sum x^2+1\right)\overset{CBS}{\geq}$

$\overset{CBS}{\geq}\left(\sum\sqrt{x^2+1}\right)^2.$

Din $\left(\sum xy-3\right)^2\geq\left(\sum\sqrt{x^2+1}\right)^2\Leftrightarrow\sum xy-3\geq\sum\sqrt{x^2+1}\Leftrightarrow\sum xy\geq 3+\sum\sqrt{x^2+1}.$

Se cunoaște identitatea în triunghi: $\tan A+\tan B+\tan C=\tan A\tan B\tan C=\dfrac{2pr}{p^2-(2R+r)^2}$.

Folosind **Lema** pentru $(x,y,z)=(\tan A,\tan B,\tan C)$ obținem:

$\sum\tan A\tan B\geq 3+\sum\sqrt{1+\tan^2 A}.$

Aplicația40.

If $a,b,c>0$, $a+b+c=3$ then

$$\sum\frac{1}{\sqrt{a+3}}\geq\frac{3}{2}.$$

Marin Chirciu

Soluție.

Lema.

If $0<a<3$,then

$$\frac{1}{\sqrt{a+3}}\geq\frac{9-a}{16}.$$

Folosim Tangent Line Method pentru funcția $f:(0,3)\to\mathbf{R}$ $f(x)=\frac{1}{\sqrt{x+3}}$ în $x_0=1$.

Avem $f(1)=\frac{1}{2}$.

Ecuația tangentei în punctul $x_0=1$ este $y-f(x_0)=f'(x_0)(x-x_0)$.

Avem $f'(x)=\frac{-1}{2(x+3)\sqrt{x+3}}$, $f'(1)=\frac{-1}{16}$.

Ecuația tangentei în punctul $x_0=1$ este $y-\frac{1}{2}=\frac{-1}{16}(x-1)\Leftrightarrow y=\frac{9-x}{16}$.

Arătăm că: $f(x)=\frac{1}{\sqrt{x+3}}\geq\frac{9-x}{16}\Leftrightarrow x^3-15x^2+27x-13\leq 0\Leftrightarrow(x-1)^2(x-13)\leq 0$,care resultă din $(x-13)<0$, pentru $0<x<3$ şi $(x-1)^2\geq 0$,cu egalitate pentru $x=1$.

$$LHS=\sum\frac{1}{\sqrt{a+3}}\overset{Lema}{\geq}\sum\frac{9-a}{16}=\frac{9\cdot 3-\sum a}{16}=\frac{27-3}{16}=\frac{24}{16}=\frac{3}{2}=RHS.$$

Remarca.

In ΔABC

$$\sum\frac{1}{\sqrt{\frac{r}{r_a}+1}}\geq\frac{3\sqrt{3}}{2}.$$

Marin Chirciu

Soluţie.

Lema.

If $x,y,z>0$, $x+y+z=3$,then

$$\sum\frac{1}{\sqrt{x+3}}\geq\frac{3}{2}.$$

.

Se cunoaşte identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind Lema pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obţinem

$$\sum\frac{1}{\sqrt{\frac{3r}{r_a}+3}}\geq\frac{3}{2}\Leftrightarrow\sum\frac{1}{\sqrt{\frac{r}{r_a}+1}}\geq\frac{3\sqrt{3}}{2}.$$

Remarca.

In ΔABC

$$\sum\frac{1}{\sqrt{\frac{r}{h_a}+1}}\geq\frac{3\sqrt{3}}{2}.$$

Remarca.

If $a,b,c>0$, $a+b+c=3$ and $\lambda\geq 0$ then

$$\sum\frac{1}{\sqrt{a+\lambda}}\geq\frac{3}{\sqrt{\lambda+1}}.$$

Marin Chirciu

Soluţie.

Lema.

If $0<a<3$, and $\lambda\geq 0$ then

$$\frac{1}{\sqrt{a+\lambda}}\geq\frac{2\lambda+3-a}{2(\lambda+1)\sqrt{\lambda+1}}.$$

Folosim Tangent Line Method pentru funcția $f:(0,3)\to\mathbf{R}$ $f(x)=\frac{1}{\sqrt{x+\lambda}}$ în $x_0=1$.

Aplicația41.

If $a,b,c>0$, $a+b+c=3$ then

$$a^2+b^2+c^2\geq a^2b+b^2c+c^2a.$$

Le Phuc Lu, Vietnam,THCS 4/2023

Remarca .

If $a,b,c>0$, $a+b+c=1$ then

$$a^2+b^2+c^2\geq 3\left(a^2b+b^2c+c^2a\right).$$

Marin Chirciu

Remarca .

In ΔABC

$$\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\ge 3r\left(\frac{1}{r_a^2r_b}+\frac{1}{r_b^2r_c}+\frac{1}{r_c^2r_a}\right).$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$x^2+y^2+z^2\ge 3\left(x^2y+y^2z+z^2x\right).$$

Soluție.

$$x^2+y^2+z^2\ge 3\left(x^2y+y^2z+z^2x\right)\Leftrightarrow (x+y+z)\left(x^2+y^2+z^2\right)\ge 3\left(x^2y+y^2z+z^2x\right)\Leftrightarrow$$

$$\Leftrightarrow \sum x^3+\sum xy^2+\sum x^2y\ge 3x^2y\Leftrightarrow \sum x^3+\sum xy^2-2\sum x^2y\ge 0\Leftrightarrow$$

$$\Leftrightarrow \sum x\left(x^2+y^2-2xy\right)\ge 0\Leftrightarrow \sum x(x-y)^2\ge 0.$$

Se cunoaşte identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obţinem:

$$\left(\frac{r}{r_a}\right)^2+\left(\frac{r}{r_b}\right)^2+\left(\frac{r}{r_c}\right)^2\ge 3\left(\left(\frac{r}{r_a}\right)^2\frac{r}{r_b}+\left(\frac{r}{r_b}\right)^2\frac{r}{r_c}+\left(\frac{r}{r_c}\right)^2\frac{r}{r_a}\right)\Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1}{r_a}\right)^2+\left(\frac{1}{r_b}\right)^2+\left(\frac{1}{r_c}\right)^2\ge 3r\left(\left(\frac{1}{r_a}\right)^2\frac{1}{r_b}+\left(\frac{1}{r_b}\right)^2\frac{1}{r_c}+\left(\frac{1}{r_c}\right)^2\frac{1}{r_a}\right)\Leftrightarrow$$

$$\Leftrightarrow \frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\ge 3r\left(\frac{1}{r_a^2r_b}+\frac{1}{r_b^2r_c}+\frac{1}{r_c^2r_a}\right).$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca .

In ΔABC

$$\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\ge 3r\left(\frac{1}{h_a^2h_b}+\frac{1}{h_b^2h_c}+\frac{1}{h_c^2h_a}\right).$$

Marin Chirciu

Aplicația42.

If $a,b,c>0$, $a+b+c=3$, then

$$\sum\frac{a}{1+b^2}\ge\frac{3}{2}.$$

THCS 4/2023

Remarca

In ΔABC

$$\sum\frac{\frac{1}{r_a}}{1+\left(\frac{3r}{r_b}\right)^2}\ge\frac{1}{2r}.$$

Marin Chirciu

Lema

If $x,y,z>0$, $x+y+z=3$, then

$$\sum\frac{x}{1+y^2}\ge\frac{3}{2}.$$

Soluție.

$$LHS=\sum\frac{x}{1+y^2}=\sum x\left(1-\frac{y^2}{1+y^2}\right)\overset{AM-GM}{\ge}\sum x\left(1-\frac{y^2}{2y}\right)=\sum x\left(1-\frac{y}{2}\right)=\sum x-\frac{1}{2}\sum xy\overset{SOS}{\ge}$$

$$\overset{SOS}{\ge}3-\frac{1}{2}\cdot\frac{1}{3}\left(\sum x\right)^2=3-\frac{1}{6}\cdot 3^2=\frac{3}{2}=RHS.$$

Se cunoaşte identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obţinem:

$$\sum\frac{\frac{3r}{r_a}}{1+\left(\frac{3r}{r_b}\right)^2}\ge\frac{3}{2}\Leftrightarrow\sum\frac{\frac{1}{r_a}}{1+\left(\frac{3r}{r_b}\right)^2}\ge\frac{1}{2r}.$$

Aplicația43.

If $a,b,c>0$, $ab+bc+ca=3$, then

$$\sum \frac{1}{a^2+2} \le 1.$$

THCS 4/2023

Soluție.

Remarca

In ΔABC

$$\sum \frac{1}{3\tan^2 \frac{A}{2} + 2} \le 1.$$

Marin Chirciu

Lema

If $x, y, z > 0$, $xy + yz + zx = 3$, then

$$\sum \frac{1}{x^2+2} \le 1.$$

Demonstrație.

