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PUTEȚI TRIMITE ARTICOLE ÎN FORMAT WORD PE REVISTA.MATEINFO@YAHOO.COM

1. Applications of J. Radon' s inequality in triangle

by D.M. Bătineţu-Giurgiu, and Neculai Stanciu

In this paper we present some new inequalities in a triangle, which follows from Radon's inequality.

1.1.) In any triangle ABC holds:

$$\frac{\sin^6 A}{\sin^2 B}+\frac{\sin^6 B}{\sin^2 C}+\frac{\sin^6 C}{\sin^2 A}\ge\frac{\left(p^2-4Rr-r^2\right)^3}{8p^2R^4};$$

1.2.) If $m\in R_+$, then in any triangle ABC holds:

$$\sin^{2m}\frac{A}{2}\sin^2 A+\sin^{2m}\frac{B}{2}\sin^2 B+\sin^{2m}\frac{C}{2}\sin^2 C\ge\frac{1}{2}\cdot\left(\frac{2R-r}{R}\right)^{m+1}\cdot\frac{p^{2m}}{(p^2+(4R+r)^2)^m};$$

1.3.) In all triangles ABC holds:

$$\cos^2\frac{A}{2}\sin^{2m}A+\cos^2\frac{B}{2}\sin^{2m}B+\cos^2\frac{C}{2}\sin^{2m}C\ge\frac{1}{2}\cdot\left(\frac{4R+r}{R}\right)^{m+1}\cdot\frac{r^{2m}}{(p^2+r^2-8Rr)^m};$$

1.4.) In all triangles ABC holds:

$$\sin^{4m+2}\frac{A}{2}+\sin^{4m+2}\frac{B}{2}+\sin^{4m+2}\frac{C}{2}\ge\left(\frac{2R-r}{2R}\right)^{m+1}\cdot\frac{r^{2m}}{(p^2+r^2-8Rr)^m};$$

1.5.) In all triangles ABC holds:

$$\cos^{4m+2}\frac{A}{2}+\cos^{4m+2}\frac{B}{2}+\cos^{4m+2}\frac{C}{2}\ge\left(\frac{4R+r}{2R}\right)^{m+1}\cdot\frac{p^{2m}}{\left((p^2+(4R+r)^2\right)^m}.$$

Proof of 1.1.) We have:

$$V=\frac{\sin^6 A}{\sin^2 B}+\frac{\sin^6 B}{\sin^2 C}+\frac{\sin^6 C}{\sin^2 A}=\frac{\left(\sin^2 A\right)^3}{\sin^2 B}+\frac{\left(\sin^2 B\right)^3}{\sin^2 C}+\frac{\left(\sin^2 C\right)^3}{\sin^2 A},$$

and by J.Radon's inequality we deduce that:

$$V \geq \frac{\left(\sin^2 A+\sin^2 B+\sin^2 C\right)^3}{\left(\sin A+\sin B+\sin C\right)^2}.$$

Since,

$$\sin^2 A+\sin^2 B+\sin^2 C=\frac{p^2-4Rr-r^2}{2R^2}, \text{ and}$$

$$\sin A+\sin B+\sin C=\frac{p}{R},$$

we obtain the conclusion.

Proof of 1.2.) We have:

$$W=\sum \sin^{2m} A \sin^2 \frac{A}{2}=2^{2m}\sum \sin^{2(m+1)}\frac{A}{2}\cos^{2m}\frac{A}{2}=2^{2m}\cdot\sum\frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(\dfrac{1}{\cos^2\frac{A}{2}}\right)^m}.$$

Using, J.Radon's inequality we obtain that:

$$W \geq 2^{2m}\cdot\frac{\left(\sum \sin^2\frac{A}{2}\right)^{m+1}}{\left(\sum\dfrac{1}{\cos^2\frac{A}{2}}\right)^m}.$$

Because,

$$\sum \sin^2\frac{A}{2}=\frac{2R-r}{2R} \text{ şi } \sum\frac{1}{\cos^2\frac{A}{2}}=\frac{(4R+r)^2+p^2}{p^2},$$

we are done.

Proof of 1.3.) We have that:

$$V=\sum\sin^{2m}A\cos^2\frac{A}{2}=2^{2m}\sum\sin^{2m}\frac{A}{2}\cos^{2(m+1)}\frac{A}{2}=2^{2m}\cdot\sum\frac{\left(\cos^2\frac{A}{2}\right)^{m+1}}{\left(\dfrac{1}{\sin^2\frac{A}{2}}\right)^m}.$$

Applying J.Radon's inequality we deduce that:

$$V\geq 2^{2m}\cdot\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left(\sum\dfrac{1}{\sin^2\frac{A}{2}}\right)^m}.$$

Using,

$$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R}, \text{ and}$$

$$\sum\frac{1}{\sin^2\frac{A}{2}}=\frac{p^2+r^2-8Rr}{r^2},$$

follows the conclusion.

Proof of 1.4.) We have:

$$W=\sum\sin^{4m+2}\frac{A}{2}=\sum\left(\sin^2\frac{A}{2}\right)^{2m+1}=\sum\frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(\dfrac{1}{\sin^2\frac{A}{2}}\right)^m}.$$

By J.Radon' s inequality, we deduce that:

$$W\geq\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left(\sum\dfrac{1}{\sin^2\frac{A}{2}}\right)^m}.$$

Since,

$$\sum \sin^2\frac{A}{2}=\frac{2R-r}{2R}\text{, and}$$

$$\sum \frac{1}{\sin^2\frac{A}{2}}=\frac{p^2+r^2-8Rr}{r^2},$$

we are done.

Proof of 1.5.) We have that:

$$U=\sum \cos^{4m+2}\frac{A}{2}=\sum\left(\cos^2\frac{A}{2}\right)^{2m+1}=\sum\frac{\left(\cos^2\frac{A}{2}\right)^{m+1}}{\left(\frac{1}{\cos^2\frac{A}{2}}\right)^m}.$$

From the inequality of J.Radon, we obtain that:

$$U\geq\frac{\left(\sum\cos^2\frac{A}{2}\right)^{m+1}}{\left(\sum\frac{1}{\cos^2\frac{A}{2}}\right)^m}.$$

It is well-known that:

$$\sum \cos^2\frac{A}{2}=\frac{4R+r}{2R}\text{, and}$$

$$\sum \frac{1}{\cos^2\frac{A}{2}}=\frac{(4R+r)^2+p^2}{p^2}.$$

From above we easily obtain the desired result.

2.1.) If $x,y,z\in R_+^*$, $m\in R_+$, then in all triangles ABC holds:

$$\frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2})^m}+\frac{\sin^{2m+2}\frac{B}{2}}{(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}+z\cos^2\frac{A}{2})^m}+$$
$$+\frac{\sin^{2m+2}\frac{C}{2}}{(x\sin^2\frac{C}{2}+y\sin^2\frac{A}{2}+z\cos^2\frac{B}{2})^m}\ge\frac{(2R-r)^{m+1}}{2R(2(x+y+2z)R+(2z-x-y)r)^m};$$

2.2.) If $x,y,z\in R_+^*$, $m\in R_+$, then in any triangle ABC holds:

$$\frac{\sin^{2m+2}\frac{A}{2}}{(x\cos^2\frac{A}{2}+y\cos^2\frac{B}{2}+z\sin^2\frac{C}{2})^m}+\frac{\sin^{2m+2}\frac{B}{2}}{(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2}+z\sin^2\frac{A}{2})^m}+$$
$$+\frac{\sin^{2m+2}\frac{C}{2}}{(x\cos^2\frac{C}{2}+y\cos^2\frac{A}{2}+z\sin^2\frac{B}{2})^m}\ge\frac{(2R-r)^{m+1}}{2R(2(2x+2y+z)R+(x+y-z)r)^m};$$

2.3.) If $x,y\in R_+^*$, $m\in R_+$, then in all triangles ABC holds:

$$\frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}+\frac{\sin^{2m+2}\frac{B}{2}}{(x\sin^2\frac{C}{2}+y\cos^2\frac{A}{2})^m}+\frac{\sin^{2m+2}\frac{C}{2}}{(x\sin^2\frac{A}{2}+y\cos^2\frac{B}{2})^m}\ge$$
$$\ge\frac{(2R-r)^{m+1}}{2R(2(x+2y)R+(y-x)r)^m};$$

2.4.) If $x,y\in R_+^*$, $m\in R_+$, then in any triangle ABC holds:

$$\frac{\sin^{2m+2}\frac{A}{2}}{(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}+\frac{\sin^{2m+2}\frac{B}{2}}{(x\cos^2\frac{C}{2}+y\cos^2\frac{A}{2})^m}+\frac{\sin^{2m+2}\frac{C}{2}}{(x\cos^2\frac{A}{2}+y\cos^2\frac{B}{2})^m}\ge$$
$$\ge\frac{(2R-r)^{m+1}}{2R(x+y)^m(4R+r)^m};$$

2.5.) If $x,y\in R_+^*$, $m\in R_+$, then in all triangles ABC holds:

$$\frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}+\frac{\sin^{2m+2}\frac{B}{2}}{(x\sin^2\frac{C}{2}+y\cos^2\frac{A}{2})^m}+\frac{\sin^{2m+2}\frac{C}{2}}{(x\sin^2\frac{A}{2}+y\cos^2\frac{B}{2})^m}\geq ;$$

$$\geq\frac{(2R-r)^{m+1}}{2R(2(x+2y)R+(y-x)r)^m}$$

2.6.) If $x,y\in R_+^*$, $m\in R_+$, then in any triangle ABC is true the inequality:

$$\frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2})^m}+\frac{\sin^{2m+2}\frac{B}{2}}{(x\sin^2\frac{C}{2}+y\sin^2\frac{A}{2})^m}+\frac{\sin^{2m+2}\frac{C}{2}}{(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2})^m}\geq\frac{2R-r}{2R(x+y)^m}.$$

Proof of 2.1.) By J.Radon's inequality we have that:

$$\sum\frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2})^m}=\sum\frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}\right)^m}\geq$$

$$\geq\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left(\sum\left(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}\right)\right)^m}=\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left((x+y)\sum\sin^2\frac{A}{2}+z\sum\cos^2\frac{A}{2}\right)^m}$$

Since,

$$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}\text{, and}$$

$$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

we obtain the conclusion.

Proof of 2.2.) From J.Radon' s inequality we have that:

$$\sum\frac{\sin^{2m+2}\frac{A}{2}}{(x\cos^2\frac{A}{2}+y\cos^2\frac{B}{2}+z\sin^2\frac{C}{2})^m}=\sum\frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\cos^2\frac{A}{2}+y\cos^2\frac{B}{2}+z\sin^2\frac{C}{2}\right)^m}\geq$$

$$\geq\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left(\sum\left(x\cos^2\frac{A}{2}+y\cos^2\frac{B}{2}+z\sin^2\frac{C}{2}\right)\right)^m}=\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left((x+y)\sum\cos^2\frac{A}{2}+z\sum\sin^2\frac{A}{2}\right)^m}$$

Since,

$$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}\text{, and}$$

$$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

we obtain the conclusion.

Proof of 2.3.) Applying J.Radon's inequality we deduce that:

$$\sum\frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}=\sum\frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)^m}\geq$$

$$\geq\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left(\sum\left(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)\right)^m}=\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sum\sin^2\frac{A}{2}+y\sum\cos^2\frac{A}{2}\right)^m}$$

Because,

$$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}\text{, and}$$

$$\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

yields the conclusion.

Proof of 2.4.) By J.Radon's inequality we obtain that:

$$\sum \frac{\sin^{2m+2}\frac{A}{2}}{(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}=\sum \frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\cos^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)^m}\geq$$

$$\geq \frac{\left(\sum \sin^2\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum \cos^2\frac{A}{2}\right)^m}$$

By the formulas,

$$\sum \sin^2\frac{A}{2}=\frac{2R-r}{2R}, \text{ and}$$

$$\sum \cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

we deduce the given inequality.

Proof of 2.5.) From the inequality of J.Radon we have:

$$\sum \frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2})^m}=\sum \frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\right)^m}\geq$$

$$\geq \frac{\left(\sum \sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sum \sin^2\frac{A}{2}+y\sum \cos^2\frac{A}{2}\right)^m}$$

Using,

$$\sum \sin^2\frac{A}{2}=\frac{2R-r}{2R}, \text{ and}$$

$$\sum \cos^2\frac{A}{2}=\frac{4R+r}{2R},$$

follows the result.

Proof of 2.6.) By J.Radon's inequality we obtain that:

$$\sum \frac{\sin^{2m+2}\frac{A}{2}}{(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2})^m}=\sum \frac{\left(\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}\right)^m}\geq$$

$$\geq \frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{\left(x\sum\sin^2\frac{A}{2}+y\sum\sin^2\frac{A}{2}\right)^m}=\frac{\left(\sum\sin^2\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum\sin^2\frac{A}{2}\right)^m}=\frac{\sum\sin^2\frac{A}{2}}{(x+y)^m}$$

and taking account that,

$$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R},$$

we get the inequality to prove.

This completes the proof.

2. Soluții trigonometrice pentru o problemă de baraj, juniori 2022

Nela Ciceu, Bacău

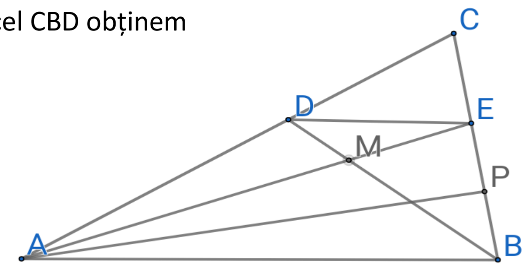
La primul baraj pentru juniori din 2022 a fost propusă următoarea problemă:

Se consideră triunghiul ABC cu <A = 30° și <B = 80°. Pe laturile AC și BC se consideră punctele D respectiv E astfel încât <ABD = <DBC și $DE||AB$. Determinați măsura unghiului EAC.

În afara celor două soluții oficiale, în [1] sunt prezentate opt soluții, șase dintre ele presupun construcția unor triunghiuri echilaterale. În această scurtă notă dorim să prezentăm patru soluții trigonometrice și să propunem alte două probleme similare.