Intorc $\sum \frac{1}{x^2+2} \le 1 \Leftrightarrow \sum \frac{x^2}{x^2+2} \ge 1$,care rezultă din:

$$\sum \frac{x^2}{x^2+2} \overset{CS}{\ge} \frac{\left(\sum x\right)^2}{\sum \left(x^2+2\right)} = \frac{\sum x^2 + 2\sum xy}{\sum x^2 + 6} = \frac{\sum x^2 + 2\cdot 3}{\sum x^2 + 6} = 1.$$

Se cunoaşte identitatea în triunghi $\sum \tan \frac{B}{2} \tan \frac{C}{2} = 1$.

Avem $\sum \tan \frac{B}{2} \tan \frac{C}{2} = 1 \Leftrightarrow \sum 3 \tan \frac{B}{2} \tan \frac{C}{2} = 3 \Leftrightarrow \sum \sqrt{3} \tan \frac{B}{2} \cdot \sqrt{3} \tan \frac{C}{2} = 3$

Folosind **Lema** pentru $(x, y, z) = \left(\sqrt{3} \tan \frac{A}{2}, \sqrt{3} \tan \frac{B}{2}, \sqrt{3} \tan \frac{C}{2}\right)$ obţinem:

$$\sum \frac{1}{3\tan^2 \frac{A}{2} + 2} \le 1.$$

Remarca

If $a, b, c > 0$, $ab + bc + ca = 3$, and $\lambda \ge 2$ then

$$\sum \frac{1}{a^2+\lambda} \le \frac{3}{\lambda+1}.$$

Marin Chirciu

Soluție.

Intorc $\sum \frac{1}{a^2+\lambda} \le \frac{3}{\lambda+1} \Leftrightarrow \sum \frac{a^2}{a^2+\lambda} \ge \frac{3}{\lambda+1}$,care reyultă din:

$$\sum \frac{a^2}{a^2+\lambda} \overset{CS}{\ge} \frac{\left(\sum a\right)^2}{\sum\left(a^2+\lambda\right)} = \frac{\sum a^2 + 2\sum ab}{\sum a^2 + 3\lambda} = \frac{\sum a^2 + 2\cdot 3}{\sum a^2 + 3\lambda} \overset{(1)}{\ge} \frac{3}{\lambda+1},$$

unde $\frac{\sum a^2 + 2\cdot 3}{\sum a^2 + 3\lambda} \ge \frac{3}{\lambda+1} \Leftrightarrow (\lambda-2)\sum a^2 \ge 3(\lambda-2)$, vezi ipoteza $\lambda \ge 2$ şi $\sum a^2 \ge 3$,

adevărată din $\sum a^2 \ge \sum ab = 3$.

Remarca

In ΔABC

$$\sum \frac{1}{3\tan^2 \frac{A}{2} + \lambda} \le \frac{3}{\lambda+1}, \text{ unde } \lambda \ge 2.$$

Marin Chirciu

Lema

If $x, y, z > 0$, $xy + yz + zx = 3$, and $\lambda \ge 2$ then

$$\sum \frac{1}{x^2+\lambda} \le \frac{3}{\lambda+1}.$$

Soluție.

Intorc $\sum \frac{1}{x^2+\lambda} \le \frac{3}{\lambda+1} \Leftrightarrow \sum \frac{x^2}{x^2+\lambda} \ge \frac{3}{\lambda+1}$,care rezultă din:

$$\sum \frac{x^2}{x^2+\lambda} \overset{CS}{\ge} \frac{\left(\sum x\right)^2}{\sum\left(x^2+\lambda\right)} = \frac{\sum x^2 + 2\sum xy}{\sum x^2 + 3\lambda} = \frac{\sum x^2 + 2\cdot 3}{\sum x^2 + 3\lambda} \overset{(1)}{\ge} \frac{3}{\lambda+1},$$

unde $\frac{\sum x^2 + 2\cdot 3}{\sum x^2 + 3\lambda} \ge \frac{3}{\lambda+1} \Leftrightarrow (\lambda-2)\sum x^2 \ge 3(\lambda-2)$, vezi ipoteza $\lambda \ge 2$ şi $\sum x^2 \ge 3$,

adevărată din $\sum x^2 \ge \sum xy = 3$.

Se cunoaşte identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$.

Avem $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1 \Leftrightarrow \sum 3\tan\frac{B}{2}\tan\frac{C}{2} = 3 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2} = 3$

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obţinem:

$$\sum\frac{1}{3\tan^2\frac{A}{2}+\lambda}\le\frac{3}{\lambda+1}.$$

Aplicația44.

If $a,b,c>0$, $ab+bc+ca=1$, then

$$\sum\frac{1}{1+a}\ge\frac{3\sqrt{3}}{\sqrt{3}+1}.$$

Math for change

Remarca

In ΔABC

$$\sum\frac{1}{1+\tan\frac{A}{2}}\ge\frac{3\sqrt{3}}{\sqrt{3}+1}.$$

Marin Chirciu

Lema

If $x,y,z>0$, $xy+yz+zx=1$, then

$$\sum\frac{1}{1+x}\ge\frac{3\sqrt{3}}{\sqrt{3}+1}.$$

Soluție.

$$\sum\frac{1}{1+x}\ge\frac{3\sqrt{3}}{\sqrt{3}+1}\Leftrightarrow\frac{3+2\sum x+\sum xy}{1+\sum x+\sum xy+xyz}\ge\frac{3\sqrt{3}}{1+\sqrt{3}}\Leftrightarrow\frac{4+2\sum x}{2+\sum x+xyz}\ge\frac{3\sqrt{3}}{1+\sqrt{3}}\Leftrightarrow$$

$\Leftrightarrow\left(2-\sqrt{3}\right)\sum x\ge 2\sqrt{3}-4+3\sqrt{3}xyz$, care rezultă din:

$x\ge\sqrt{3}$,(vezi$\left(\sum x\right)^2\ge 3\sum xy=3$)şi $xyz\le\frac{1}{3\sqrt{3}}$,(vezi: $1=\sum xy\ge 3\sqrt[3]{x^2y^2z^2}$).

Se cunoaşte identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$

Folosind **Lema** pentru $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obţinem:

$$\sum \frac{1}{1+\tan\frac{A}{2}} \geq \frac{3\sqrt{3}}{\sqrt{3}+1}.$$

Remarca

If $a,b,c>0$, $ab+bc+ca=1$, and $0\leq\lambda\leq\frac{2}{\sqrt{3}}$ then

$$\sum \frac{1}{\lambda+a} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}}.$$

Marin Chirciu

Soluție.

$$\sum \frac{1}{\lambda+a} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}} \Leftrightarrow \frac{3\lambda^2+2\lambda\sum a+\sum ab}{\lambda^3+\lambda^2\sum a+\lambda\sum ab+abc} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}} \Leftrightarrow$$

$$\Leftrightarrow \frac{3\lambda^2+2\lambda\sum a+1}{\lambda^3+\lambda^2\sum a+\lambda+abc} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}} \Leftrightarrow \qquad \Leftrightarrow$$

$\Leftrightarrow \lambda\left(2-\lambda\sqrt{3}\right)\sum a \geq 2\lambda\sqrt{3}-3\lambda^2-1+3\sqrt{3}abc$, care rezultă din:

$\sum a \geq \sqrt{3}$,(vezi$\left(\sum a\right)^2 \geq 3\sum ab=3$)şi $abc\leq\frac{1}{3\sqrt{3}}$,(vezi: $1=\sum ab\geq 3\sqrt[3]{a^2b^2c^2}$).

Obţinem:

$$\lambda\left(2-\lambda\sqrt{3}\right)\cdot\sqrt{3} \geq 2\lambda\sqrt{3}-3\lambda^2-1+3\sqrt{3}\cdot\frac{1}{3\sqrt{3}} \Leftrightarrow \left(2-\lambda\sqrt{3}\right)\cdot\sqrt{3}\geq 2\sqrt{3}-3\lambda \Leftrightarrow$$

$\Leftrightarrow 2-\lambda\sqrt{3}\geq 2-\sqrt{3}\lambda$, vezi ipoteza $0\leq\lambda\leq\frac{2}{\sqrt{3}}$.

Remarca

In ΔABC

$$\sum \frac{1}{\lambda+\tan\frac{A}{2}} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}}, \text{unde } 0\leq\lambda\leq\frac{2}{\sqrt{3}}.$$

Marin Chirciu

Lema

If $x,y,z>0$, $xy+yz+zx=1$, and $0\leq\lambda\leq\frac{2}{\sqrt{3}}$ then

$$\sum \frac{1}{\lambda + x} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}}.$$

Soluție.

$$\sum \frac{1}{\lambda + x} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}} \Leftrightarrow \frac{3\lambda^2 + 2\lambda\sum x + \sum xy}{\lambda^3 + \lambda^2\sum x + \lambda\sum xy + xyz} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}} \Leftrightarrow$$

$$\Leftrightarrow \frac{3\lambda^2 + 2\lambda x + 1}{\lambda^3 + \lambda^2\sum x + \lambda + xyz} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}} \Leftrightarrow \qquad \Leftrightarrow$$

$\Leftrightarrow \lambda\left(2-\lambda\sqrt{3}\right)\sum x \geq 2\lambda\sqrt{3} - 3\lambda^2 - 1 + 3\sqrt{3}xyz$, care rezultă din:

$\sum x \geq \sqrt{3}$,(vezi $\left(\sum x\right)^2 \geq 3\sum xy = 3$)și $xyz \leq \frac{1}{3\sqrt{3}}$,(vezi: $1 = xy \geq 3\sqrt[3]{x^2y^2z^2}$).

Obţinem:

$$\lambda\left(2-\lambda\sqrt{3}\right)\cdot\sqrt{3} \geq 2\lambda\sqrt{3} - 3\lambda^2 - 1 + 3\sqrt{3}\cdot\frac{1}{3\sqrt{3}} \Leftrightarrow \left(2-\lambda\sqrt{3}\right)\cdot\sqrt{3} \geq 2\sqrt{3} - 3\lambda \Leftrightarrow$$

$\Leftrightarrow 2-\lambda\sqrt{3} \geq 2-\sqrt{3}\lambda$, vezi ipoteza $0 \leq \lambda \leq \frac{2}{\sqrt{3}}$.

Se cunoaşte identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$

Folosind **Lema** pentru $(x, y, z) = \left(\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right)$ obţinem:

$$\sum \frac{1}{\lambda + \tan\frac{A}{2}} \geq \frac{3\sqrt{3}}{1+\lambda\sqrt{3}}.$$

Aplicația45.

If $a, b, c > 0$, $ab + bc + ca = 3$ then

$$\sum \frac{ab}{a^2+b^2} + \frac{5(a+b+c)}{6abc} \geq 4.$$

Nguyen Thai An, Vietnam, THCS 4/2023

Remarca.

If $a, b, c > 0$, $ab + bc + ca = 3$ and $\lambda \geq \frac{3}{4}$ then

$$\sum \frac{ab}{a^2+b^2} + \frac{\lambda(a+b+c)}{abc} \geq \frac{3}{2}(2\lambda + 1).$$

Marin Chirciu

Remarca.

In ΔABC

$$\sum\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda(4R+r)}{3r}\geq\frac{3}{2}(2\lambda+1)\text{, unde }\lambda\geq\frac{3}{4}.$$

Marin Chirciu

Soluţie

Lema

If $x,y,z>0$, $xy+yz+zx=3$ and $\lambda\geq\frac{3}{4}$ then

$$\sum\frac{xy}{x^2+y^2}+\frac{\lambda(x+y+z)}{xyz}\geq\frac{3}{2}(2\lambda+1).$$

Demonstraţie.

If $x,y,z>0$ then

$$\frac{x+y+z}{xyz}\geq 3.$$

$$3(x+y+z)=(x+y+z)(xy+yz+zx)\overset{AM-GM}{\geq}3\sqrt[3]{xyz}\cdot 3\sqrt[3]{x^2y^2z^2}=9xyz\Rightarrow\frac{x+y+z}{xyz}\geq 3,$$

cu egalitate pentru $a=b=c=1$.

Din AM-GM rezultă $\frac{xy}{x^2+y^2}+\frac{x^2+y^2}{4xy}\geq 1\Rightarrow\sum\frac{xy}{x^2+y^2}+\sum\frac{x^2+y^2}{4xy}\geq 3\Rightarrow$

$$\Rightarrow\sum\frac{xy}{x^2+y^2}+\frac{\sum z(x^2+y^2)}{4xyz}\geq 3\Rightarrow\sum\frac{xy}{x^2+y^2}+\frac{(x+y+z)(xy+yz+zx)-3xyz}{4xyz}\geq 3\Rightarrow$$

$$\Rightarrow\sum\frac{xy}{x^2+y^2}+\frac{3(x+y+z)-3xyz}{4xyz}\geq 3\Rightarrow\sum\frac{xy}{x^2+y^2}+\frac{3(x+y+z)}{4xyz}\geq\frac{15}{4}.$$

Din $\sum\frac{xy}{x^2+y^2}+\frac{3(x+y+z)}{4xyz}\geq\frac{15}{4}$, $\frac{x+y+z}{xyz}\geq 3$ şi $\lambda\geq\frac{3}{4}$ obţinem

$$\sum\frac{xy}{x^2+y^2}+\frac{\lambda(x+y+z)}{xyz}\geq\frac{3}{2}(2\lambda+1).$$

Se cunoaşte identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$.

Avem $\sum \tan\frac{B}{2}\tan\frac{C}{2}=1 \Leftrightarrow \sum 3\tan\frac{B}{2}\tan\frac{C}{2}=3 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obţinem

$$\sum\frac{\sqrt{3}\tan\frac{A}{2}\cdot\sqrt{3}\tan\frac{B}{2}}{3\tan^2\frac{A}{2}+3\tan^2\frac{B}{2}}+\frac{\lambda\left(\sqrt{3}\tan\frac{A}{2}+\sqrt{3}\tan\frac{B}{2}+\sqrt{3}\tan\frac{C}{2}\right)}{\sqrt{3}\tan\frac{A}{2}\cdot\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}}\geq\frac{3}{2}(2\lambda+1)\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)}{3\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}\geq\frac{3}{2}(2\lambda+1)\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda\cdot\frac{4R+r}{p}}{3\cdot\frac{r}{p}}\geq\frac{3}{2}(2\lambda+1)\Leftrightarrow\sum\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{\lambda(4R+r)}{3r}\geq\frac{3}{2}(2\lambda+1)$$

Remarca.

Cazul $\lambda=\frac{3}{4}$.

In ΔABC

$$\sum\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}+\frac{R}{r}\geq\frac{7}{2}.$$

Marin Chirciu

Aplicația46.

If $a,b,c>0$, $a+b+c=1$then

$$\sum(a+b)\sqrt{\frac{a}{b}}\geq 2.$$

Panagiotis Danousis, Greece, MathAtelier 4/2023

Remarca.

In ΔABC

$$\sum\left(\frac{r}{r_a}+\frac{r}{r_b}\right)\sqrt{\frac{r_b}{r_a}}\geq 2$$

Marin Chirciu

Soluție.

Lema

If $x, y > 0$ then

$$\sum (x+y)\sqrt{\frac{x}{y}} \geq 2.$$

Demonstraţie

Avem $(x+y)\sqrt{\frac{x}{y}} \geq 2x$, care rezultă din:

$(x+y)\sqrt{\frac{x}{y}} \geq 2x \Leftrightarrow \left(\sqrt{x}-\sqrt{y}\right)^2 \sqrt{\frac{x}{y}} \geq 0$, cu egalitate pentru $x = y$.

Obţinem $\sum (x+y)\sqrt{\frac{x}{y}} \overset{Lema}{\geq} \sum 2x = 2\sum x = 2 \cdot 1 = 2$.

Se cunoaşte identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{r}{r_a}, \frac{r}{r_b}, \frac{r}{r_c}\right)$ obţinem:

$$\sum \left(\frac{r}{r_a}+\frac{r}{r_b}\right)\sqrt{\frac{\frac{r}{r_a}}{\frac{r}{r_b}}} \geq 2 \Leftrightarrow \sum \left(\frac{r}{r_a}+\frac{r}{r_b}\right)\sqrt{\frac{r_b}{r_a}} \geq 2 \Leftrightarrow \sum \left(\frac{1}{r_a}+\frac{1}{r_b}\right)\sqrt{\frac{r_b}{r_a}} \geq \frac{2}{r}.$$

Aplicaţia47.

If $a, b, c > 0$, $a+b+c=1$then

$$\sum \frac{(a+b)^3}{a+b+ab} \geq \frac{8}{7}.$$

George Apostolopoulos, Greece, Mathematical Inequalities 4/2023

Remarca.