Avem <C = 70°, <ABD = <DBC = 40°; deducem că triunghiul BDC este isoscel cu BC = BD.

Soluția 1. Presupunem că raza cercului circumscris triunghiului ABC este $R = \frac{1}{2}$. Atunci a = sin30°, b = sin80° = cos10°, c = sin70° = cos20°. Din triunghiul isoscel CBD obținem



CD = 2asin20° = sin20°. Cum $DE||AB$, rezultă că

$$\frac{CE}{CB} = \frac{CD}{CA} \Leftrightarrow \frac{CE}{a} = \frac{2a\sin 20°}{b},$$

de unde

$$CE = \frac{2a^2\sin 20°}{\mathrm{b}} = \frac{2\sin 20°}{4\cos 10°} = \sin 10°\ .$$

Aplicând teorema cosinusului în triunghiul AEC avem

$$AE^2 = \cos^2 10° + \sin^2 10° - 2\sin 10°\cos 10°\cos 70° = 1 - \sin^2 20° = \cos^2 20°\ ,$$

deci AE = AB. Atunci <EAB = 20°, <EAC = 10°.

Soluția 2. Deoarece CD = 2BCsin20° = 2asin20° și $DE||AB$, avem $\frac{CE}{CB} = \frac{CD}{CA} \Leftrightarrow \frac{CE}{a} = \frac{2a\sin 20°}{b}$. Atunci putem scrie

$$\frac{CE}{EB} = \frac{2a\sin 20°}{b - 2a\sin 20°} = \frac{2\sin 30°\sin 20°}{\sin 80° - 2\sin 30°\sin 20°} = \frac{\sin 20°}{\sin 80° - \sin 20°} = \frac{\sin 20°}{2\sin 30°\cos 50°} = \frac{\sin 20°}{\sin 40°}$$
$$= \frac{\sin 10°\sin 80°}{\sin 20°\sin 70°} = \frac{b\sin 10°}{c\sin 20°}.$$

Considerăm punctul E' situat pe latura BC astfel încât <E'AC = 10°. Folosind rapoarte de arii deducem că

$$\frac{CE'}{E'B} = \frac{A_{ACE'}}{A_{ABE'}} = \frac{\mathrm{b}\sin 10°}{\mathrm{c}\sin 20°} = \frac{\mathrm{CE}}{\mathrm{EB}}.$$

Rezultă că punctele E și E' coincid, deci <EAC = 10°.

Soluția 3. Ca în soluția 1, presupunem că raza cercului circumscris triunghiului ABC este $R = \frac{1}{2}$. Fie P proiecția punctului A pe latura BC. Obținem imediat <PAB = 10°, <PAC= 20°, PB = sin70°sin10°, CE = sin10°, CP = sin80°sin20°. Atunci

$$PE = CP - CE = \sin 20°\cos 10° - \sin 10° = \sin 10°(2\cos^2 10° - 1) = \sin 10°\cos 20° = \mathrm{PC},$$

deci P este mijlocul segmentului BE; triunghiul ABP este isoscel și rezultă că <PAE = 10°, <CAE = 10°.

Observație: Din această soluție deducem că AP și AE sunt ceviene izogonale; deoarece înălțimea din A și diametrul ce trece prin A sunt izogonale, rezultă că centrul cercului circumscris triunghiului ABC aparține segmentului AE. Dreptele AE și AP sunt trisectoarele unghiului A.

Variantă de finalizare: Deoarece AP = sin70°cos10°, obținem

$$\frac{AC}{AP}=\frac{\cos 10°}{\cos 20°\cos 10°}=\frac{\sin 10°}{\sin 10°\cos 20°}=\frac{CE}{EP}.$$

Conform reciprocei teoremei bisectoarei, Rezultă că <EAC = <EAP = 10°.

Soluția 4. Fie M intersecția dreptelor AE și BD. Notăm x = < EAB = <AED; atunci <EAC = 30° - x, <AEB = 110° - x. Folosind rapoarte de arii și teorema sinusurilor, avem

$$\frac{ME}{MA}=\frac{EB\sin 40°}{AB\sin 40°}=\frac{\sin x}{\sin{(100°-x)}}$$

$$\frac{ME}{MA}=\frac{DE\sin 40°}{AD\sin 110°}=\frac{\sin 40°\sin{(30°-x)}}{\cos 20°\sin x}.$$

Obținem ecuația $\sin 40°\sin{(30°-x)}\cos{(x-10°)}=\cos 20°\sin^2 x$, care se scrie succesiv

$$2\sin 20°\sin(30°-x)\cos(x-10°)=\sin^2 x \Leftrightarrow 2\sin 20°\big(\sin 20°+\sin(40°-2x)\big)=1-\cos 2x$$

$$2\sin^2 20°+2\sin 20°\sin(40°-2x)=1-\cos 2x \Leftrightarrow 2\sin 20°\sin(40°-2x)=\cos 40°-\cos 2x$$

$$2\sin 20°\sin(40°-2x)=2\sin(20°+x)\sin(x-20°)$$

$$2\sin 20°\sin(x-20°)\cos(x-20°)+\sin(x+20°)\sin(x-20°)=0.$$

Pentru $0<x<30°$ avem $2\sin 20°\cos(x-20°)+\sin(x+20°)>0$; rezultă x = 20°, deci <EAC = 10°.

În încheiere, propunem următoarele două probleme similare:

1. *Se consideră triunghiul ABC cu <A = 80° și <B = 40°. Pe laturile AC și BC se consideră punctele D respectiv E astfel încât <ABD = <DBC și $DE||AB$. Determinați măsura unghiului EAC.* (**<EAC = 50°**)
2. *Se consideră triunghiul ABC cu <A = 54° și <B = 84°. Pe laturile AC și BC se consideră punctele D respectiv E astfel încât <ABD = <DBC și $DE||AB$. Determinați măsura unghiului EAC.* (**<EAC = 24°**)

Bibliografie:

[1] Adrian Bud, *O problemă propusă la barajul pentru juniori 2022*, G.M.-B nr. 10/2022

Art 5200

3. Generarea unor inegalităti în triunghi din inegalități algebrice

Generating triangle inequalities from algebraic inequalities (II)

Marin Chirciu [1]

Articolul prezintă inegalităti în triunghi obținute din inegalități algebrice, selectate din diverse publicații de specialitate.

Aplicația50.

In acute ΔABC

$$\sum \frac{\sec B+\sec C}{\sec^2 A} \geq 3.$$

George Apostolopoulos, Greece, Mathematical Inequalites, 11/2023

Soluție.

Lema.

If $x, y, z>0$ then

$$\sum \frac{y+z}{x^2} \geq 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Demonstrație.

$$\sum \frac{y+z}{x^2}=\sum x\left(\frac{1}{y^2}+\frac{1}{z^2}\right) \stackrel{AM-GM}{\geq} \sum x \cdot \frac{2}{yz}=2 \sum \frac{x}{yz}=\frac{2 \sum x^2}{xyz} \stackrel{SOS}{\geq} \frac{2 \sum yz}{xyz}=2 \sum \frac{1}{x}.$$

Egalitatea are loc dacă și numai dacă $x=y=z$.

Folosind **Lema** pentru $(x, y, z)=(\sec A, \sec B, \sec C)$ obținem:

$$\sum \frac{\sec B+\sec C}{\sec^2 A} \geq 2 \sum \frac{1}{\sec A}=2 \sum \cos A=2\left(1+\frac{r}{R}\right)=2+\frac{2r}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral .

Remarca.

[1] *Profesor, Colegiul Național „Zinca Golescu"Pitești*

In ΔABC

$$\sum \frac{\csc B + \csc C}{\csc^2 A} \geq \frac{2p}{R}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x, y, z) = (\csc A, \csc B, \csc C)$ obținem:

$$\sum \frac{\csc B + \csc C}{\csc^2 A} \geq 2\sum \frac{1}{\csc A} = 2\sum \sin A = 2 \cdot \frac{p}{R}.$$

Remarca.

In acute ΔABC

$$\sum \frac{\tan B + \tan C}{\tan^2 A} \geq 2\sqrt{3}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x, y, z) = (\tan A, \tan B, \tan C)$ obținem:

$$\sum \frac{\tan B + \tan C}{\tan^2 A} \overset{Lema}{\geq} 2\sum \frac{1}{\tan A} = 2\sum \cot A = 2 \cdot \frac{p^2 - r^2 - 4Rr}{2pr} \geq 2 \cdot \sqrt{3} = 2\sqrt{3}.$$

Remarca.

In acute ΔABC

$$\sum \frac{\cot B + \cot C}{\cot^2 A} \geq 6\sqrt{3}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x, y, z) = (\cot A, \cot B, \cot C)$ obținem:

$$\sum \frac{\cot B + \cot C}{\cot^2 A} \overset{Lema}{\geq} 2\sum \frac{1}{\cot A} = 2\sum \tan A = 2 \cdot \frac{2pr}{p^2 - (2R + r)^2} \geq 2 \cdot 3\sqrt{3} = 6\sqrt{3}.$$

Remarca.

In acute ΔABC

$$\sum \frac{\cos B + \cos C}{\cos^2 A} \geq 6.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x,y,z)=(\cos A,\cos B,\cos C)$ obținem:

$$\sum\frac{\cos B+\cos C}{\cos^2 A}\overset{Lema}{\geq}2\sum\frac{1}{\cos A}=\frac{p^2+r^2-4R^2}{p^2-(2R+r)^2}\geq 6.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\frac{\sin B+\sin C}{\sin^2 A}\geq 2\sqrt{3}.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x,y,z)=(\sin A,\sin B,\sin C)$ obținem:

$$\sum\frac{\sin B+\sin C}{\sin^2 A}\overset{Lema}{\geq}2\sum\frac{1}{\sin A}=\frac{p^2+r^2+4Rr}{2pr}\geq 2\sqrt{3}.$$

Aplicația51.

S.2635. In ΔABC

$$\sum\frac{(b^4+c^4)h_a}{b+c}\geq 8\sqrt{3}F^2.$$

D.M.Bătinețu-Giurgiu, Daniel Sitaru, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b>0$

$$\frac{a^4+b^4}{a+b}\geq\frac{(a+b)^3}{8}.$$

Demonstrație

$$\frac{a^4+b^4}{a+b}\overset{Holder}{\geq}\frac{\frac{(a+b)^4}{8}}{a+b}=\frac{(a+b)^3}{8}$$, cu egalitate pentru $b=c$.

$$LHS=\sum\frac{(b^4+c^4)h_a}{b+c}\overset{Lema}{\geq}\sum\frac{(b+c)^3h_a}{8}\overset{AM-GM}{\geq}\frac{1}{8}\cdot 3\sqrt[3]{\prod(b+c)^3h_a}=\frac{3}{8}\prod(b+c)\sqrt[3]{\prod h_a}\overset{(1)}{\geq}$$

$\overset{(1)}{\geq}\frac{3}{8}\cdot 4F(9R-2r)\cdot(3r)\geq 8\sqrt{3}F^2=RHS$ unde (1)rezultă din $\prod(b+c)\geq 4F(9R-2r)$ și $\prod h_a\geq(3r)^3$.

Remarca.

In ΔABC

$$\sum\frac{(b^{3n+1}+c^{3n+1})h_a}{b+c}\geq 9r(4FR)^n,n\in\mathbf{N}.$$

Marin Chirciu

Soluție

Lema

If $a,b>0$

$$\frac{a^{3n+1}+b^{3n+1}}{a+b}\geq\frac{(a+b)^{3n}}{2^{3n}}.$$

Demonstrație

$\frac{a^{3n+1}+b^{3n+1}}{a+b}\overset{Holder}{\geq}\frac{\frac{(a+b)^{3n+1}}{2^{3n}}}{a+b}=\frac{(b+c)^{3n}}{2^{3n}}$, cu egalitate pentru $b=c$.

Să trecem la rezolvarea problemei din enunț.

Folosind **Lema** obținem:

$$LHS=\sum\frac{b^{3n+1}+c^{3n+1}}{b+c}h_a\overset{Lema}{\geq}\sum\frac{(b+c)^{3n}}{2^{3n}}h_a\overset{AM-GM}{\geq}\frac{1}{2^{3n}}\cdot 3\sqrt[3]{\prod(b+c)^{3n}h_a}=$$

$$=\frac{3}{2^{3n}}\prod(b+c)^n\cdot\sqrt[3]{\prod h_a}\overset{(1)}{\geq}\frac{3}{2^{3n}}\cdot(32FR)^n\cdot(3r)=9r\left(\frac{32FR}{8}\right)^n=9r(4FR)^n=RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația52.

S.2626. In ΔABC then

$$\prod\left(\frac{1}{a^2}+2\right)\geq\frac{9}{R^2}.$$

D.M.Bătinețu-Giurgiu, Oana Simona Dascălu, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Demonstrație

Demonstrăm mai întâi inegalitatea:

If $a,b,t\geq 0$

$$\left(a^2+t\right)\left(b^2+t\right)\geq\frac{3}{4}t\left(t+(a+b)^2\right),(*).$$

Într-adevăr:

Avem $\left(a^2+t\right)\left(b^2+t\right)\geq\frac{3}{4}t\left(t+(a+b)^2\right)\Leftrightarrow 4\left(a^2+t\right)\left(b^2+t\right)\geq 3t\left(t+a^2+b^2+2ab\right)\Leftrightarrow$

$\Leftrightarrow t\left(a^2+b^2-2ab\right)+t^2-2abt+4a^2b^2\geq 0\Leftrightarrow t(a-b)^2+(t-2ab)^2\geq 0,$

cu egalitate pentru $a=b$ și $t=2ab$.

În continuare obținem:

$\left(a^2+t\right)\left(b^2+t\right)\left(c^2+t\right)\overset{(*)}{\geq}\frac{3}{4}t\left(t+(a+b)^2\right)\left(c^2+t\right)\overset{(1)}{\geq}\frac{3}{4}(a+b+c)^2t^2$, unde (1) $\Leftrightarrow$

$\Leftrightarrow\frac{3}{4}t\left(t+(a+b)^2\right)\left(c^2+t\right)\geq\frac{3}{4}(a+b+c)^2t^2\Leftrightarrow\left(t+(a+b)^2\right)\left(c^2+t\right)\geq(a+b+c)^2t\Leftrightarrow$

$\Leftrightarrow(t-a(b+c))^2\geq 0$, cu egalitate pentru $t=a(b+c)$.