If $a, b, c > 0$, $a+b+c=1$and $\lambda \geq 0$ then

$$\sum \frac{(a+b)^3}{a+b+\lambda ab} \geq \frac{8}{\lambda+6}.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{(a+b)^3}{a+b+\lambda ab}\overset{Holder}{\geq}\frac{\left(\sum(a+b)\right)^3}{3\sum(a+b+\lambda ab)}=\frac{\left(2\sum a\right)^3}{3\left(2\sum a+\lambda\sum ab\right)}=\frac{8}{3\left(2+\lambda\sum ab\right)}\overset{(1)}{\geq}\frac{8}{\lambda+6}=$$

$=RHS$,unde

$\frac{8}{3\left(2+\lambda\sum ab\right)}\geq\frac{8}{\lambda+6}\Leftrightarrow\lambda+6\geq3\left(2+\lambda\sum ab\right)\Leftrightarrow\lambda\geq3\lambda\sum ab\Leftrightarrow\left(\sum a\right)^2\geq3\sum ab$, inegalitate cunoscută.

Se cunoaşte identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obţinem o inegalitate în triunghi.

Remarca.

If $a,b,c>0$, $a+b+c=1$ and $n\geq0$, $\lambda\geq0$ then

$$\sum\frac{(a+nb)^3}{a+b+\lambda ab}\geq\frac{(n+1)^3}{\lambda+6}.$$

Marin Chirciu

Aplicația48.

If $a,b,c>0$, $a+b+c\geq3abc$ then

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\geq\frac{3}{4}.$$

Nguyen Thau An, Vietnam,THCS 4/2023

Remarca.

Problema se poate dezvolta.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\frac{x^2}{(x+1)^2}+\frac{y^2}{(y+1)^2}+\frac{z^2}{(z+1)^2}\geq\frac{3}{4}.$$

Marin Chirciu

Demonstrație.

$$\sum\frac{x^2}{(x+1)^2}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum(x+1)^2}=\frac{\left(\sum x\right)^2}{\sum x^2+2\sum x+3}\overset{\sum xy=3}{\geq}\frac{\left(\sum x\right)^2}{\sum x^2+2\sum x+\sum xy}\overset{(1)}{\geq}\frac{3}{4},$$

unde $\frac{\left(\sum x\right)^2}{\sum x^2+2\sum x+\sum xy}\geq\frac{3}{4}\Leftrightarrow 4\left(\sum x\right)^2\geq 3\sum x^2+6\sum x+3\sum xy\Leftrightarrow$

$\left(\sum x\right)^2+3\sum x^2+6\sum xy\geq 3\sum x^2+6\sum x+3\sum xy\Leftrightarrow\left(\sum x\right)^2-6\sum x+3\sum xy\geq 0,$

care rezultă din $\sum xy=3$.

Obţinem $\left(\sum x\right)^2-6\sum x+3\sum xy=\left(\sum x\right)^2-6\sum x+9=\left(\sum x-3\right)^2\geq 0$.

Remarca.

In ΔABC

$$\frac{\tan^2\frac{A}{2}}{\left(\sqrt{3}\tan\frac{A}{2}+1\right)^2}+\frac{\tan^2\frac{B}{2}}{\left(\sqrt{3}\tan\frac{B}{2}+1\right)^2}+\frac{\tan^2\frac{C}{2}}{\left(\sqrt{3}\tan\frac{C}{2}+1\right)^2}\geq\frac{1}{4}.$$

Marin Chirciu

Soluţie

Lema

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\frac{x^2}{\left(x^2+1\right)^2}+\frac{y^2}{\left(y^2+1\right)^2}+\frac{z^2}{\left(z^2+1\right)^2}\geq\frac{3}{4}.$$

Demonstraţie.

$$\sum\frac{x^2}{(x+1)^2}\overset{CS}{\geq}\frac{\left(\sum x\right)^2}{\sum(x+1)^2}=\frac{\left(\sum x\right)^2}{\sum x^2+2\sum x+3}\overset{\sum xy=3}{\geq}\frac{\left(\sum x\right)^2}{\sum x^2+2\sum x+\sum xy}\overset{(1)}{\geq}\frac{3}{4},$$

unde $\frac{\left(\sum x\right)^2}{\sum x^2+2\sum x+\sum xy}\geq\frac{3}{4}\Leftrightarrow 4\left(\sum x\right)^2\geq 3\sum x^2+6\sum x+3\sum xy\Leftrightarrow$

$\left(\sum x\right)^2+3\sum x^2+6\sum xy\geq 3\sum x^2+6\sum x+3\sum xy\Leftrightarrow\left(\sum x\right)^2-6\sum x+3\sum xy\geq 0,$

care rezultă din $\sum xy=3$.

Obţinem $\left(\sum x\right)^2-6\sum x+3\sum xy=\left(\sum x\right)^2-6\sum x+9=\left(\sum x-3\right)^2\geq 0$.

Se cunoaşte identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2}=1$.

Avem $\sum 3\tan\frac{B}{2}\tan\frac{C}{2}=3 \Leftrightarrow \sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obţinem:

$$\frac{3\tan^2\frac{A}{2}}{\left(\sqrt{3}\tan\frac{A}{2}+1\right)^2}+\frac{3\tan^2\frac{B}{2}}{\left(\sqrt{3}\tan\frac{B}{2}+1\right)^2}+\frac{3\tan^2\frac{C}{2}}{\left(\sqrt{3}\tan\frac{C}{2}+1\right)^2}\geq\frac{3}{4}\Leftrightarrow$$

$$\Leftrightarrow\frac{\tan^2\frac{A}{2}}{\left(\sqrt{3}\tan\frac{A}{2}+1\right)^2}+\frac{\tan^2\frac{B}{2}}{\left(\sqrt{3}\tan\frac{B}{2}+1\right)^2}+\frac{\tan^2\frac{C}{2}}{\left(\sqrt{3}\tan\frac{C}{2}+1\right)^2}\geq\frac{1}{4}.$$

Aplicația49.

In ΔABC

$$\sum\left(\frac{a}{b+c}\right)^2\frac{y+z}{x}\geq\frac{81r^2}{2p^2}.$$

Mehmet Şahin, Turkey

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\sum\left(\frac{h_a}{b+c}\right)^2\frac{y+z}{x}\geq\frac{9r^2}{2R^2}.$$

Marin Chirciu

Soluție.

Lemă.

Fie $x,y,z>0$ și $f:\mathrm{D}\to\boldsymbol{R}$ o funcție . Are loc relația

$$\sum\frac{y+z}{x}f^2(a)\geq 2\sum f(b)f(c).$$

Demonstrație.

Avem $\sum\frac{y+z}{x}f^2(a)=\sum\left(\frac{y+z}{x}+1-1\right)f^2(a)=\sum\frac{x+y+z}{x}f^2(a)-\sum f^2(a)\overset{CS}{\geq}$

$$\overset{CS}{\geq}(x+y+z)\frac{\left(\sum f(a)\right)^2}{x+y+z}-\sum f^2(a)=(x+y+z)\frac{\left(\sum f(a)\right)^2}{(x+y+z)}-\sum f^2(a)=$$

$$=\left(\sum f(a)\right)^2-\sum f^2(a)=\sum f^2(a)+2\sum f(b)f(c)-\sum f^2(a)=2\sum f(b)f(c).$$

Folosind **Lema** pentru $f(a)=\frac{h_a}{2p-a}=\frac{h_a}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{h_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{h_b}{c+a}\cdot\frac{h_c}{a+b}=2\sum\frac{h_bh_c}{(a+b)(a+c)}=$$

$$=2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9r^2}{2R^2}=RHS,$$

unde $2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9r^2}{2R^2}\Leftrightarrow p^2(4R-9r)+r\left(16R^2-14Rr-9r^2\right)\geq 0$.

Distingem cazurile:

Cazul1).Dacă $(4R-9r)\geq 0$, inegalitatea este evidentă.

Cazul2).Dacă $(4R-9r)<0$, inegalitatea se rescrie: $r\left(16R^2-14Rr-9r^2\right)\geq p^2(9r-4R)$,

care rezultă din inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$$r\left(16R^2-14Rr-9r^2\right)\geq\left(4R^2+4Rr+3r^2\right)(9r-4R)\Leftrightarrow 8R^3-2R^2r-19Rr^2-18r^3\geq 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)\left(8R^2+14Rr+9r^2\right)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Am folosit mai sus: $\sum\frac{h_bh_c}{(a+b)(a+c)}=\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$\sum\left(\frac{m_a}{b+c}\right)^2\frac{y+z}{x}\geq\frac{9r^2}{2R^2}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a)=\frac{m_a}{2p-a}=\frac{m_a}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{m_a}{b+c}\right)^2\frac{y+z}{x}\overset{m_a\geq h_a}{\geq}\sum\left(\frac{h_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{h_b}{c+a}\cdot\frac{h_c}{a+b}=$$

$$2\sum\frac{h_bh_c}{(a+b)(a+c)}=2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9r^2}{2R^2}=RHS,$$

unde $2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9r^2}{2R^2}\Leftrightarrow p^2(4R-9r)+r\left(16R^2-14Rr-9r^2\right)\geq 0$.

Distingem cazurile:

Cazul1).Dacă $(4R-9r)\geq 0$, inegalitatea este evidentă.

Cazul2).Dacă $(4R-9r)<0$, inegalitatea se rescrie: $r\left(16R^2-14Rr-9r^2\right)\geq p^2(9r-4R)$,

care rezultă din inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$$r\left(16R^2-14Rr-9r^2\right)\geq\left(4R^2+4Rr+3r^2\right)(9r-4R)\Leftrightarrow 8R^3-2R^2r-19Rr^2-18r^3\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)\left(8R^2+14Rr+9r^2\right)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Am folosit mai sus: $\sum\frac{h_bh_c}{(a+b)(a+c)}=\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$\sum\left(\frac{w_a}{b+c}\right)^2\frac{y+z}{x}\geq\frac{9r^2}{2R^2}.$$

Marin Chirciu

Soluţie.

Folosind **Lema** pentru $f(a)=\frac{w_a}{2p-a}=\frac{w_a}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{h_a}{b+c}\right)^2\frac{y+z}{x}\overset{w_a\geq h_a}{\geq}\sum\left(\frac{h_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{h_b}{c+a}\cdot\frac{h_c}{a+b}=$$

$$2\sum \frac{h_b h_c}{(a+b)(a+c)} = 2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)} \overset{(1)}{\geq} \frac{9r^2}{2R^2} = RHS,$$

unde $2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)} \overset{(1)}{\geq} \frac{9r^2}{2R^2} \Leftrightarrow p^2(4R-9r)+r\left(16R^2-14Rr-9r^2\right) \geq 0$.

Distingem cazurile:

Cazul1).Dacă $(4R-9r) \geq 0$, inegalitatea este evidentă.

Cazul2).Dacă $(4R-9r) < 0$, inegalitatea se rescrie: $r\left(16R^2-14Rr-9r^2\right) \geq p^2(9r-4R)$,

care rezultă din inegalitatea lui Gerretsen $p^2 \leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$r\left(16R^2-14Rr-9r^2\right) \geq \left(4R^2+4Rr+3r^2\right)(9r-4R) \Leftrightarrow 8R^3-2R^2r-19Rr^2-18r^3 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)\left(8R^2+14Rr+9r^2\right) \geq 0$, evident din inegalitatea lui Euler $R \geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Am folosit mai sus: $\sum \frac{h_b h_c}{(a+b)(a+c)} = \frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$\sum \left(\frac{s_a}{b+c}\right)^2 \frac{y+z}{x} \geq \frac{9r^2}{2R^2}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a) = \frac{s_a}{2p-a} = \frac{s_a}{b+c}$ obţinem:

$$LHS = \sum \left(\frac{s_a}{b+c}\right)^2 \frac{y+z}{x} \overset{s_a \geq h_a}{\geq} \sum \left(\frac{h_a}{b+c}\right)^2 \frac{y+z}{x} \overset{Lema}{\geq} 2\sum f(b) f(c) = 2\sum \frac{h_b}{c+a} \cdot \frac{h_c}{a+b} =$$

$$= 2\sum \frac{h_b h_c}{(a+b)(a+c)} = 2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)} \overset{(1)}{\geq} \frac{9r^2}{2R^2} = RHS,$$

unde $2\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)} \overset{(1)}{\geq} \frac{9r^2}{2R^2} \Leftrightarrow p^2(4R-9r)+r\left(16R^2-14Rr-9r^2\right) \geq 0$.

Distingem cazurile:

Cazul1).Dacă $(4R-9r)\geq 0$, inegalitatea este evidentă.

Cazul2).Dacă $(4R-9r)<0$, inegalitatea se rescrie: $r\left(16R^2-14Rr-9r^2\right)\geq p^2(9r-4R)$,

care rezultă din inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$r\left(16R^2-14Rr-9r^2\right)\geq\left(4R^2+4Rr+3r^2\right)(9r-4R)\Leftrightarrow 8R^3-2R^2r-19Rr^2-18r^3\geq 0\Leftrightarrow$

$\Leftrightarrow(R-2r)\left(8R^2+14Rr+9r^2\right)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Am folosit mai sus: $\sum\frac{h_b h_c}{(a+b)(a+c)}=\frac{r\left(p^2+r^2+4Rr\right)}{R\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$\sum\left(\frac{r_a}{b+c}\right)^2\frac{y+z}{x}\geq\frac{9}{8}.$$

Marin Chirciu

Soluție.

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Folosind **Lema** pentru $f(a)=\frac{r_a}{2p-a}=\frac{r_a}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{r_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{r_b}{c+a}\cdot\frac{r_c}{a+b}=2\sum\frac{r_b r_c}{(a+b)(a+c)}=$$

$$=2\frac{p^2-r^2-4Rr}{p^2+r^2+2Rr}\overset{(1)}{\geq}\frac{9}{8}=RHS,$$

unde $2\frac{p^2-r^2-4Rr}{p^2+r^2+2Rr}\overset{(1)}{\geq}\frac{9}{8}\Leftrightarrow 7p^2\geq 82Rr+25r^2$,

care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.

Rămâne să arătăm că:

$7\left(16Rr-5r^2\right)\geq 82Rr+25r^2\Leftrightarrow R\geq 2r$,(Euler).

Am folosit mai sus: $\sum \frac{r_b r_c}{(a+b)(a+c)} = \frac{p^2 - r^2 - 4Rr}{p^2 + r^2 + 2Rr}$.

Remarca.

În ΔABC

$$\sum \frac{h_b h_c}{(a+b)(a+c)} \le \sum \frac{r_b r_c}{(a+b)(a+c)}.$$

Marin Chirciu

Soluție.

Avem sumele:

$$\sum \frac{h_b h_c}{(a+b)(a+c)} = \frac{r(p^2 + r^2 + 4Rr)}{R(p^2 + r^2 + 2Rr)} \text{ şi } \sum \frac{r_b r_c}{(a+b)(a+c)} = \frac{p^2 - r^2 - 4Rr}{p^2 + r^2 + 2Rr}.$$

Inegalitatea se scrie:

$\frac{r(p^2 + r^2 + 4Rr)}{R(p^2 + r^2 + 2Rr)} \le \frac{p^2 - r^2 - 4Rr}{p^2 + r^2 + 2Rr} \Leftrightarrow p^2(R - r) \ge r(4R^2 + 5Rr + r^2)$, care reyultă din inegalitatea lui Gerretsen $p^2 \ge 16Rr - 5r^2$.

Rămâne să arătăm că:

$(16Rr - 5r^2)(R - r) \ge r(4R^2 + 5Rr + r^2) \Leftrightarrow 6R^2 - 13Rr + 2r^2 \ge 0 \Leftrightarrow$

$\Leftrightarrow (R - 2r)(6R - r) \ge 0$, evident din inegalitatea lui Euler $R \ge 2r$.

Remarca.

In ΔABC

$$\sum \left(\frac{\tan \frac{A}{2}}{b+c} \right)^2 \frac{y+z}{x} \ge \frac{9}{8}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a) = \frac{\tan \frac{A}{2}}{2p - a} = \frac{\tan \frac{A}{2}}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{r_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{\tan\frac{B}{2}}{c+a}\cdot\frac{\tan\frac{C}{2}}{a+b}=2\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}=$$

$$=2\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9}{8p^2}=RHS,$$

unde $2\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9}{8p^2}\Leftrightarrow 7p^2\geq 82Rr+25r^2$,

care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.

Rămâne să arătăm că:

$7\left(16Rr-5r^2\right)\geq 82Rr+25r^2\Leftrightarrow R\geq 2r$,(Euler).

Am folosit mai sus: $\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}=\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$\sum\left(\frac{\tan\frac{A}{2}}{b+c}\right)^2\frac{y+z}{x}\geq\frac{9}{8}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a)=\frac{\tan\frac{A}{2}}{2p-a}=\frac{\tan\frac{A}{2}}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{r_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{\tan\frac{B}{2}}{c+a}\cdot\frac{\tan\frac{C}{2}}{a+b}=2\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}=$$

$$=2\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9}{8p^2}=RHS,$$

unde $2\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{9}{8p^2}\Leftrightarrow 7p^2\geq 82Rr+25r^2$,

care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.

Rămâne să arătăm că:

$7\left(16Rr-5r^2\right)\geq 82Rr+25r^2 \Leftrightarrow R\geq 2r$,(Euler).