Folosind **Lema** pentru $(a,b,c,t)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},2\right)$ obținem:

$$LHS=\prod\left(\frac{1}{a^2}+2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum\frac{1}{a}\right)^2 2^2\overset{Leuenberger}{\geq}\frac{3}{4}\left(\frac{\sqrt{3}}{R}\right)^2 2^2=\frac{9}{R^2}=RHS.$$

Am folosit mai sus inegalitatea lui Leuenberger $\sum\frac{1}{a}\geq\frac{\sqrt{3}}{R}$.

Remarca.

Problema se poate dezvolta.

If $\lambda>0$, then in ΔABC

$$\prod\left(\frac{1}{a^2}+\lambda\right)\geq\frac{9\lambda^2}{4R^2}.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\lambda\right)$obținem:

$$LHS=\prod\left(\frac{1}{a^2}+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum\frac{1}{a}\right)^2\lambda^2\overset{Leuenberger}{\geq}\frac{3}{4}\left(\frac{\sqrt{3}}{R}\right)^2\lambda^2=\frac{9\lambda^2}{4R^2}=RHS.$$

Am folosit mai sus inegalitatea lui Leuenberger $\sum\frac{1}{a}\geq\frac{\sqrt{3}}{R}$.

Aplicația53.

If $a,b,c>0$ then

$$\sum\frac{\sqrt{b+c}}{a}\geq\frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}.$$

Nasa Neverdie, Vietnam,THCS 11/2023

Soluție.

$$\sum\frac{\sqrt{b+c}}{a}\geq\frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}\Leftrightarrow\sum\frac{(b+c)\sqrt{(a+b)(a+c)}}{a}\geq4\sum a,$$

care rezultă din $\sqrt{(a+b)(a+c)}\overset{CBS}{\geq}a+\sqrt{bc}$.

Rămâne să arătăm că:

$$\sum\frac{(b+c)\left(a+\sqrt{bc}\right)}{a}\geq4\sum a\Leftrightarrow\sum(b+c)+\sum\frac{(b+c)\sqrt{bc}}{a}\geq4\sum a\Leftrightarrow$$

$\Leftrightarrow 2\sum a+\sum\frac{(b+c)\sqrt{bc}}{a}\geq4\sum a\Leftrightarrow\sum\frac{(b+c)\sqrt{bc}}{a}\geq2\sum a$, adevărată din $b+c\geq2\sqrt{bc}$.

Este suficient să arătăm că:

$\sum\frac{2\sqrt{bc}\cdot\sqrt{bc}}{a}\geq2\sum a\Leftrightarrow\sum\frac{bc}{a}\geq\sum a$, vezi

$\sum\frac{bc}{a}\geq\sum a\Leftrightarrow\sum b^2c^2\geq abc\sum a$,vezi $\sum x^2\geq\sum yz$, pentru $(x,y,z)=(bc,ca,ab)$.

Remarca.

Problema se poate dezvolta.

În ΔABC

$$\sum \frac{\sqrt{r_b + r_c}}{r_a} \geq 2\sqrt{\frac{3}{R}}.$$

Marin Chirciu

Lema.

If $a, b, c > 0$ then

$$\sum \frac{\sqrt{b+c}}{a} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}.$$

Soluție.

Se cunosc identitățile în triunghi $\sum r_a = 4R + r$ și $\prod(r_b + r_c) = 4Rp^2$.

Folosind **Lema** pentru $(x, y, z) = (r_a, r_b, r_c)$ obținem:

$$\sum \frac{\sqrt{r_b + r_c}}{r_a} \geq \frac{4\sum r_a}{\sqrt{\prod(r_b + r_c)}} \Leftrightarrow \sum \frac{\sqrt{r_b + r_c}}{r_a} \geq \frac{4(4R+r)}{\sqrt{4Rp^2}} \Leftrightarrow \sum \frac{\sqrt{r_b + r_c}}{r_a} \geq \frac{2(4R+r)}{p\sqrt{R}}.$$

Folosind inegalitatea lui Doucet $4R + r \geq p\sqrt{3}$ rezultă:

$$\sum \frac{\sqrt{r_b + r_c}}{r_a} \geq \frac{2(4R+r)}{p\sqrt{R}} \overset{Doucet}{\geq} \frac{2 \cdot p\sqrt{3}}{p\sqrt{R}} = \frac{2\sqrt{3}}{\sqrt{R}} = 2\sqrt{\frac{3}{R}} \Rightarrow \sum \frac{\sqrt{r_b + r_c}}{r_a} \geq 2\sqrt{\frac{3}{R}}.$$

Aplicația54.

S.2604. In $a, b, c, x, y, t > 0$ then

$$\prod\left(\frac{a^2}{(bx+cy)^2} + t^2\right) \geq \frac{27t^4}{4(x+y)^2}.$$

D.M.Bătinețu-Giurgiu, Gheorghe Boroica, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a, b, c, t \geq 0$

$$\prod(a^2 + t) \geq \frac{3}{4}(a+b+c)^2 t^2.$$

Hojoo Lee Inequality

Folosind **Lema** pentru $(x,y,z,t)=\left(\frac{a}{bx+cy},\frac{b}{cx+ay},\frac{c}{ax+by},t^2\right)$ obținem:

$$LHS=\prod\left(\frac{a^2}{(bx+cy)^2}+t^2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum\frac{a}{bx+cy}\right)^2\left(t^2\right)^2\overset{Nesbitt}{\geq}\frac{3}{4}\left(\frac{3}{b+c}\right)^2 t^4=\frac{27t^4}{4(x+y)^2}=RHS.$$

Am folosit mai sus inegalitatea lui Nesbitt extinsă: $\sum\frac{a}{bx+cy}\geq\frac{3}{x+y}$.

Aplicația55.

S.2572. In ΔABC then

$$\prod\left(a^2+t\right)\geq 9\sqrt{3}t^2F.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}(a+b+c)^2 t^2.$$

Hojoo Lee Inequality

Folosind **Lema** obținem:

$$LHS=\prod\left(a^2+t\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum a\right)^2 t^2=\frac{3}{4}(2p)^2 t^2=3p^2t^2\overset{Hadwiger}{\geq}3\cdot 3\sqrt{3}F\cdot t^2=RHS.$$

Am folosit mai sus inegalitatea lui Hadwiger $p^2\geq 3\sqrt{3}F$.

Aplicația56.

S.2595. If $x,y>0$, $xp>y\max\{a,b,c\}$ then :

$$\sum\frac{a^3}{xp-ya}\geq\frac{8\sqrt{3}}{3x-2y}F.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție.

$$LHS=\sum\frac{a^3}{xp-ya}\overset{Holder}{\geq}\frac{\left(\sum a\right)^3}{3\sum(xp-ya)}=\frac{(2p)^3}{3(3x-2y)p}=\frac{8p^3}{3(3x-2y)p}=\frac{8p^2}{3(3x-2y)}\overset{Hadwiger}{\geq}$$

$$\overset{Hadwiger}{\geq}\frac{8\cdot 3\sqrt{3}F}{3(3x-2y)}=\frac{8\sqrt{3}}{3x-2y}F=RHS\ .$$

Am folosit mai sus inegalitatea Hadwiger: $p^2\geq 3F\sqrt{3}$.

Remarca.

If $x,y>0$, $xp>y\max\{a,b,c\}$ and $n\in\mathbf{N}$ then :

$$\sum\frac{a^{2n+1}}{xp-ya}\geq\frac{6}{3x-2y}\left(\frac{4F}{\sqrt{3}}\right)^n .$$

Marin Chirciu

Soluție.

Pentru $n=0$ se folosește inegalitatea lui Nesbitt.

Pentru $n\in\mathbf{N}^*$ se folosește inegalitatea lui Holder.

$$LHS=\sum\frac{a^{2n+1}}{xp-ya}\overset{Holder}{\geq}\frac{\left(\sum a\right)^{2n+1}}{3^{2n-1}\sum(xp-ya)}=\frac{(2p)^{2n+1}}{3^{2n-1}(3x-2y)p}=\frac{(2p)^{2n}\cdot 2p}{3^{2n-1}(3x-2y)p}=\frac{(2p)^{2n}\cdot 2}{3^{2n-1}(3x-2y)}=$$

$$=\frac{\left(4p^2\right)^n\cdot 2}{3^{2n-1}(3x-2y)}\overset{Hadwiger}{\geq}\frac{\left(4\cdot 3\sqrt{3}F\right)^n\cdot 2\cdot 3}{9^n(3x-2y)}=\frac{6}{3x-2y}\left(\frac{4\cdot 3\sqrt{3}F}{9}\right)^n=\frac{6}{3x-2y}\left(\frac{4\sqrt{3}F}{3}\right)^n=RHS\ .$$

Aplicația57.

In ΔABC

$$\frac{4a^2+b^2+c^2}{2a^2+bc}\leq\frac{R}{r}.$$

Adil Abdullayev, Azerbaijan,RMM, 11/2023

Soluție.

Lema.

If $a,b,c>0$ then

$$\frac{4a^2+b^2+c^2}{2a^2+bc}\leq\frac{b^2+c^2}{bc}.$$

Demonstrație.

$\dfrac{4a^2+b^2+c^2}{2a^2+bc}\leq\dfrac{b^2+c^2}{bc}\Leftrightarrow a^2(b-c)^2\geq 0$, cu egalitate pentru $b=c$.

Folosind **Lema** obținem:

$$LHS=\frac{4a^2+b^2+c^2}{2a^2+bc}\overset{Lema}{\leq}\frac{b^2+c^2}{bc}=\frac{b}{c}+\frac{c}{b}\overset{Bandila}{\leq}\frac{R}{r}=RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este isoscel cu $b=c$.

Remarca.

În aceeași clasă de probleme.

In ΔABC

$$\frac{4m_a^2+m_b^2+m_c^2}{2m_a^2+m_bm_c}\leq\frac{R}{r}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\frac{4x^2+y^2+z^2}{2x^2+yz}\leq\frac{y^2+z^2}{yz}.$$

Folosind **Lema** pentru $(x,y,z)=(m_a,m_b,m_c)$ obținem:

$$LHS=\frac{4m_a^2+m_b^2+m_c^2}{2m_a^2+m_bm_c}\overset{Lema}{\leq}\frac{m_b^2+m_c^2}{m_bm_c}=\frac{m_b}{m_c}+\frac{m_c}{m_b}\overset{Abdullayev}{\leq}\frac{R}{r}=RHS.$$

Am folosit mai sus: $\frac{m_a}{m_b}+\frac{m_b}{m_a}\leq\frac{R}{r}$ (Adil Abdullayev, Azerbaijan).

Remarca.

În aceeași clasă de probleme.

In ΔABC

$$\frac{4h_a^2+h_b^2+h_c^2}{2h_a^2+h_bh_c}\leq\frac{R}{r}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\frac{4x^2+y^2+z^2}{2x^2+yz}\leq\frac{y^2+z^2}{yz}.$$

Folosind **Lema** pentru $(x,y,z)=(h_a,h_b,h_c)$ obținem:

$$LHS=\frac{4h_a^2+h_b^2+h_c^2}{2h_a^2+h_bh_c}\overset{Lema}{\leq}\frac{h_b^2+h_c^2}{h_bh_c}=\frac{h_b}{h_c}+\frac{h_c}{h_b}=\frac{\frac{2S}{b}}{\frac{2S}{c}}+\frac{\frac{2S}{c}}{\frac{2S}{b}}=\frac{c}{b}+\frac{b}{c}\overset{Bandila}{\leq}\frac{R}{r}=RHS.$$

Am folosit mai sus: $\frac{b}{c}+\frac{c}{b}\leq\frac{R}{r}$ (Băndilă, 1985).

Remarca.

In ΔABC

$$\frac{4r_a^2+r_b^2+r_c^2}{2r_a^2+r_br_c}\leq\frac{R}{r}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\frac{4x^2+y^2+z^2}{2x^2+yz}\leq\frac{y^2+z^2}{yz}.$$

Folosind **Lema** pentru $(x,y,z)=(r_a,r_b,r_c)$ obținem:

$$LHS=\frac{4r_a^2+r_b^2+r_c^2}{2r_a^2+r_br_c}\overset{Lema}{\leq}\frac{r_b^2+r_c^2}{r_br_c}=\frac{r_b}{r_c}+\frac{r_c}{r_b}=\frac{\frac{S}{p-b}}{\frac{S}{p-c}}+\frac{\frac{S}{p-c}}{\frac{S}{p-b}}=\frac{p-c}{p-b}+\frac{p-b}{p-c}\overset{SOS}{\leq}$$

$$\overset{SOS}{\leq}\frac{c}{b}+\frac{b}{c}\overset{Bandila}{\leq}\frac{R}{r}=RHS,$$

unde(SOS) $\Leftrightarrow\frac{p-c}{p-b}+\frac{p-b}{p-c}\leq\frac{c}{b}+\frac{b}{c}\Leftrightarrow(b-c)^2\geq 0$, cu egalitate pentru $b=c$.

Aplicația58.

S.2568. In ΔABC

$$\sum\frac{m_a^3}{Rm_a+rm_c}\geq\frac{2\sqrt{3}}{R}F.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție.

$$LHS=\sum\frac{m_a^3}{Rm_b+rm_c}=\sum\frac{m_a^4}{Rm_am_b+rm_am_c}\overset{CS}{\geq}\frac{\left(\sum m_a^2\right)^2}{\sum\left(Rm_am_b+rm_am_c\right)}=\frac{\left(\sum m_a^2\right)^2}{(R+r)\sum m_am_b}\overset{SOS}{\geq}$$
$$\overset{SOS}{\geq}\frac{\sum m_a^2}{R+r}\overset{Hadwiger}{\geq}\frac{3\sqrt{3}F}{R+r}\overset{Euler}{\geq}\frac{2\sqrt{3}}{R}F=RHS,$$

Am folosit mai sus inegalitatea Hadwiger: $\sum m_a^2\geq 3\sqrt{3}F$.