Am folosit mai sus: $\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}=\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$\sum\left(\frac{\cot\frac{A}{2}}{b+c}\right)^2\frac{y+z}{x}\geq\frac{81}{8p^2}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a)=\frac{\cot\frac{A}{2}}{2p-a}=\frac{\cot\frac{A}{2}}{b+c}$ obţinem:

$$LHS=\sum\left(\frac{r_a}{b+c}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{\cot\frac{B}{2}}{c+a}\cdot\frac{\cot\frac{C}{2}}{a+b}=2\sum\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{(a+b)(a+c)}=$$

$$=2\cdot\frac{2(R+r)}{r\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{81}{8p^2}=RHS,$$

unde $2\cdot\frac{2(R+r)}{r\left(p^2+r^2+2Rr\right)}\overset{(1)}{\geq}\frac{81}{8p^2}\Leftrightarrow p^2(32R-49r)\geq 81r\left(r^2+2Rr\right)$,

care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.

Rămâne să arătăm că:

$\left(16Rr-5r^2\right)(32R-49r)\geq 81r\left(r^2+2Rr\right)\Leftrightarrow 256R^2-553Rr+82r^2\geq 0\Leftrightarrow$

$\Leftrightarrow (R-2r)(6R-r)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Am folosit mai sus: $\sum\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{(a+b)(a+c)}=\frac{2(R+r)}{r\left(p^2+r^2+2Rr\right)}$.

Remarca.

In ΔABC

$$9\cdot\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}\le\sum\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{(a+b)(a+c)}.$$

Marin Chirciu

Soluţie

Lema1

In ΔABC

$$\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}=\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}.$$

Demonstraţie

$$\sum\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{(a+b)(a+c)}=\sum\frac{\sqrt{\frac{(p-a)(p-c)}{p(p-b)}}\sqrt{\frac{(p-a)(p-b)}{p(p-c)}}}{(a+b)(a+c)}=\frac{1}{p}\sum\frac{p-a}{(a+b)(a+c)}=$$

$$=\frac{1}{p}\frac{\sum(p-a)(b+c)}{\prod(b+c)}=\frac{1}{p}\cdot\frac{2\left(p^2-r^2-4Rr\right)}{2p\left(p^2+r^2+2Rr\right)}=\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}.$$

Lema2

In ΔABC

$$\sum\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{(a+b)(a+c)}=\frac{2(R+r)}{r\left(p^2+r^2+2Rr\right)}.$$

Demonstraţie

$$\sum\frac{\cot\frac{B}{2}\cot\frac{C}{2}}{(a+b)(a+c)}=\sum\frac{\sqrt{\frac{p(p-b)}{(p-a)(p-c)}}\sqrt{\frac{p(p-c)}{(p-a)(p-b)}}}{(a+b)(a+c)}=p\sum\frac{1}{(p-a)(a+b)(a+c)}=$$

$$=\frac{\sum(p-b)(p-c)(b+c)}{\prod(p-a)\prod(b+c)}=\frac{4rp(R+r)}{r^2p\cdot 2p\left(p^2+r^2+2Rr\right)}=\frac{2(R+r)}{r\left(p^2+r^2+2Rr\right)}.$$

Inegalitatea se scrie:

$$9\cdot\frac{p^2-r^2-4Rr}{p^2\left(p^2+r^2+2Rr\right)}\le\frac{2(R+r)}{r\left(p^2+r^2+2Rr\right)}\Leftrightarrow p^2(2R-7r)+9r^2(4R+r)\ge 0.$$

Distingem cazurile:

Cazul1). Dacă $(2R-7r)\geq 0$, inegalitatea este evidentă.

Cazul2). Dacă $(2R-7r)<0$, inegalitatea se rescrie $9r^2(4R+r)\geq p^2(7r-2R)$,

care rezultă din inegalitatea lui Gerretsen: $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$$9r^2(4R+r)\geq(4R^2+4Rr+3r^2)(7r-2R)\Leftrightarrow 4R^3-10R^2r+7Rr^2-6r^3\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(4R^2-2Rr+3r^2)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

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Art 4800

1 Septembrie 2023

3. Metoda polinoamelor reciproce

De Gheorghe Ghiţă, Buzău

Articolul propune utilizarea metodei polinoamelor reciproce (polinoame care au coeficienţii egali depărtaţi de extreme egali între ei) pentru demonstrarea unor inegalităţi, metodă care se bazează pe pozitivitatea unor polinoame reciproce cu coeficienţi întregi ce admit pe 1 ca rădăcină dublă. Fie P(t), $t > 0$ polinomul reciproc ataşat inegalităţii respective.

1) $x, y > 0 \Rightarrow \sqrt{\frac{x^2+y^2}{2}} \leq \sqrt[3]{\frac{x^3+y^3}{2}} \leq \sqrt[4]{\frac{x^4+y^4}{2}} \leq \frac{x^3+y^3}{2xy}$

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Soluţie. $\frac{x^2+y^2}{x+y} \geq \sqrt[3]{\frac{x^3+y^3}{2}} \Leftrightarrow \left(\frac{t^2+1}{t+1}\right)^3 \geq \frac{t^3+1}{2}, t = \frac{x}{y} \Leftrightarrow P(t) = t^6 - 3t^5 + 3t^4 - 2t^3 + 3t^2 - 3t + 1 = \quad (t-1)^4(t^2+t+1) \geq 0;$

$\sqrt{\frac{x^2+y^2}{2}} \leq \sqrt[3]{\frac{x^3+y^3}{2}} \Leftrightarrow \sqrt{\frac{t^2+1}{2}} \leq \sqrt[3]{\frac{t^3+1}{2}}, t = \frac{x}{y} \Leftrightarrow \; P(t) = t^6 - 3t^4 + 4t^3 - 3t^2 + 1 = (t-1)^2(t^4 + 2t^3 + 2t + 1) \geq 0;$

$\sqrt[3]{\frac{x^3+y^3}{2}} \leq \sqrt[4]{\frac{x^4+y^4}{2}} \Leftrightarrow \sqrt[3]{\frac{t^3+1}{2}} \leq \sqrt[4]{\frac{t^4+1}{2}}, t = \frac{x}{y} \Leftrightarrow \; P(t) = t^{12} - 4t^9 + 6t^8 - 6t^6 * 6t^4 - 4t^3 + 1 = (t-1)^2(t^{10} + 2t^9 + 3t^8 + 3t^6 + 6t^5 + 3t^4 + 3t^2 + 2t + 1) \geq 0;$

$\sqrt[4]{\frac{x^4+y^4}{2}} \leq \frac{x^3+y^3}{2xy} \Leftrightarrow \left(\frac{t^3+1}{2t}\right)^4 \geq \frac{t^4+1}{2}, t = \frac{x}{y} \Leftrightarrow P(t) = t^{12} + 4t^9 - 8t^8 + 6t^6 - 8t^4 + 4t^3 + 1 = (t-1)^2(t^{10} + 2t^9 + 3t^8 + 8t^7 + 5t^6 + 2t^5 + 5t^4 + 8t^3 + 3t^2 2t + 1) \geq 0.$

2) $\forall x, y > 0; n \in N^*, k \in N \Rightarrow \frac{x^{2n+k}+y^{2n+k}}{x^n y^n (x^{k+1}+y^{k+1})} + \frac{y^{2n+k}+z^{2n+k}}{y^n z^n (y^{k+1}+z^{k+1})} + \frac{z^{2n+k}+x^{2n+k}}{z^n x^n (z^{k+1}+x^{k+1})} \geq \boldsymbol{\frac{9}{x+y+z}}.$

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Soluţie. **Lemă**: $\forall x, y > 0; n \in N^*, k \in N \Rightarrow \frac{x^{2n+k}+y^{2n+k}}{x^n y^n (x^{k+1}+y^{k+1})} \geq \frac{2}{x+y}.$

Pentru $n = 1, k \in N \Rightarrow \boldsymbol{\frac{x^{k+2}+y^{k+2}}{xy(x^{k+1}+y^{k+1})}} \geq \frac{2}{x+y} \Leftrightarrow \boldsymbol{\frac{t^{k+2}+1}{t(t^{k+1}+1)}} \geq \frac{2}{t+1}, t = \frac{x}{y} \Leftrightarrow P(t) = (t-1)^2(t^{k+1} + t^k + \cdots + t + 1) \geq 0$; pentru $n \geq 2, k \geq 1, \boldsymbol{\frac{x^{2n+k}+y^{2n+k}}{x^n y^n (x^{k+1}+y^{k+1})}} \geq \frac{2}{x+y} \quad \Leftrightarrow$ $\boldsymbol{\frac{t^{2n+k}+1}{t^n(t^{k+1}+1)}} \geq \boldsymbol{\frac{2}{t+1}}, \boldsymbol{t} = \frac{x}{y} \Leftrightarrow$ P$(t) = t^{2n+k+1} + t^{2n+k} - 2t^{n+k+1} - 2t^n + t + 1 =$
$t^{k+1}(t^n-1)^2 + t^{2n+k} - 2t^n - t^{k+1} + t + 1 = t^{k+1}(t^n-1)^2 + t^k(t^n-1)^2 + 2t^{n+k} - 2t^n - t^{k+1} - t^k + t + 1 = (t^n-1)^2(t^{k+1}+t^k) + 2t^n(t^k-1) - t(t^k-1) - (t^k-1) = (t^n-1)^2(t^{k+1}+t^k) + (t^k-1)\big(t^n - 1 + t(t^{n-1}-1)\big) = (t^n-1)^2(t^{k+1}+t^k) + (t^k-1)(t-1)(t^{n-1}+t^{n-2}+\ldots t+1) + t(t^k-1)(t-1)(t^{n-2}+t^{n-3}+\ldots t+1) =$
$(t-1)^2(t^{n-1}+t^{n-2}+\ldots t+1)^2(t^{k+1}+t^k) + (t-1)^2(t^{k-1}+t^{k-2}+\cdots+t+1)(t^{n-1}+$

$t^{n-2}+\ldots t+1)+t(t-1)^2(t^{k-1}+t^{k-2}+\cdots+t+1)(t^{n-2}+t^{n-3}+\ldots t+1)=$
$(t-1)^2[(t^{n-1}+t^{n-2}+\cdots+t+1)^2(t^{k+1}+t^k)+(t^{k-1}+t^{k-2}+\cdots+t+1)(2t^{n-1}+$
$2t^{n-2}+\ldots+2t+1)]\geq 0;$

$$\sum\frac{x^{2n+k}+y^{2n+k}}{x^ny^n(x^{k+1}+y^{k+1})}\overset{Lemă}{\geq}\sum\frac{2}{x+y}\overset{MA-MH}{\geq}\frac{18}{\sum(x+y)}=\frac{18}{2\sum x}=\frac{9}{\sum x}.$$

3) $x,y,z,\mathrm{m}>0;k,n\in N^*\Rightarrow\frac{x^{n+k}+my^{n+k}}{x^n+my^n}+\frac{y^{n+k}+mz^{n+k}}{y^n+mz^n}+\frac{z^{n+k}+mx^{n+k}}{z^n+mx^n}\geq\frac{(\sum x)^k}{3^{k-1}}.$

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Soluţie. **Lemă.** $\forall x,y>0;k,n\in N^*,\mathrm{m}>0\Rightarrow\frac{x^{n+k}+my^{n+k}}{x^n+my^n}\geq\frac{x^k+my^k}{1+m};$
$\frac{x^{n+k}+my^{n+k}}{x^n+my^n}\geq\frac{x^k+my^k}{1+m}\Leftrightarrow\frac{t^{n+k}+m}{t^n+m}\geq\frac{t^k+m}{1+m},t=\frac{x}{y}>0\Leftrightarrow P(t)=(t^k-1)(t^n-1)=$
$(t-1)^2(t^{k-1}+t^{k-2}+\cdots+t+1)(t^{n-1}+t^{n-2}+\cdots+t+1)\geq 0;$

$$\sum\frac{x^{n+k}+my^{n+k}}{x^n+my^n}\overset{Lemă}{\geq}\sum\frac{x^k+my^k}{1+m}=\frac{\sum x^k+m\sum y^k}{m+1}=\frac{(m+1)\sum x^k}{m+1}=\sum x^k=\sum\frac{x^k}{1^{k-1}}\overset{Radon}{\geq}\frac{(\sum x)^k}{3^{k-1}}.$$

4) $\Delta ABC\Rightarrow\frac{m(r_a^{2n}+r_b^{2n})+2r_a^nr_b^n}{r_a^{2n+2}+r_b^{2n+2}}+\frac{m(r_b^{2n}+r_c^{2n})+2r_b^nr_c^n}{r_b^{2n+2}+r_c^{2n+2}}+\frac{m(r_c^{2n}+r_a^{2n})+2r_c^nr_a^n}{r_c^{2n+2}+r_a^{2n+2}}\leq\frac{R(m+1)}{6r^3}.$

Soluţie. **Lemă.** $\forall x,y,m>0;n\in N^*\Rightarrow\frac{m(x^{2n}+y^{2n})+2x^ny^n}{x^{2n+2}+y^{2n+2}}\leq\frac{m+1}{xy}.$

$\frac{m(x^{2n}+y^{2n})+2x^ny^n}{x^{2n+2}+y^{2n+2}}\leq\frac{m+1}{xy}\Leftrightarrow\frac{m(t^{2n}+1)+2t^n}{t^{2n+2}+1}\leq\frac{m+1}{t},t=\frac{x}{y}\Leftrightarrow\mathrm{P}(t)=(\mathrm{m}+1)t^{2n+2}-mt^{2n+1}-$
$2t^{n+1}-mt+m+1=mt^{2n+1}(t-1)+(t^{n+1}-1)^2-m(t-1)=m(t-1)(t^{2n+1}-1)+$
$(t-1)^2(t^n+t^{n-1}+\cdots+t+1)^2=(t-1)^2(m(t^{2n}+t^{2n-1}+\cdots+t+1)+$
$(t^n+t^{n-1}+\cdots+t+1)^2)\geq 0,\forall t,m>0;$

$$\sum\frac{m(r_a^{2n}+r_b^{2n})+2r_a^nr_b^n}{r_a^{2n+2}+r_b^{2n+2}}\overset{Lemă}{\leq}\sum\frac{m+1}{r_ar_b}=(m+1)\frac{\sum r_a}{r_ar_br_c}\overset{\sum r_a=4R+r}{=}\frac{(m+1)(4R+r)}{\frac{S^3}{(p-a)(p-b)(p-c)}}=\frac{(m+1)(4R+r)}{\frac{pS^3}{S^2}}=$$
$$\frac{(m+1)(4R+r)}{p^2r}\overset{Mitrinovic}{\leq}\frac{(m+1)(4R+r)}{27r^3}\overset{Euler}{\leq}\frac{(m+1)R}{6r^3}$$

5) $\forall x,y>0;n\in N\Rightarrow\frac{x^{n+1}+y^{n+1}}{x^n+y^n}\leq\sqrt{\frac{x^{n+2}+y^{n+2}}{x^n+y^n}}\leq\sqrt[3]{\frac{x^{n+3}+y^{n+3}}{x^n+y^n}}\leq\frac{x^{n+2}+y^{n+2}}{x^{n+1}+y^{n+1}}.$

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Soluţie. $\frac{x^{n+1}+y^{n+1}}{x^n+y^n}\leq\sqrt{\frac{x^{n+2}+y^{n+2}}{x^n+y^n}}\Leftrightarrow\left(\frac{t^{n+1}+1}{t^n+1}\right)^2\leq\frac{t^{n+2}+1}{t^n+1},t=\frac{x}{y}\Leftrightarrow P(t)=t^n(t-1)^2(t^n+$
$1)\geq 0;$

$\sqrt{\frac{x^{n+2}+y^{n+2}}{x^n+y^n}}\leq\sqrt[3]{\frac{x^{n+3}+y^{n+3}}{x^n+y^n}},t=\frac{x}{y}\Leftrightarrow\left(\frac{t^{n+2}+1}{t^n+1}\right)^3\leq\left(\frac{t^{n+3}+1}{t^n+1}\right)^2,t=\frac{x}{y}\Leftrightarrow P(t)=$
$t^n(t^n+1)^2(t^{n+6}-3t^{n\pm4}+2t^{n+3}+2t^3-3t^2+1)=t^n(t-1)^2(t^n+1)^2(t^{n\pm4}+2t^{n+3}+$
$2t+1)\geq 0;$

$\sqrt[3]{\frac{x^{n+3}+y^{n+3}}{x^n+y^n}} \leq \frac{x^{n+2}+y^{n+2}}{x^{n+1}+y^{n+1}} \Leftrightarrow \frac{t^{n+3}+1}{t^n+1} \leq \left(\frac{t^{n+2}+1}{t^{n+1}+1}\right)^3, t=\frac{x}{y} \Leftrightarrow P(t)=t^n(t^{2n+6}-3t^{2n+5}+3t^{2n+4}-t^{2n+3}-t^3+3t^2-3t+1)=t^n(t^{2n+3}(t-1)^3-(t-1)^3)=t^n(t-1)^3(t^{2n+3}-1)=t^n(t-1)^4(t^{2n+2}+t^{2n+1}+\cdots+t+1)\geq 0.$

6) $\forall x,y>0 \Rightarrow \frac{x+y}{2} \leq \sqrt{\frac{x^2+xy+y^2}{3}} \leq \sqrt{\frac{x^2+y^2}{2}} \leq \sqrt{x^2-xy+y^2} \leq \frac{x^{n+1}+y^{n+1}}{x^n+y^n}, n \in N, n \geq 2.$