Remarca.

In ΔABC

$$\sum\frac{m_a^{2n+1}}{Rm_a+rm_c}\geq\frac{2}{R}\left(\sqrt{3}F\right)^n, n\in\mathbf{N}.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{m_a^{2n+1}}{Rm_b+rm_c}=\sum\frac{m_a^{2n+2}}{Rm_am_b+rm_am_c}\overset{Holder}{\geq}\frac{\left(\sum m_a^2\right)^{n+1}}{3^{n-1}\sum\left(Rm_am_b+rm_am_c\right)}=$$
$$=\frac{\left(\sum m_a^2\right)^{n+1}}{3^{n-1}(R+r)\sum m_am_b}\overset{SOS}{\geq}\frac{\left(\sum m_a^2\right)^n}{3^{n-1}(R+r)}\overset{(1)}{\geq}\frac{\left(3\sqrt{3}F\right)^n}{3^{n-1}(R+r)}=\frac{3}{R+r}\left(\sqrt{3}F\right)^n\overset{Euler}{\geq}\frac{2}{R}\left(\sqrt{3}F\right)^n=RHS,$$

unde (1) $\Leftrightarrow\sum m_a^2\geq 3\sqrt{3}F$,(Hadwiger).

Aplicația59.

S.2576. In ΔABC then

$$\sum\left(\frac{1}{r_a^2}+p\right)\geq\frac{81}{4}.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Demonstrație

Folosind **Lema** pentru $(a,b,c,t)=\left(\frac{1}{r_a},\frac{1}{r_b},\frac{1}{r_c},p\right)$ obținem:

$$LHS=\sum\left(\frac{1}{r_a^2}+p\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum\frac{1}{r_a}\right)^2 p^2=\frac{3}{4}\left(\frac{1}{r}\right)^2 p^2\overset{Mitrinovic}{\geq}\frac{3}{4}\frac{1}{r^2}27r^2=\frac{81}{4}=RHS.$$

Remarca.

In ΔABC then

$$\sum\left(\frac{1}{r_a^2}+\lambda p\right)\geq\frac{81\lambda^2}{4},\lambda>0.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=\left(\frac{1}{r_a},\frac{1}{r_b},\frac{1}{r_c},\lambda p\right)$ obținem:

$$LHS=\sum\left(\frac{1}{r_a^2}+\lambda p\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum\frac{1}{r_a}\right)^2\lambda^2p^2=\frac{3}{4}\left(\frac{1}{r}\right)^2\lambda^2p^2\overset{Mitrinovic}{\geq}\frac{3}{4}\frac{1}{r^2}27\lambda^2r^2=\frac{81\lambda^2}{4}=RHS.$$

Aplicația60.

S.2581. If $x,y>0$, $M\in Int(\Delta ABC), d_a=d(M,BC), d_b=d(M,CA), d_c=d(M,AB)$ then :

$$\sum\frac{a^3}{xh_a+yd_a}\geq\frac{24}{3x+y}F.$$

D.M.Bătinețu-Giurgiu, Dan Nănuți, Romania, RMM-43, Winter-2024

Soluție.

Lema.

If $M\in Int(\Delta ABC), d_a=d(M,BC), F_1=Aria[MBC]$ then :

$$ad_a+bd_b+cd_c=2S.$$

Demonstrație.

Avem $ad_a+bd_b+cd_c=2F_1+2F_2+2F_3=2(F_1+F_2+F_3)=2F$;

$$LHS=\sum\frac{a^3}{xh_a+yd_a}=\sum\frac{a^4}{xah_a+yad_a}=\sum\frac{a^4}{x\cdot 2F+y\cdot 2F_1}\overset{CS}{\geq}\frac{\left(\sum a^2\right)^2}{\sum(x\cdot 2F+y\cdot 2F_1)}\overset{I-W}{\geq}\frac{\left(4\sqrt{3}F\right)^2}{6xF+2yF}=$$
$$=\frac{16\cdot 3F^2}{2F(3x+y)}=\frac{24}{3x+y}F=RHS.$$

Am folosit mai sus inegalitatea Ionescu-Weitzenbock: $\sum a^2\geq 4F\sqrt{3}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral și $M\equiv G$.

Remarca.

Problema se poate dezvolta. If $x, y > 0$, $M \in Int(\Delta ABC)$,
$d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$ then :

$$\sum \frac{a^{2n+1}}{xh_a + yd_a} \geq \frac{2\sqrt{3}}{3x+y}\left(4\sqrt{3}F\right)^n, n \in \mathbf{N}.$$

Marin Chirciu

Soluție.

Lema.

If $M \in Int(\Delta ABC)$, $d_a = d(M, BC)$, $F_1 = Aria[MBC]$ then :

$$ad_a + bd_b + cd_c = 2S.$$

Demonstrație.

Avem $ad_a + bd_b + cd_c = 2F_1 + 2F_2 + 2F_3 = 2(F_1 + F_2 + F_3) = 2F$;

Folosind **Lema** obținem:

$$LHS = \sum \frac{a^{2n+1}}{xh_a + yd_a} = \sum \frac{a^{2n+2}}{xah_a + yad_a} = \sum \frac{a^{2n+2}}{x \cdot 2F + y \cdot 2F_1} \overset{Holder}{\geq} \frac{\left(\sum a^2\right)^{n+1}}{3^{n-1}\sum\left(x \cdot 2F + y \cdot 2F_1\right)} \overset{I-W}{\geq}$$

$$\overset{I-W}{\geq} \frac{\left(4\sqrt{3}F\right)^{n+1}}{3^{n-1}\left(6xF + 2yF\right)} = \frac{\left(4\sqrt{3}F\right)^{n+1}}{3^{n-1}2F\left(3x+y\right)} = \frac{6\sqrt{3}}{3x+y}\left(\frac{4\sqrt{3}F}{3}\right)^n = RHS.$$

Aplicația61.

If $x, y, z > 0$, $x + y + z = 1$ then

$$\sum x\sqrt{x+2y} \leq 1.$$

Do Dinh Quang Khanh, Vietnam,THCS 11/2023

Soluție.

$$LHS = \sum x\sqrt{x+2y} = \sum x\sqrt{(x+2y)\cdot 1} \overset{AM-GM}{\leq} \sum x\frac{(x+2y)+1}{2} = \frac{1}{2}\sum\left(x^2 + 2xy + x\right) =$$

$$= \frac{1}{2}\left(\sum\left(x^2 + 2xy\right) + \sum x\right) = \frac{1}{2}\left(\left(\sum x\right)^2 + \sum x\right) = \frac{1}{2}\left(1^2 + 1\right) = 1 = RHS.$$

Remarca.

In ΔABC

$$\sum \frac{1}{r_a}\sqrt{\frac{1}{r_a}+\frac{2}{r_b}} \le \frac{1}{r\sqrt{r}}.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $x+y+z=1$ then

$$\sum x\sqrt{x+2y} \le 1.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x, y, z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obținem:

$$\sum \frac{r}{r_a}\sqrt{\frac{r}{r_a}+2\frac{r}{r_b}} \le 1 \Leftrightarrow \sum \frac{1}{r_a}\sqrt{\frac{1}{r_a}+\frac{2}{r_b}} \le \frac{1}{r\sqrt{r}}.$$

Remarca.

In ΔABC

$$\sum \frac{1}{h_a}\sqrt{\frac{1}{h_a}+\frac{2}{h_b}} \le \frac{1}{r\sqrt{r}}.$$

Marin Chirciu

Aplicația62.

S.2562. If ΔABC then

$$\prod\left(a^2 r_a^2+1\right) \ge 27F^2.$$

D.M.Bătinețu-Giurgiu, Claudia Nănuți, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a, b, c, t \ge 0$

$$\prod\left(a^2+t\right) \ge \frac{3}{4}(a+b+c)^2 t^2.$$

Hojoo Lee Inequality

Folosind **Lema** pentru $(a,b,c,t)=(ar_a, br_b, cr_c, 1)$ obținem:

$$LHS=\prod\left(a^2r_a^2+1\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum ar_a\right)^2\cdot 1^2\overset{(1)}{\geq}\frac{3}{4}(6F)^2\cdot 1^2=27F^2=RHS,$$

unde (1) $\Leftrightarrow$ $\sum ar_a\geq 6F$, vezi: $\sum ar_a=2p(2R-r)\overset{Euler}{\geq}2p\cdot 3r=6F$.

Remarca.

Problema se poate dezvolta.

In ΔABC and $\lambda>0$ then

$$\prod\left(a^2r_a^2+\lambda\right)\geq 27\lambda^2F^2.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=(ar_a,br_b,cr_c,\lambda)$ obținem:

$$LHS=\prod\left(a^2r_a^2+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum ar_a\right)^2\cdot\lambda^2\overset{(1)}{\geq}\frac{3\lambda^2}{4}(6F)^2=27\lambda^2F^2=RHS,$$

unde (1) $\Leftrightarrow$ $\sum ar_a\geq 6F$, vezi: $\sum ar_a=2p(2R-r)\overset{Euler}{\geq}2p\cdot 3r=6F$.

Aplicația63.

S.2620. In ΔABC then

$$\sum\left(a^2+2\right)^3\geq 108F.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Folosind $x^3+y^3+z^3\geq 3xyz$ pentru $(x,y,z)=\left(a^2+2,b^2+2,c^2+2\right)$ obținem

$\sum\left(a^2+2\right)^3\geq 3\sum\left(a^2+2\right)\geq 108F$.

Rămâne să arătăm că $3\sum\left(a^2+2\right)\geq 108F\Leftrightarrow\sum\left(a^2+2\right)\geq 36F$.

Folosind **Lema** pentru $t=2$ obținem:

$$LHS=\sum\left(a^2+2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum a\right)^2 2^2=3(2p)^2=12p^2\overset{Hadwiger}{\geq}12\cdot 3\sqrt{3}F.$$

Remarca.

In ΔABC then

$$\sum\left(a^2+\lambda\right)^3\geq 27\lambda^2\sqrt{3}F.$$

Marin Chirciu

Aplicația64.

4890. If $a,b,c>0$, $ab+bc+ca=3$ then

$$\left(a^3+\frac{2}{3}\right)\left(b^3+\frac{2}{3}\right)\left(c^3+\frac{2}{3}\right)\geq\frac{125}{27}.$$

George Apostolopoulos, Greece, Crux Mathematicorum

Soluție.

$$\prod\left(a^3+\frac{2}{3}\right)\geq\frac{125}{27}\Leftrightarrow\prod\left(3a^3+2\right)\geq 125.$$

Lema.

If $a,b,c>0$, $ab+bc+ca=3$ then

$$\prod\left(3a^3+2\right)\geq(ab+c+3)^3.$$

Demonstrație.

$$\prod\left(3a^3+2\right)=\left(a^3+a^3+a^3+1+1\right)\left(b^3+b^3+1+b^3+1\right)\left(1+1+c^3+c^3+c^3\right)\overset{Holder}{\geq}$$
$$\overset{Holder}{\geq}(ab+ab+ac+bc+c)=(ab+c+3)^3.$$

$$LHS=\prod\left(3a^3+2\right)\overset{Lema}{\geq}\frac{\sum(ab+c+3)^3}{3}\overset{Holder}{\geq}\frac{\frac{\left(\sum(ab+c+3)\right)^3}{3^2}}{3}=\frac{\left(\sum ab+\sum c+\sum 3\right)^3}{27}\overset{SOS}{\geq}$$
$$\overset{SOS}{\geq}\frac{(3+3+9)^3}{27}=125=RHS \text{ ,vezi(SOS): } \sum a\geq\sqrt{3\sum ab}=\sqrt{3\cdot 3}=3.$$

Remarca.

If $a,b,c>0$, $ab+bc+ca=3$ then

$$\left(a^3+\frac{1}{2}\right)\left(b^3+\frac{1}{2}\right)\left(c^3+\frac{1}{2}\right)\geq\frac{27}{8}.$$

Marin Chirciu

Soluție.

$$\prod\left(a^3+\frac{1}{2}\right)\geq\frac{27}{8}\Leftrightarrow\prod\left(2a^3+1\right)\geq 27.$$

Lema.

If $a,b,c>0$, $ab+bc+ca=3$ then

$$\prod\left(2a^3+1\right)\geq\left(ab+bc+ca\right)^3.$$

Demonstrație.

$$\prod\left(2a^3+1\right)=\left(a^3+a^3+1\right)\left(b^3+1+b^3\right)\left(1+c^3+c^3\right)\overset{Holder}{\geq}\left(ab+ac+bc\right)^3.$$

$$LHS=\prod\left(2a^3+1\right)\overset{Lema}{\geq}\left(ab+bc+ca\right)^3=3^3=27=RHS.$$

Remarca.

In ΔABC

$$\prod\left(3\sqrt{3}\tan^3\frac{A}{2}+\frac{1}{2}\right)\geq\frac{27}{8}.$$

Marin Chirciu

Soluție.

Lema.

If $a,b,c>0$, $ab+bc+ca=3$ then

$$\left(a^3+\frac{1}{2}\right)\left(b^3+\frac{1}{2}\right)\left(c^3+\frac{1}{2}\right)\geq\frac{27}{8}.$$

Demonstrație.

$$\prod\left(a^3+\frac{1}{2}\right)\geq\frac{27}{8}\Leftrightarrow\prod\left(2a^3+1\right)\geq 27.$$

$$\prod\left(2a^3+1\right)=\left(a^3+a^3+1\right)\left(b^3+1+b^3\right)\left(1+c^3+c^3\right)\overset{Holder}{\geq}\left(ab+ac+bc\right)^3=3^3=27.$$

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\Leftrightarrow\sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(a,b,c)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\prod\left(3\sqrt{3}\tan^3\frac{A}{2}+\frac{1}{2}\right)\geq\frac{27}{8}.$$

Remarca.

In ΔABC

$$\prod\left(3\sqrt{3}\tan^3\frac{A}{2}+\frac{2}{3}\right)\geq\frac{125}{27}.$$

Marin Chirciu

Soluție.

Lema.

If $a,b,c>0$, $ab+bc+ca=3$ then

$$\left(a^3+\frac{2}{3}\right)\left(b^3+\frac{2}{3}\right)\left(c^3+\frac{2}{3}\right)\geq\frac{125}{27}.$$

$$LHS=\prod\left(3a^3+2\right)\overset{Lema}{\geq}\frac{\sum\left(ab+c+3\right)^3}{3}\overset{Holder}{\geq}\frac{\frac{\left(\sum\left(ab+c+3\right)\right)^3}{3^2}}{3}=\frac{\left(\sum ab+\sum c+\sum 3\right)^3}{27}\overset{SOS}{\geq}$$

$\overset{SOS}{\geq}\frac{(3+3+9)^3}{27}=125=RHS$,vezi(SOS):. $\sum a\geq\sqrt{3\sum ab}=\sqrt{3\cdot 3}=3$.

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1\Leftrightarrow\sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\sqrt{3}\tan\frac{A}{2},\sqrt{3}\tan\frac{B}{2},\sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$\prod\left(3\sqrt{3}\tan^3\frac{A}{2}+\frac{2}{3}\right)\geq\frac{125}{27}.$

Remarca.

In ΔABC

$$\prod\left(h_a^3+2\right)\geq(9r)^3$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Demonstrație.

$\prod\left(x^3+2\right)=\left(x^3+1+1\right)\left(1+y^3+1\right)\left(1+1+z^3\right)\overset{Holder}{\geq}\left(x+y+z\right)^3.$

Folosind **Lema** pentru $(x,y,z)=(h_a,h_b,h_c)$ și $\sum h_a=\dfrac{p^2+r^2+4Rr}{2R}\geq 9r$ obținem:

$\prod\left(h_a^3+2\right)\overset{Lema}{\geq}\left(\sum h_a\right)^3\geq(9r)^3.$

Remarca.

In ΔABC

$$\prod\left(m_a^3+2\right)\geq(9r)^3$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $(x,y,z)=(m_a,m_b,m_c)$ și $\sum m_a\geq\sum h_a=\dfrac{p^2+r^2+4Rr}{2R}\geq 9r$ obținem:

$\prod\left(m_a^3+2\right)\overset{Lema}{\geq}\left(\sum m_a\right)^3\geq(9r)^3.$

Remarca.

In ΔABC

$$\prod\left(w_a^3+2\right)\geq(9r)^3$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x,y,z)=(w_a,w_b,w_c)$ și $\sum w_a\geq\sum h_a=\dfrac{p^2+r^2+4Rr}{2R}\geq 9r$ obținem:

$\prod\left(w_a^3+2\right)\overset{Lema}{\geq}\left(\sum w_a\right)^3\geq(9r)^3.$

Remarca.

In ΔABC

$$\prod\left(s_a^3+2\right)\geq(9r)^3$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$ then

Folosind **Lema** pentru $(x, y, z) = (s_a, s_b, s_c)$ și $\sum s_a \geq \sum h_a = \frac{p^2 + r^2 + 4Rr}{2R} \geq 9r$ obținem:

$\prod(s_a^3 + 2) \overset{Lema}{\geq} (\sum s_a)^3 \geq (9r)^3$.

Remarca.

In ΔABC

$$\prod(r_a^3 + 2) \geq (4R + r)^3$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$ then

$$\prod(x^3 + 2) \geq (x + y + z)^3.$$

Folosind **Lema** pentru $(x, y, z) = (r_a, r_b, r_c)$ și $\sum r_a = 4R + r$ obținem:

$\prod(r_a^3 + 2) \overset{Lema}{\geq} (\sum r_a)^3 \geq (4R + r)^3$.

Remarca.

In ΔABC

$$\prod\left(\sin^3 \frac{A}{2} + 2\right) \geq \left(1 + \frac{r}{R}\right)^3.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$ then

$$\prod(x^3 + 2) \geq (x + y + z)^3.$$

Folosind **Lema** pentru $(x, y, z) = \left(\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}\right)$ și $\sum \sin\frac{A}{2} \geq 1 + \frac{r}{R}$ obținem:

$$\prod\left(\sin^3\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\sin\frac{A}{2}\right)^3\geq\left(1+\frac{r}{R}\right)^3.$$

Remarca.

In ΔABC

$$\prod\left(\cos^3\frac{A}{2}+2\right)\geq\frac{27r^3}{R^3}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $(x,y,z)=\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right)$ și $\sum\cos\frac{A}{2}\geq\frac{27r}{2p}\geq\frac{3\sqrt{3}r}{R}$ obținem:

$$\prod\left(\cos^3\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\cos\frac{A}{2}\right)^3\geq\left(\frac{3\sqrt{3}r}{R}\right)^3=\frac{27r^3}{R^3}.$$

Remarca.

In ΔABC

$$\prod\left(\tan^3\frac{A}{2}+2\right)\geq3\sqrt{3}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ și $\sum\tan\frac{A}{2}=\frac{4R+r}{p}\overset{Doucet}{\geq}\sqrt{3}$ obținem:

$$\prod\left(\tan^3\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\tan\frac{A}{2}\right)^3\geq\left(\sqrt{3}\right)^3=3\sqrt{3}.$$

Remarca.

In ΔABC

$$\prod\left(\cot^3\frac{A}{2}+2\right)\geq 81\sqrt{3}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ și $\sum\cot\frac{A}{2}=\frac{p}{r}\overset{Mitrinovic}{\geq}3\sqrt{3}$ obținem:

$$\prod\left(\cot^3\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\cot\frac{A}{2}\right)^3\geq\left(3\sqrt{3}\right)^3=81\sqrt{3}.$$

Remarca.

In ΔABC

$$\prod\left(\sec^3\frac{A}{2}+2\right)\geq 24\sqrt{3}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $(x,y,z)=\left(\sec\frac{A}{2},\sec\frac{B}{2},\sec\frac{C}{2}\right)$ și $\sum\sec\frac{A}{2}=\sum\frac{1}{\cos\frac{A}{2}}\geq 2\sqrt{3}$ obținem:

$$\prod\left(\sec^3\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\sec\frac{A}{2}\right)^3\geq\left(2\sqrt{3}\right)^3=24\sqrt{3}.$$

Remarca.

In ΔABC

$$\prod\left(\csc^3\frac{A}{2}+2\right)\geq 216.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$ then

$$\prod\left(x^3+2\right) \geq (x+y+z)^3.$$

Folosind **Lema** pentru $(x, y, z)=\left(\csc\frac{A}{2}, \csc\frac{B}{2}, \csc\frac{C}{2}\right)$ și $\sum\csc\frac{A}{2}=\sum\frac{1}{\sin\frac{A}{2}} \geq 6$ obținem:

$$\prod\left(\sec^3\frac{A}{2}+2\right) \overset{Lema}{\geq} \left(\sum\sec\frac{A}{2}\right)^3 \geq 6^3=216.$$

Remarca.

In ΔABC

$$\prod\left(\sin^6\frac{A}{2}+2\right) \geq \frac{27}{64}.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$ then

$$\prod\left(x^3+2\right) \geq (x+y+z)^3.$$

Folosind **Lema** pentru $(x, y, z)=\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right)$ și $\sum\sin^2\frac{A}{2}=1-\frac{r}{2R} \overset{Euler}{\geq} \frac{3}{4}$ obținem:

$$\prod\left(\sin^6\frac{A}{2}+2\right) \overset{Lema}{\geq} \left(\sum\sin^2\frac{A}{2}\right)^3 \geq \left(\frac{3}{4}\right)^3=\frac{27}{64}.$$

Remarca.

In ΔABC

$$\prod\left(\cos^6\frac{A}{2}+2\right) \geq \left(\frac{9r}{2R}\right)^3.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $(x, y, z)=\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right)$ și $\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}=\frac{4R+r}{2R} \overset{Euler}{\geq} \frac{9r}{2R}$ obținem:

$$\prod\left(\cos^6\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\cos^2\frac{A}{2}\right)^3\geq\left(\frac{9r}{2R}\right)^3.$$

Remarca.

In ΔABC

$$\prod\left(\tan^6\frac{A}{2}+2\right)\geq 1.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $\left(x,y,z\right)=\left(\tan^2\frac{A}{2},\tan^2\frac{B}{2},\tan^2\frac{C}{2}\right)$ și $\sum\tan^2\frac{A}{2}=\frac{\left(4R+r\right)^2-2p^2}{p^2}\overset{Doucet}{\geq}1$

obținem:

$$\prod\left(\tan^6\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\tan^2\frac{A}{2}\right)^3\geq 1^3=1.$$

Remarca.

In ΔABC

$$\prod\left(\cot^6\frac{A}{2}+2\right)\geq 729.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$ then

$$\prod\left(x^3+2\right)\geq\left(x+y+z\right)^3.$$

Folosind **Lema** pentru $\left(x,y,z\right)=\left(\cot^2\frac{A}{2},\cot^2\frac{B}{2},\cot^2\frac{C}{2}\right)$ și $\sum\cot^2\frac{A}{2}=\frac{p^2-2r^2-8Rr}{r^2}\overset{Gerretsen}{\geq}9$

obținem:

$$\prod\left(\cot^6\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\cot^2\frac{A}{2}\right)^3\geq 9^3=729.$$

Remarca.

In ΔABC

$$\prod\left(\sec^6\frac{A}{2}+2\right)\geq 64.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z>0$ then

$$\prod\left(x^3+2\right)\geq(x+y+z)^3.$$

Folosind **Lema** pentru $(x, y, z)=\left(\sec^2\frac{A}{2}, \sec^2\frac{B}{2}, \sec^2\frac{C}{2}\right)$ și

$\sum\sec^2\frac{A}{2}=\sum\frac{1}{\cos^2\frac{A}{2}}=\frac{p^2+(4R+r)^2}{p^2}\overset{Doucet}{\geq}4$ obținem:

$$\prod\left(\sec^6\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\sec^2\frac{A}{2}\right)^3\geq 4^3=64.$$

Remarca.

In ΔABC

$$\prod\left(\csc^6\frac{A}{2}+2\right)\geq 1728.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z>0$ then

$$\prod\left(x^3+2\right)\geq(x+y+z)^3.$$

Folosind **Lema** pentru $(x, y, z)=\left(\csc^2\frac{A}{2}, \csc^2\frac{B}{2}, \csc^2\frac{C}{2}\right)$ și

$\sum\csc^2\frac{A}{2}=\sum\frac{1}{\sin^2\frac{A}{2}}=\frac{p^2+r^2-8Rr}{r^2}\overset{Gerretsen}{\geq}12$ obținem:

$$\prod\left(\sec^6\frac{A}{2}+2\right)\overset{Lema}{\geq}\left(\sum\sec^2\frac{A}{2}\right)^3\geq 12^3=1728.$$

Aplicația65.

4887. If $a,b,c>0$ then

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\geq 3\sqrt{\frac{2(ab+bc+ca)}{a+b+c}}$$

Nguyen Viet Hung, Vietnam, Crux Mathematicorum, November 2023

Soluție.

$$LHS=\sum\sqrt{a+b}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\sqrt{a+b}}=3\sqrt[3]{\sqrt{\prod(a+b)}}\overset{Lema8/9}{\geq}3\sqrt[3]{\sqrt{\frac{8}{9}\sum a\sum ab}}\overset{(1)}{\geq}$$

$\overset{(1)}{\geq}3\sqrt{\frac{2(ab+bc+ca)}{a+b+c}}=RHS$, unde $3\sqrt[3]{\sqrt{\frac{8}{9}\sum a\sum ab}}\geq 3\sqrt{\frac{2(ab+bc+ca)}{a+b+c}}\Leftrightarrow$

$$\Leftrightarrow\sqrt[3]{\frac{8}{9}\sum a\sum ab}\geq\frac{2(ab+bc+ca)}{a+b+c}\Leftrightarrow\sqrt[3]{\frac{8}{9}pq}\geq\frac{2q}{p}\Leftrightarrow\frac{8}{9}pq\geq\frac{8q^3}{p^3}\Leftrightarrow p^4\geq 9q^2\Leftrightarrow$$

$\Leftrightarrow p^2\geq 3q$, vezi $p^2=(x+y+z)^2\geq 3(xy+yz+zx)=3q$.

Lema 8/9 .

If $a,b,c>0$ then

$$(a+b)(b+c)(c+a)\geq\frac{8}{9}(a+b+c)(ab+bc+ca).$$

Lema8/9

Demonstrație.

$9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca)\Leftrightarrow$

$\Leftrightarrow 9\left(\sum bc(b+c)+2abc\right)\geq\left(\sum bc(b+c)+3abc\right)\Leftrightarrow\sum bc(b+c)\geq 6abc\Leftrightarrow$

$\Leftrightarrow\sum a(b-c)^2\geq 0$, evident cu egalitate pentru $a=b=c$.

Remarca.

In ΔABC

$$\sum\sqrt{\tan\frac{A}{2}+\tan\frac{B}{2}}\geq 3\sqrt{\frac{2p}{4R+r}}.$$

Marin Chirciu

Lema

If $a,b,c>0$ then

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\geq 3\sqrt{\frac{2(ab+bc+ca)}{a+b+c}}.$$

Se cunosc identitățile în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$ și $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$.

Folosind **Lema** pentru $(a,b,c)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$\sum\sqrt{\tan\frac{A}{2}+\tan\frac{B}{2}}\geq 3\sqrt{\frac{2\cdot 1}{\frac{4R+r}{p}}}\Leftrightarrow\sum\sqrt{\tan\frac{A}{2}+\tan\frac{B}{2}}\geq 3\sqrt{\frac{2p}{4R+r}}.$$

<u>Aplicația66.</u>

If $x,y,z>0$, $x+y+z=3$ then

$$xyz\left(x^2+y^2+z^2\right)\leq 3.$$

Vu Long, Vietnam,THCS 11/2023

<u>Soluție.</u>

Folosind pqr -Method.

Notăm $p=x+y+z,q=xy+yz+zx,r=xyz$.

Avem $p=3$, $9=p^2=(x+y+z)^2=\sum x^2+2\sum xy=\sum x^2+2q\Rightarrow\sum x^2=9-2q$.