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Soluţie. $\frac{x+y}{2} \leq \sqrt{\frac{x^2+xy+y^2}{3}} \Leftrightarrow \frac{t+1}{2} \leq \sqrt{\frac{t^2+t+1}{3}}, t=\frac{x}{y} \Leftrightarrow \ P(t)=(t-1)^2\geq 0;$

$\sqrt{\frac{x^2+xy+y^2}{3}} \leq \sqrt{\frac{x^2+y^2}{2}} \Leftrightarrow \sqrt{\frac{t^2+t+1}{3}} \leq \sqrt{\frac{t^2+1}{2}}, t=\frac{x}{y} \Leftrightarrow \ P(t)=(t-1)^2\geq 0;$

$\sqrt{\frac{x^2+y^2}{2}} \leq \sqrt{x^2-xy+y^2} \Leftrightarrow \sqrt{\frac{t^2+1}{2}} \leq \sqrt{t^2-t+1}, t=\frac{x}{y} \Leftrightarrow P(t)=(t-1)^2\geq 0;$

$\sqrt{x^2-xy+y^2} \leq \frac{x^{n+1}+y^{n+1}}{x^n+y^n} \Leftrightarrow t^2-t+1 \leq \left(\frac{t^{n+1}+1}{t^n+1}\right)^2 \ t=\frac{x}{y} \Leftrightarrow P(t)=t^{2n+1}-t^{2n}-2t^{n+2}+4t^{n+1}-2t^n-t^2+t= \ t(t-1)(t^{2n-1}-2t^{n-1}(t-1)-1)=t(t-1)^2(t^{2n-2}+t^{2n-3}+\cdots+t^{n+1}+t^{n-2}(t^2-t+1)+t^{n-3}+\cdots+t^2+t+1)\geq 0.$

7) $\forall x,y>0, n\in N, n\geq 2 \Rightarrow \frac{x^{n+2}+y^{n+2}}{x^n+y^n} \geq \sqrt{\frac{x^4+y^4}{2}}$

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Soluţie. $\frac{x^{n+2}+y^{n+2}}{x^n+y^n} \geq \sqrt{\frac{x^4+y^4}{2}} \Leftrightarrow \left(\frac{t^{n+2}+1}{t^n+1}\right)^2 \geq \frac{t^4+1}{2}, t=\frac{x}{y} \Leftrightarrow P(t)=t^{2n+4}-t^{2n}-2t^{n+4}+4t^{n+2}-2t^n-t^4+1=t^{2n}(t^4-1)-2t^n(t^2-1)^2-(t^4-1)=(t^2-1)\big(t^{2n}(t^2+1)-2t^n(t^2-1)-(t^2+1)\big)=(t-1)(t+1)\big((t^2+1)(t^{2n}-1)-2t^n(t-1)(t+1)\big)=(t-1)^2(t+1)(t^{2n+1}+t^{2n}+2t^{2n-1}+\cdots+2t^{n+2}+2t^{n-1}+2t^{n-2}+\cdots+2t^3+2t^2+t+1)\geq 0;$

8) $\Delta ABC \Rightarrow \frac{a^{n+5}+b^{n+5}}{ab(a^n+b^n)}+\frac{b^{n+5}+c^{n+5}}{bc(b^n+c^n)}+\frac{c^{n+5}+a^{n+5}}{ca(c^n+a^n)} \geq 144\sqrt{3}r^3.$

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Soluţie. **Lemă.** $\forall x. y>0 \Rightarrow \frac{x^{n+5}+y^{n+5}}{x^n+y^n} \geq \frac{2x^3y^3}{x+y};$

$\frac{x^{n+5}+y^{n+5}}{x^n+y^n} \geq \frac{2x^3y^3}{x+y} \Leftrightarrow \frac{t^{n+5}+1}{t^n+1} \geq \frac{2t^3}{t+1}, t=\frac{x}{y}>0 \Leftrightarrow P(t)=t^{n+6}+t^{n+5}-2t^{n+3}-2t^3+t+1=t^{n+3}(t^3+t^2-2)-(2t^3-t-1)= \ t^{n+3}(t-1)(t^2+2t+2)-(t-1)(2t^2+2t+1)=(t-1)(t^{n+5}+2t^{n+4}+2t^{n+3}-2t^2-2t-1)=(t-1)\big(t^{n+5}-1+2t(t+1)(t^{n+2}-1)\big)=(t-1)^2\big(t^{n+4}+t^{n+3}+\cdots+t+1+2t(t+1)(t^{n+1}+t^n+\cdots+t+1)\big)\geq 0;$

$\sum \frac{a^{n+5}+b^{n+5}}{ab(a^n+b^n)} \overset{Lemă}{\geq} \sum \frac{2a^3b^3}{ab(a+b)} = 2\sum \frac{(ab)^2}{a+b} \overset{Bergström}{\geq} \frac{2(\sum ab)^2}{\sum(a+b)} \overset{\sum ab \geq 18Rr}{\geq} \frac{2(18Rr)^2}{2\sum a} = \frac{324R^2r^2}{p} \overset{Mitrinovic}{\geq}$ $\frac{648R^2r^2}{3\sqrt{3}R} = 72\sqrt{3}Rr^2 \overset{Euler}{\geq} 144\sqrt{3}r^3$.

9) $\Delta ABC, n \in N^*, m > 0 \Rightarrow \frac{a^{n+1}b^{n+1}}{(a^{n+2}-b^{n+2})^2+ma^3b^3(a^{n-1}+b^{n-1})^2} + \frac{b^{n+1}c^{n+1}}{(b^{n+2}-c^{n+2})^2+mb^3c^3(b^{n-1}+c^{n-1})^2} + \frac{c^{n+1}a^{n+1}}{(c^{n+2}-a^{n+2})^2+mc^3a^3(c^{n-1}+a^{n-1})^2} \leq \frac{1}{16mr^2}$.

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Soluţie.**Lemă.** $n \in N^*, m > 0; a, b > 0 \Rightarrow (a^{n+2}-b^{n+2})^2 + ma^3b^3(a^{n-1}+b^{n-1})^2 \geq 4ma^{n+2}b^{n+2} \Leftrightarrow (t^{n+2}-1)^2 + mt^3(t^{n-1}+1)^2 \geq 4mt^{n+2}, t = \frac{a}{b} > 0 \Leftrightarrow (t^{n+2}-1)^2 + mt^3(t^{n-1}-1)^2 = (t-1)^2((t^{n+1}+t^n+\cdots+t+1)^2 + mt^3(t^{n-2}+t^{n-3}+\cdots+t+1)^2) \geq 0$;

$\sum \frac{a^{n+1}b^{n+1}}{(a^{n+2}-b^{n+2})^2+ma^3b^3(a^{n-1}+b^{n-1})^2} \overset{Lemă.}{\leq} \sum \frac{a^{n+1}b^{n+1}}{4ma^{n+2}b^{n+2}} = \sum \frac{1}{4mab} = \frac{\sum a}{4mabc} = \frac{2p}{16mpRr} = \frac{1}{8mRr} \overset{Euler}{\leq} \frac{1}{16mr^2}$.

10) $p \in N, n \in N^* \Rightarrow$ şirul $a_n = \frac{1^{p-1}+1}{1^{p+2}+1} + \frac{2^{p-1}+1}{2^{p+2}+1} + \cdots + \frac{n^{p-1}+1}{n^{p+2}+1}$ este convergent.

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Soluţie. **Lemă.** $p \in N, t > 0 \Rightarrow \frac{t^{p-1}+1}{t^{p+2}+1} \leq \frac{2}{t(t+1)} \Leftrightarrow P(t) = 2t^{p+2} - t^{p+1} - t^p - t^2 - t + 2 = 2t^p(t-1)^2 + 3t^{p+1} - 3t^p - t^2 - t + 2 = 2t^p(t-1)^2 + 3t^p(t-1) - (t-1)(t+2) = 2t^p(t-1)^2 + (t-1)(t^p - t + 2t^p - 2) = (t-1)^2(2t^p + 3t^{p-1} + 3t^{p-2} + \cdots + 3t + 2) \geq 0$;

$a_n = \sum_{k=1}^{n} \frac{k^{p-1}+1}{k^{p+2}+1} \overset{Lemă}{\leq} \sum_{k=1}^{n} \frac{2}{k(k+1)} = 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$.

$a_{n+1} - a_n = \frac{(n+1)^{p-1}+1}{(n+1)^{p+2}+1} > 0 \Rightarrow (a_n)_{n\geq 1}$ strict crescător $\Rightarrow a_n \geq a_1 = 1 \Rightarrow 1 \leq a_n \leq \frac{2n}{n+1} < 2 \Rightarrow (a_n)_{n\geq 1}$ mărginit $\Rightarrow (a_n)_{n\geq 1}$ convergent.