$q^2=(xy+yz+zx)^2\geq 3xyz(x+y+z)=3rp=9r\Rightarrow r\leq\frac{q^2}{9}$.

Inegalitatea $xyz\left(x^2+y^2+z^2\right)\leq 3$ se scrie $r(9-2q)\leq 3$, care rezultă din $r\leq\frac{q^2}{9}$.

Rămâne să arătăm că $\frac{q^2}{9}(9-2q)\leq 3\Leftrightarrow 2q^3-9q^2+27\geq 0\Leftrightarrow(q-3)^2(2q+3)\geq 0$.

<u>Remarca.</u>

In ΔABC

$$\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\leq\frac{R^2}{12r^4}.$$

Marin Chirciu

<u>Soluție.</u>

<u>Lema.</u>

If $x,y,z>0$, $x+y+z=3$ then

$$xyz\left(x^2+y^2+z^2\right)\le 3.$$

Se cunoaște identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obținem:

$$\frac{3r}{r_a}\cdot\frac{3r}{r_b}\cdot\frac{3r}{r_c}\left(\left(\frac{3r}{r_a}\right)^2+\left(\frac{3r}{r_b}\right)^2+\left(\frac{3r}{r_c}\right)^2\right)\le 3\Leftrightarrow\frac{27r^3}{r_ar_br_c}\cdot 9r^2\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\le 3\Leftrightarrow$$

$$\Leftrightarrow\frac{27r^3}{rp^2}\cdot 9r^2\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\le 3\Leftrightarrow\frac{81r^4}{p^2}\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\le 1\Leftrightarrow\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\le\frac{p^2}{81r^4}.$$

Folosind inegalitatea lui Mitrinovic $p^2\le\frac{27R^2}{4}$ rezultă:

$$\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\le\frac{p^2}{81r^4}\le\frac{\frac{27R^2}{4}}{81r^4}=\frac{R^2}{12r^4}.$$

<u>Aplicația67.</u>

S.2534. If $x,y,z>0$ then in ΔABC hods:

$$\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{h_a^2}\ge\frac{2\sqrt{3}}{F}.$$

D.M.Bătinețu-Giurgiu, Claudia Nănuți, Romania, RMM-43, Winter-2024

<u>Soluție</u>

$$LHS=\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{h_a^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{h_a^2}}=3\sqrt[3]{\frac{\prod(y+z)}{xyz}\prod\frac{1}{h_a^2}}\overset{Cesaro}{\ge}3\sqrt[3]{8\prod\frac{1}{h_a^2}}=$$

$=6\sqrt[3]{\prod\frac{1}{h_a^2}}=6\cdot\frac{1}{\sqrt{3}F}=\frac{2\sqrt{3}}{F}=RHS$, unde (1) $\Leftrightarrow\sqrt[3]{\prod\frac{1}{h_a^2}}\ge\frac{1}{\sqrt{3}F}$, vezi:

$$\sqrt[3]{\prod\frac{1}{h_a^2}}=\frac{1}{\sqrt[3]{\prod h_a^2}}=\frac{1}{\sqrt[3]{\prod\frac{4F^2}{a^2}}}=\sqrt[3]{\prod\frac{a^2}{4F^2}}=\sqrt[3]{\frac{(abc)^2}{\left(4F^2\right)^3}}=\frac{1}{4F^2}(abc)^{\frac{2}{3}}\overset{Carlitz}{\ge}\frac{1}{4F^2}\cdot\frac{4F}{\sqrt{3}}=\frac{1}{\sqrt{3}F}.$$

<u>Remarca.</u>

If $x,y,z>0$ then in ΔABC hods:

$$\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{r_a^2}\ge\frac{8}{3R^2}.$$

Marin Chirciu

Soluție

$$LHS=\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{r_a^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{r_a^2}}=3\sqrt[3]{\frac{\prod(y+z)}{xyz}\prod\frac{1}{r_a^2}}\overset{Cesaro}{\ge}3\sqrt[3]{8\prod\frac{1}{r_a^2}}=$$

$$=6\sqrt[3]{\prod\frac{1}{r_a^2}}\ge 6\cdot\left(\frac{2}{3R}\right)^2=\frac{8}{3R^2}=RHS$$, unde (1) $\Leftrightarrow\ \sqrt[3]{\prod\frac{1}{r_a^2}}\ge\left(\frac{2}{3R}\right)^2$, vezi:

$$\sqrt[3]{\prod\frac{1}{r_a^2}}=\frac{1}{\sqrt[3]{\prod r_a^2}}=\frac{1}{\sqrt[3]{(rp^2)^2}}\overset{Euler\&Mitrinovic}{\ge}\frac{1}{\sqrt[3]{\left(\frac{R}{2}\cdot\frac{27R^2}{4}\right)^2}}=\frac{1}{\sqrt[3]{\left(\frac{27R^3}{8}\right)^2}}=\left(\frac{2}{3R}\right)^2.$$

Remarca.

If $x,y,z>0$ then in ΔABC hods:

$$\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{m_a^2}\ge\frac{8}{3R^2}.$$

Marin Chirciu

Soluție

$$LHS=\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{m_a^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{m_a^2}}=3\sqrt[3]{\frac{\prod(y+z)}{xyz}\prod\frac{1}{m_a^2}}\overset{Cesaro}{\ge}3\sqrt[3]{8\prod\frac{1}{m_a^2}}=$$

$$=6\sqrt[3]{\prod\frac{1}{m_a^2}}\ge 6\cdot\left(\frac{2}{3R}\right)^2=\frac{8}{3R^2}=RHS$$, unde (1) $\Leftrightarrow\ \sqrt[3]{\prod\frac{1}{m_a^2}}\ge\left(\frac{2}{3R}\right)^2$, vezi:

$$\sqrt[3]{\prod\frac{1}{m_a^2}}=\frac{1}{\sqrt[3]{\prod m_a^2}}\ge\frac{1}{\sqrt[3]{\left(\frac{Rp^2}{2}\right)^2}}\overset{Mitrinovic}{\ge}\frac{1}{\sqrt[3]{\left(\frac{R}{2}\cdot\frac{27R^2}{4}\right)^2}}=\frac{1}{\sqrt[3]{\left(\frac{27R^3}{8}\right)^2}}=\left(\frac{2}{3R}\right)^2.$$

Am folosit mai sus $\prod m_a\le\frac{Rp^2}{2}$.

Remarca.

If $x,y,z>0$ then in ΔABC hods:

$$\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{s_a^2}\ge\frac{8}{3R^2}.$$

Marin Chirciu

Soluție

$$LHS=\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{s_a^2}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{s_a^2}}=3\sqrt[3]{\frac{\prod(y+z)}{xyz}\prod\frac{1}{s_a^2}}\overset{Cesaro}{\geq}3\sqrt[3]{8\prod\frac{1}{s_a^2}}=$$

$$=6\sqrt[3]{\prod\frac{1}{s_a^2}}\geq 6\cdot\left(\frac{2}{3R}\right)^2=\frac{8}{3R^2}=RHS$$, unde (1) $\Leftrightarrow \sqrt[3]{\prod\frac{1}{s_a^2}}\geq\left(\frac{2}{3R}\right)^2$, vezi:

$$\sqrt[3]{\prod\frac{1}{s_a^2}}=\frac{1}{\sqrt[3]{\prod s_a^2}}\geq\frac{1}{\sqrt[3]{\prod m_a^2}}\geq\frac{1}{\sqrt[3]{\left(\frac{Rp^2}{2}\right)^2}}\overset{Mitrinovic}{\geq}\frac{1}{\sqrt[3]{\left(\frac{R}{2}\cdot\frac{27R^2}{4}\right)^2}}=\frac{1}{\sqrt[3]{\left(\frac{27R^3}{8}\right)^2}}=\left(\frac{2}{3R}\right)^2.$$

Am folosit mai sus $\prod s_a\leq\prod m_a\leq\frac{Rp^2}{2}$.

Remarca.

If $x,y,z>0$ then in ΔABC hods:

$$\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{w_a^2}\geq\frac{8}{3R^2}.$$

Marin Chirciu

Soluție

$$LHS=\sum\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{w_a^2}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\left(\frac{1}{y}+\frac{1}{z}\right)\frac{x}{w_a^2}}=3\sqrt[3]{\frac{\prod(y+z)}{xyz}\prod\frac{1}{w_a^2}}\overset{Cesaro}{\geq}3\sqrt[3]{8\prod\frac{1}{w_a^2}}=$$

$$=6\sqrt[3]{\prod\frac{1}{w_a^2}}\geq 6\cdot\left(\frac{2}{3R}\right)^2=\frac{8}{3R^2}=RHS$$, unde (1) $\Leftrightarrow \sqrt[3]{\prod\frac{1}{w_a^2}}\geq\left(\frac{2}{3R}\right)^2$, vezi:

$$\sqrt[3]{\prod\frac{1}{w_a^2}}=\frac{1}{\sqrt[3]{\prod w_a^2}}\geq\frac{1}{\sqrt[3]{\prod m_a^2}}\geq\frac{1}{\sqrt[3]{\left(\frac{Rp^2}{2}\right)^2}}\overset{Mitrinovic}{\geq}\frac{1}{\sqrt[3]{\left(\frac{R}{2}\cdot\frac{27R^2}{4}\right)^2}}=\frac{1}{\sqrt[3]{\left(\frac{27R^3}{8}\right)^2}}=\left(\frac{2}{3R}\right)^2.$$

Am folosit mai sus $\prod w_a\leq\prod m_a\leq\frac{Rp^2}{2}$.

Aplicația68.

S.2534. If $x,y,z>0$ then in ΔABC hods:

$$\left(\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}\right)^2+1\right)\left(\frac{z^2c^4}{(x+y)^2}+1\right)\geq 12F^2.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,t\geq 0$

$$\left(a^2+t\right)\left(b^2+t\right)\geq \frac{3}{4}t\left(t+\left(a+b\right)^2\right).$$

Hojoo Lee Inequality

Folosind **Lema** pentru $\left(a,b,t\right)=\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x},\frac{zc^2}{x+y},1\right)$ obținem:

$$LHS=\left(\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}\right)^2+1\right)\left(\frac{z^2c^4}{\left(x+y\right)^2}+1\right)\overset{Lema}{\geq}\frac{3}{4}1\left(1+\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}+\frac{zc^2}{x+y}\right)^2\right)\overset{(1)}{\geq}$$

$\geq 12F^2=RHS$,unde(1) $\Leftrightarrow \left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}+\frac{zc^2}{x+y}\right)^2\geq 12F^2$, vezi

$$\sum\frac{xa^2}{y+z}=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\left(\frac{x+y+z}{y+z}-1\right)a^2=\left(x+y+z\right)\sum\frac{a^2}{y+z}-\sum a^2\overset{CS}{\geq}$$

$$\overset{CS}{\geq}\left(x+y+z\right)\frac{\left(\sum a\right)^2}{\sum\left(y+z\right)}-\sum a^2=\left(x+y+z\right)\frac{\left(\sum a\right)^2}{2\left(x+y+z\right)}-\sum a^2=\frac{\left(\sum a\right)^2}{2}-\sum a^2=$$

$$=\frac{\sum a^2+2\sum ab}{2}-\sum a^2=\sum ab-\frac{1}{2}\sum a^2=p^2+r^2+4Rr-\frac{1}{2}\cdot 2\left(p^2+r^2+4Rr\right)=$$

$$=2r\left(4R+r\right)\overset{Doucet}{\geq}2r\cdot p\sqrt{3}=2\sqrt{3}F.$$

Remarca.

If $x,y,z,\lambda>0$ then in ΔABC hods:

$$\left(\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}\right)^2+\lambda\right)\left(\frac{z^2c^4}{\left(x+y\right)^2}+\lambda\right)\geq 9\lambda F^2.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $\left(a,b,t\right)=\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x},\frac{zc^2}{x+y},\lambda\right)$ obținem:

$$LHS=\left(\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}\right)^2+\lambda\right)\left(\frac{z^2c^4}{\left(x+y\right)^2}+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\lambda\left(\lambda+\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}+\frac{zc^2}{x+y}\right)^2\right)\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{3}{4}\lambda\left(\lambda+12F^2\right)\geq\frac{3}{4}\lambda\cdot 12F^2=9\lambda F^2=RHS,$$

unde(1) $\Leftrightarrow\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}+\frac{zc^2}{x+y}\right)^2\geq 12F^2$, vezi

$\geq 12F^2=RHS$,unde(1) $\Leftrightarrow\left(\frac{xa^2}{y+z}+\frac{yb^2}{z+x}+\frac{zc^2}{x+y}\right)^2\geq 12F^2$, vezi

$$\sum\frac{xa^2}{y+z}=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\left(\frac{x+y+z}{y+z}-1\right)a^2=(x+y+z)\sum\frac{a^2}{y+z}-\sum a^2\overset{CS}{\geq}$$

$$\overset{CS}{\geq}(x+y+z)\frac{\left(\sum a\right)^2}{\sum(y+z)}-\sum a^2=(x+y+z)\frac{\left(\sum a\right)^2}{2(x+y+z)}-\sum a^2=\frac{\left(\sum a\right)^2}{2}-\sum a^2=$$

$$=\frac{\sum a^2+2\sum ab}{2}-\sum a^2=\sum ab-\frac{1}{2}\sum a^2=p^2+r^2+4Rr-\frac{1}{2}\cdot 2\left(p^2+r^2+4Rr\right)=$$

$$=2r(4R+r)\overset{Doucet}{\geq}2r\cdot p\sqrt{3}=2\sqrt{3}F.$$

Aplicația69.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$\sum\sqrt{x^2+1}\leq 2(x+y+z).$$

Vu Long, Vietnam,THCS 11/2023

Soluție.

Lema.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$\sqrt{x^2+1}\leq\frac{2x+y+z}{2}.$$

Demonstrație.

$$\sqrt{x^2+1}=\sqrt{x^2+xy+yz+zx}=\sqrt{(x+y)(x+z)}\overset{AM-GM}{\leq}\frac{(x+y)+(x+z)}{2}=\frac{2x+y+z}{2}.$$

$$LHS=\sum\sqrt{x^2+1}\overset{Lema}{\leq}\sum\frac{2x+y+z}{2}=\frac{4\sum x}{2}=2\sum x=RHS.$$

Remarca.

In ΔABC

$$\sum\sqrt{\tan^2\frac{A}{2}+1}\leq\frac{2(4R+r)}{p}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $xy+yz+zx=1$ then

$$\sum\sqrt{x^2+1}\le 2(x+y+z).$$

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$

Folosind **Lema** pentru $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$\sum\sqrt{\tan^2\frac{A}{2}+1}\le 2\sum\tan\frac{A}{2}\Leftrightarrow\sum\sqrt{\tan^2\frac{A}{2}+1}\le 2\cdot\frac{4R+r}{p}.$$

Aplicația70.

S.2518. In ΔABC

$$\prod\left(\frac{\sin^2 A}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}+1\right)\ge 81.$$

D.M.Bătinețu-Giurgiu, Claudia Nănuți, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\ge 0$

$$\prod\left(a^2+t\right)\ge\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Folosind **Lema** pentru $(a,b,c,t)=\left(\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}},\frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}},\frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}},1\right)$ obținem:

$$LHS=\prod\left(\frac{\sin^2 A}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}+1\right)\overset{Lema}{\ge}\frac{3}{4}\left(\sum\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}\right)^2\cdot 1^2\overset{(2)}{\ge}\frac{3}{4}\left(6\sqrt{3}\right)^2=81=RHS,$$

unde(2) $\Leftrightarrow \sum \frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq 6\sqrt{3}$, vezi :

$$\sum \frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}} = 3\sqrt[3]{\prod \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}} = 3\cdot 2\sqrt[3]{\frac{\prod \sin\frac{A}{2}\prod\cos\frac{A}{2}}{\prod \sin^2\frac{A}{2}}} =$$

$$= 6\sqrt[3]{\prod \cot\frac{A}{2}} = 6\sqrt[3]{\frac{p}{r}} \overset{Mitrinovic}{\geq} 6\sqrt[3]{\frac{3r\sqrt{3}}{r}} = 6\sqrt{3}$$

Remarca.

If $\lambda > 0$, In ΔABC

$$\prod \left(\frac{\sin^2 A}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}} + \lambda \right) \geq 81\lambda^2.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t) = \left(\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}, \frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}}, \frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}}, \lambda \right)$ obținem:

$$LHS = \prod \left(\frac{\sin^2 A}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}} + \lambda \right) \overset{Lema}{\geq} \frac{3}{4}\left(\sum \frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} \right)^2 \cdot \lambda^2 \overset{(2)}{\geq} \frac{3}{4}\left(6\sqrt{3}\right)^2 \cdot \lambda^2 = 81\lambda^2 = RHS,$$

unde(2) $\Leftrightarrow \sum \frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq 6\sqrt{3}$,

Aplicația71.

S.2516. If $m,n > 0$, In ΔABC

$$\sum \frac{\cot^3\frac{A}{2}}{m\tan\frac{B}{2} + n\tan\frac{C}{2}} \geq \frac{p^2}{(m+n)r^2}.$$

D.M.Bătinețu-Giurgiu, Neculai Stanciu, Romania, RMM-43, Winter-2024

Soluție

$$LHS=\sum\frac{\cot^3\frac{A}{2}}{m\tan\frac{B}{2}+n\tan\frac{C}{2}}\overset{Holder}{\geq}\frac{\left(\sum\cot\frac{A}{2}\right)^3}{3\sum\left(m\tan\frac{B}{2}+n\tan\frac{C}{2}\right)}=\frac{\left(\sum\cot\frac{A}{2}\right)^3}{3(m+n)\sum\tan\frac{A}{2}}=$$

$$=\frac{\left(\frac{p}{r}\right)^3}{3(m+n)\frac{4R+r}{p}}\overset{(1)}{\geq}\frac{p^2}{(m+n)r^2}=RHS,$$

unde (1) $\Leftrightarrow \dfrac{\left(\frac{p}{r}\right)^3}{3(m+n)\frac{4R+r}{p}}\geq\dfrac{p^2}{(m+n)r^2}\Leftrightarrow p^2\geq r(4R+r)$, vezi $p^2\geq 16Rr-5r$.

Remarca.

If $x,y>0$ and $n\in\mathbf{N}^*$, In ΔABC

$$\sum\frac{\cot^{n+1}\frac{A}{2}}{x\tan\frac{B}{2}+n\tan\frac{C}{2}}\geq\frac{3}{x+y}\left(\frac{p}{3r}\right)^n.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{\cot^{n+1}\frac{A}{2}}{x\tan\frac{B}{2}+y\tan\frac{C}{2}}\overset{Holder}{\geq}\frac{\left(\sum\cot\frac{A}{2}\right)^{n+1}}{3^{n-1}\sum\left(x\tan\frac{B}{2}+\tan\frac{C}{2}\right)}=\frac{\left(\sum\cot\frac{A}{2}\right)^{n+1}}{3^{n-1}(x+y)\sum\cot\frac{A}{2}}=$$

$$=\frac{\left(\frac{p}{r}\right)^{n+1}}{3^{n-1}(x+y)\frac{4R+r}{p}}=\frac{\left(\frac{p}{r}\right)^n\frac{p}{r}}{3^{n-1}(x+y)\frac{4R+r}{p}}=\frac{3}{x+y}\left(\frac{p}{3r}\right)^n\frac{p^2}{r(4R+r)}\overset{Gerretsen}{\geq}\frac{3}{x+y}\left(\frac{p}{3r}\right)^n\cdot 3=$$

$$=\frac{9}{x+y}\left(\frac{p}{3r}\right)^n=RHS.$$

Aplicația72.

J.2630. If $x,y,z>0$, $x+y+z=3$, In ΔABC

$$\sum\frac{xa^4}{(y+z)^2}\geq 4F^2.$$

D.M.Bătinețu-Giurgiu, Claudia Nănuți, Romania, RMM-43, Winter-2024

Soluție

$$LHS=\sum\frac{xa^4}{(y+z)^2}=\sum\frac{\frac{x^2a^4}{(y+z)^2}}{x}\overset{CS}{\geq}\frac{\left(\sum\frac{xa^2}{y+z}\right)^2}{\sum x}=\frac{\left(\sum\frac{xa^2}{y+z}\right)^2}{3}\overset{(1)}{\geq}\frac{\left(2\sqrt{3}F\right)^2}{3}=4F^2,$$

unde(1) $\Leftrightarrow\sum\frac{xa^2}{y+z}\geq 2\sqrt{3}F$ și $\frac{xyz}{\prod(y+z)}\leq\frac{1}{8}$, vezi:

1). $\sum\frac{xa^2}{y+z}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{xa^2}{y+z}}=3\sqrt[3]{\frac{xyz}{\prod(y+z)}\prod a^2}\overset{(1)}{\geq}3\sqrt[3]{\frac{1}{8}\prod a^2}=3\cdot\frac{1}{2}\sqrt[3]{\prod a^2}=$

$=\frac{3}{2}(abc)^{\frac{2}{3}}\overset{Carlitz}{\geq}\frac{3}{2}\cdot\frac{4F}{\sqrt{3}}=2\sqrt{3}F$.

2). $\frac{xyz}{\prod(y+z)}\leq\frac{1}{8}$, din: $\frac{xyz}{\sum x\sum yz-xyz}\leq\frac{1}{8}\Leftrightarrow xyz\leq 1$.

Am folosit mai sus $xyz\leq 1$, din: $3=x+y+z\geq 3\sqrt[3]{xyz}\Rightarrow xyz\leq 1$.

Inegalitatea lui Carlitz $abc\geq\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}$, cu egalitate pentru triunghiul echilateral.

Remarca.

Problema se poate dezvolta.

If $x,y,z>0$, $x+y+z=3$ and $n\in\mathbf{N}$, In ΔABC

$$\sum\frac{xa^{2n}}{(y+z)^2}\geq\frac{3}{4}\left(\frac{4F}{\sqrt{3}}\right)^n.$$

Marin Chirciu

Soluție

$$\sum\frac{xa^{2n}}{(y+z)^2}=\sum\frac{\frac{x^2a^{2n}}{(y+z)^2}}{x}\overset{CS}{\geq}\frac{\left(\sum\frac{xa^n}{y+z}\right)^2}{\sum x}=\frac{\left(\sum\frac{xa^n}{y+z}\right)^2}{3}\overset{(1)}{\geq}\frac{\left(\frac{3}{2}\left(\frac{4F}{\sqrt{3}}\right)^{\frac{n}{2}}\right)^2}{3}=\frac{3}{4}\left(\frac{4F}{\sqrt{3}}\right)^n.$$

Remarca.

ProbleIf $x,y,z>0$, $x+y+z=3$ and $n\in\mathbf{N}$ In ΔABC

$$\sum\frac{x^2a^{2n}}{(y+z)^2}\ge\frac{3}{8}\left(\frac{4F}{\sqrt{3}}\right)^n.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{x^2a^{2n}}{(y+z)^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{x^2a^{2n}}{(y+z)^2}}=3\sqrt[3]{\frac{(xyz)^2}{\prod(y+z)^2}\prod a^{2n}}\overset{(1)}{\ge}3\sqrt[3]{\frac{1}{64}\prod a^4}=$$

$$=3\cdot\frac{1}{8}\sqrt[3]{\prod a^{2n}}=\frac{3}{8}(abc)^{\frac{2n}{3}}\overset{Carlitz}{\ge}\frac{3}{8}\left[\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}\right]^{\frac{2n}{3}}=\frac{3}{8}\left(\frac{4F}{\sqrt{3}}\right)^n=RHS.$$

Remarca.

If $x,y,z>0$, $x+y+z=3$ and $n\in\mathbf{N}$ In ΔABC

$$\sum\frac{xa^{2n}}{y+z}\ge\frac{3}{8}\left(\frac{4F}{\sqrt{3}}\right)^n.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{xa^{2n}}{y+z}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{xa^{2n}}{(y+z)}}=3\sqrt[3]{\frac{xyz}{\prod(y+z)}\prod a^{2n}}\overset{(1)}{\ge}3\sqrt[3]{\frac{1}{8}\prod a^4}=$$

$$=3\cdot\frac{1}{2}\sqrt[3]{\prod a^{2n}}=\frac{3}{2}(abc)^{\frac{2n}{3}}\overset{Carlitz}{\ge}\frac{3}{2}\left[\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}\right]^{\frac{2n}{3}}=\frac{3}{2}\left(\frac{4F}{\sqrt{3}}\right)^n=RHS.$$

Aplicația73.

J.2624. If $x,y,z>0$, $x+y+z\le 3$, In ΔABC

$$\sum\frac{(x+y)a^4}{z^2}\ge 32F^2.$$

D.M.Bătinețu-Giurgiu, Nicolae Mușuroia, Romania, RMM-43, Winter-2024

Soluție

$$LHS=\sum\frac{(x+y)a^4}{z^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{(x+y)a^4}{z^2}}=3\sqrt[3]{\frac{\prod(x+y)}{x^2y^2z^2}\prod a^4}\overset{Cesaro}{\ge}3\sqrt[3]{\frac{8xyz}{x^2y^2z^2}\prod a^4}=$$

$$=3\cdot 2\sqrt[3]{\frac{1}{xyz}\prod a^4}\overset{xyz\le 1}{\ge}6\sqrt[3]{\prod a^4}=6(abc)^{\frac{4}{3}}\overset{Carlitz}{\ge}6\left[\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}\right]^{\frac{4}{3}}=6\left(\frac{4F}{\sqrt{3}}\right)^2=32F^2=RHS.$$

Am folosit mai sus $xyz\le 1$, vezi $3\ge x+y+z\ge 3\sqrt[3]{xyz}\Rightarrow xyz\le 1$.

Inegalitatea lui Carlitz $abc\ge\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}$, cu egalitate pentru triunghiul echilateral.

Remarca.

Problema se poate dezvolta.

If $x,y,z>0$, $x+y+z\le 3$ and $n\in\mathbf{N}$ In ΔABC

$$\sum\frac{(y+z)a^{2n}}{x^2}\ge 6\left(\frac{4F}{\sqrt{3}}\right)^n.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{(x+y)a^{2n}}{z^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{(x+y)a^{2n}}{z^2}}=3\sqrt[3]{\frac{\prod(x+y)}{x^2y^2z^2}\prod a^{2n}}\overset{Cesaro}{\ge}3\sqrt[3]{\frac{8xyz}{x^2y^2z^2}\prod a^{2n}}=$$

$$=3\cdot 2\sqrt[3]{\frac{1}{xyz}\prod a^{2n}}\overset{xyz\le 1}{\ge}6\sqrt[3]{\prod a^{2n}}=6(abc)^{\frac{2n}{3}}\overset{Carlitz}{\ge}6\left[\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}\right]^{\frac{2n}{3}}=6\left(\frac{4F}{\sqrt{3}}\right)^n=RHS.$$

Aplicația74.

J.2626. If ΔABC then

$$\prod\left(a^2+2s\right)\ge 324F^2.$$

D.M.Bătinețu-Giurgiu, Daniel Sitaru, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\ge 0$

$$\prod\left(a^2+t\right)\ge\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Folosind **Lema** pentru $t=2s$ obținem:

$$LHS=\prod\left(a^2+2s\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum a\right)^2\cdot 4s^2=3\left(2s\right)^2\cdot s^2=12s^4\overset{Mitrinovic}{\geq}324F^2=RHS,$$

Remarca.

In ΔABC and $\lambda>0$ then

$$\prod\left(a^2+\lambda s\right)\geq 81\lambda^2F^2.$$

D.M.Bătinețu-Giurgiu, Daniel Sitaru, Romania

Soluție

Folosind **Lema** pentru $t=\lambda s$ obținem:

$$LHS=\prod\left(a^2+\lambda s\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum a\right)^2\cdot\left(\lambda s\right)^2=\frac{3}{4}\left(2s\right)^2\cdot\left(\lambda s\right)^2=3\lambda^2s^2\cdot s^2\overset{Mitrinovic}{\geq}3\lambda^2s^2\cdot 27r^2=$$

$$=81\lambda^2F^2=RHS.$$

Aplicația75.

J.2627. If ΔABC then

$$\prod\left(a^2r_a^2+2\right)\geq 108F^2.$$

D.M.Bătinețu-Giurgiu, Daniel Sitaru, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}\left(a+b+c\right)^2t^2.$$

Hojoo Lee Inequality

Demonstrație

Folosind **Lema** pentru $\left(a,b,c,t\right)=\left(ar_a,br_b,cr_c,2\right)$ obținem:

$$LHS=\prod\left(a^2r_a^2+2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum ar_a\right)^2\cdot 2^2\overset{(1)}{\geq}\frac{3}{4}\left(6F\right)^2\cdot 2^2=108F^2=RHS,$$

unde (1) $\Leftrightarrow$ $\sum ar_a\geq 6F$, vezi: $\sum ar_a=2p\left(2R-r\right)\overset{Euler}{\geq}2p\cdot 3r=6F$.

Remarca.

In ΔABC and $\lambda>0$ then

$$\prod\left(a^2r_a^2+\lambda\right)\geq 27\lambda^2F^2.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=(ar_a,br_b,cr_c,\lambda)$ obținem:

$$LHS=\prod\left(a^2r_a^2+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum ar_a\right)^2\cdot\lambda^2\overset{(1)}{\geq}\frac{3\lambda^2}{4}(6F)^2=27\lambda^2F^2=RHS,$$

unde (1) $\Leftrightarrow$ $\sum ar_a\geq 6F$, vezi: $\sum ar_a=2p(2R-r)\overset{Euler}{\geq}2p\cdot 3r=6F$.

Aplicația76.

J.2628. If ΔABC then

$$\prod\left(a^4r_a^2+2\right)\geq 432r^3.$$

D.M.Bătinețu-Giurgiu, Ionuț Ivănescu, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Folosind **Lema** pentru $(a,b,c,t)=(a^2r_a,b^2r_b,c^2r_c,2)$ obținem:

$$LHS=\prod\left(a^4r_a^2+2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum a^2r_a\right)^2\cdot 2^2\overset{(1)}{\geq}\frac{3}{4}\left(\frac{27R^3}{2}\right)^2\cdot 2^2=\frac{2187R^6}{4}=RHS,$$

unde (1) $\Leftrightarrow$ $\sum a^2r_a\geq\frac{27R^3}{2}$, vezi:

$$\sum a^2r_a=4p^2(R-r)\overset{Gerretsen}{\geq}4\left(4R^2+4Rr+3r^2\right)(R-r)=4\left(4R^3-Rr^2-3r^3\right)\overset{Euler}{\geq}\frac{27R^3}{2}$$

Remarca.

If ΔABC and $\lambda>0$ then

$$\prod\left(a^4r_a^2+\lambda\right)\geq\frac{2187\lambda^2R^6}{16}.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=(a^2r_a,b^2r_b,c^2r_c,\lambda)$ obținem:

$$LHS=\prod\left(a^4r_a^2+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum a^2r_a\right)^2\cdot\lambda^2\overset{(1)}{\geq}\frac{3}{4}\left(\frac{27R^3}{2}\right)^2\cdot\lambda^2=\frac{2187\lambda^2R^6}{16}=RHS,$$

unde (1) $\Leftrightarrow$ $\sum a^2r_a\geq\frac{27R^3}{2}$, vezi:

$$\sum a^2r_a=4p^2(R-r)\overset{Gerretsen}{\geq}4\left(4R^2+4Rr+3r^2\right)(R-r)=4\left(4R^3-Rr^2-3r^3\right)\overset{Euler}{\geq}\frac{27R^3}{2}.$$

Aplicația77.

J.2615. If $x,y,z>0$, In ΔABC

$$\prod\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2a^4+2\right)\geq 900F^2.$$

D.M.Bătinețu-Giurgiu, Daniel Sitaru, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t\geq 0$

$$\prod\left(a^2+t\right)\geq\frac{3}{4}(a+b+c)^2t^2.$$

Hojoo Lee Inequality

Demonstrație

Folosind **Lema** pentru $(a,b,c,t)=\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)a^2,\left(\frac{y}{z+x}+\frac{y+z}{x}\right)b^2,\left(\frac{z}{x+y}+\frac{z+x}{y}\right)c^2,2\right)$

obținem:

$$LHS=\prod\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2a^4+2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum\left(\frac{x}{y+z}+\frac{x+y}{z}\right)a^2\right)^2\cdot 2^2\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{3}{4}\left(10\sqrt{3}F\right)^2\cdot 2^2=900F^2=RHS.$$

unde (1) $\Leftrightarrow$ $\sum\left(\frac{x}{y+z}+\frac{x+y}{z}\right)a^2=\sum\frac{x}{y+z}a^2+\sum\frac{x+y}{z}a^2\overset{T\sin tsifas}{\geq}2\sqrt{3}F+8\sqrt{3}F=10\sqrt{3}F.$

$$\sum \frac{x+y}{z}a^2 \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{x+y}{z}a^2} = 3\sqrt[3]{\frac{\prod(x+y)}{xyz}a^2} \overset{Cesaro}{\geq} 3\sqrt[3]{\frac{8xyz}{xyz}a^2} = 3\cdot 2(abc)^{\frac{2}{3}} \overset{Carlitz}{\geq} 6\cdot\frac{4F}{\sqrt{3}} =$$

$$=8\sqrt{3}F.$$

Am folosit mai sus $\sum \frac{x}{y+z}a^2 \geq 2\sqrt{3}F$,(G.Tsintsifas)și $\sum \frac{x+y}{z}a^2 \geq 8\sqrt{3}F$.

Remarca.

If $x, y, z, \lambda > 0$, In ΔABC

$$\prod\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2 a^4+\lambda\right) \geq (15\lambda F)^2.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)a^2, \left(\frac{y}{z+x}+\frac{y+z}{x}\right)b^2, \left(\frac{z}{x+y}+\frac{z+x}{y}\right)c^2, \lambda\right)$

obținem:

$$LHS = \prod\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2 a^4+\lambda\right) \overset{Lema}{\geq} \frac{3}{4}\left(\sum\left(\frac{x}{y+z}+\frac{x+y}{z}\right)a^2\right)^2\cdot\lambda^2 \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} \frac{3}{4}\left(10\sqrt{3}F\right)^2\cdot\lambda^2 = 225\lambda^2F^2 = RHS.$$

Aplicația78.

J.2617. If ΔABC then

$$\prod\left(r_a^2+2\right) \geq 27\sqrt{3}F.$$

D.M.Bătinețu-Giurgiu, Alecu Orlando, Romania, RMM-43, Winter-2024

Soluție

Lema

If $a,b,c,t \geq 0$

$$\prod\left(a^2+t\right) \geq \frac{3}{4}(a+b+c)^2 t^2.$$

Hojoo Lee Inequality

Folosind **Lema** pentru $(a,b,c,t)=(r_a,r_b,r_c,2)$ obținem:

$$LHS=\prod\left(r_a^2+2\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum r_a\right)^2\cdot 2^2=\frac{3}{4}(4R+r)^2\cdot 2^2\overset{(1)}{\geq}27\sqrt{3}F=RHS,$$

unde (1) $\Leftrightarrow\ \frac{3}{4}(4R+r)^2\cdot 2^2\geq 27\sqrt{3}F\Leftrightarrow(4R+r)^2\geq 9\sqrt{3}pr$,vezi $p\leq\frac{3R\sqrt{3}}{2}$.

Rămâne să arătăm că:

$$(4R+r)^2\geq 9\sqrt{3}\cdot\frac{3R\sqrt{3}}{2}\cdot r\Leftrightarrow 2(4R+r)^2\geq 81Rr\Leftrightarrow 32R^2-65Rr+2r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(32R-r)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Remarca.

In ΔABC then

$$\prod\left(r_a^2+\lambda\right)\geq\frac{27\lambda^2\sqrt{3}F}{4},\lambda>0.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=(r_a,r_b,r_c,\lambda)$ obținem:

$$LHS=\prod\left(r_a^2+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum r_a\right)^2\cdot\lambda^2=\frac{3}{4}(4R+r)^2\cdot\lambda^2\overset{(1)}{\geq}\frac{3}{4}\cdot 9\sqrt{3}pr\cdot\lambda^2=\frac{27\lambda^2\sqrt{3}F}{4}=RHS,$$

unde (1) $\Leftrightarrow\ (4R+r)^2\geq 9\sqrt{3}pr$ care rezultă din inegalitatea lui Mitrinovic $p\leq\frac{3R\sqrt{3}}{2}$.

Rămâne să arătăm că:

$$(4R+r)^2\geq 9\sqrt{3}\cdot\frac{3R\sqrt{3}}{2}\cdot r\Leftrightarrow 2(4R+r)^2\geq 81Rr\Leftrightarrow 32R^2-65Rr+2r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(32R-r)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Remarca.

If ΔABC then

$$\prod\left(h_a^2+\lambda\right)\geq\frac{243\lambda^2r^2}{4},\lambda>0.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t)=(h_a,h_b,h_c,\lambda)$ obținem:

$$LHS = \prod\left(h_a^2+\lambda\right) \overset{Lema}{\geq} \frac{3}{4}\left(\sum h_a\right)^2 \cdot \lambda^2 \overset{(1)}{\geq} \frac{3}{4}(9r)^2 \cdot \lambda^2 = \frac{243\lambda^2 r^2}{4} = RHS, (1) \Leftrightarrow \sum h_a \geq 9r.$$

Remarca.

If ΔABC then

$$\prod\left(w_a^2+\lambda\right) \geq \frac{243\lambda^2 r^2}{4}, \lambda > 0.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t) = (w_a, w_b, w_c, \lambda)$ obținem:

$$LHS = \prod\left(w_a^2+\lambda\right) \overset{Lema}{\geq} \frac{3}{4}\left(\sum w_a\right)^2 \cdot \lambda^2 \overset{(1)}{\geq} \frac{3}{4}(9r)^2 \cdot \lambda^2 = \frac{243\lambda^2 r^2}{4} = RHS,$$

unde (1) $\Leftrightarrow \sum w_a \geq 9r$.

Remarca.

If ΔABC then

$$\prod\left(m_a^2+\lambda\right) \geq \frac{243\lambda^2 r^2}{4}, \lambda > 0.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t) = (m_a, m_b, m_c, \lambda)$ obținem:

$$LHS = \prod\left(m_a^2+\lambda\right) \overset{Lema}{\geq} \frac{3}{4}\left(\sum m_a\right)^2 \cdot \lambda^2 \overset{(1)}{\geq} \frac{3}{4}(9r)^2 \cdot \lambda^2 = \frac{243\lambda^2 r^2}{4} = RHS,$$

unde (1) $\Leftrightarrow \sum m_a \geq 9r$.

Remarca.

If ΔABC then

$$\prod\left(s_a^2+\lambda\right) \geq \frac{243\lambda^2 r^2}{4}, \lambda > 0.$$

Marin Chirciu

Soluție

Folosind **Lema** pentru $(a,b,c,t) = (s_a, s_b, s_c, \lambda)$ obținem:

$$LHS=\prod\left(s_a^2+\lambda\right)\overset{Lema}{\geq}\frac{3}{4}\left(\sum s_a\right)^2\cdot\lambda^2\overset{(1)}{\geq}\frac{3}{4}(9r)^2\cdot\lambda^2=\frac{243\lambda^2r^2}{4}=RHS,$$

unde (1) $\Leftrightarrow$ $\sum s_a\geq 9r$.

Aplicația79.

J648. If $a,b,c>0$, $ab+bc+ca=1$ then

$$a+b+c+3abc\geq\frac{4}{a+b+c}.$$

Marius Stănean, Zalău, Romania, Mathematical Reflections 6/2023

Solution

Using the identity in the triangle $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$ substitution can be made

$$(a,b,c)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right).$$

It is enough to show that:

$$\sum\tan\frac{A}{2}+3\prod\tan\frac{A}{2}\geq\frac{4}{\sum\tan\frac{A}{2}}.$$

Using the identitities in the triangle $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ and $\prod\tan\frac{A}{2}=\frac{r}{p}$, the above inequality is written:

$$\frac{4R+r}{p}+3\frac{r}{p}\geq\frac{4}{\frac{4R+r}{p}}\Leftrightarrow\frac{4R+4r}{p}\geq\frac{4p}{4R+r}\Leftrightarrow\frac{R+r}{p}\geq\frac{p}{4R+r}\Leftrightarrow(4R+r)(R+r)\geq p^2,$$

which results from Gerretsen Inequality: $p^2\leq 4R^2+4Rr+3r^2$.

It remains to show that:

$(4R+r)(R+r)\geq 4R^2+4Rr+3r^2\Leftrightarrow R\geq 2r$,(Euler Inequality).

Equality occurs if and only if $a=b=c=\frac{1}{\sqrt{3}}$.

Remark.

The problem can develop.

If $x,y,z>0$, $xy+yz+zx=1$ and $\lambda\leq 3$ then

$$x+y+z+\lambda xyz \geq \frac{\lambda+9}{3(x+y+z)}.$$

Marin Chirciu

Aplicația80.

J.2607. If $x, y, z > 0$ then in ΔABC

$$\sum \frac{x}{y+z} m_a^2 \geq \frac{16Fp-27R^3}{8R}.$$

Mehmet Şahin, Turkey, RMM-43, Winter-2024

Soluție.

Lemă.

Fie $x, y, z > 0$ și $f : \mathrm{D} \to \boldsymbol{R}$ o funcție . Are loc relația

$$\sum \frac{x}{y+z} f^2(t) \geq \frac{1}{2}\left(\sum f(t)\right)^2 - \sum f^2(t).$$

Demonstrație.

Avem $\sum \frac{x}{y+z} f^2(t) = \sum\left(\frac{x}{y+z}+1-1\right) f^2(t) = \sum \frac{x+y+z}{y+z} f^2(t) - \sum f^2(t) \overset{CS}{\geq}$

$$\overset{CS}{\geq}(x+y+z)\frac{\left(\sum f(t)\right)^2}{\sum(y+z)} - \sum f^2(t) = (x+y+z)\frac{\left(\sum f(t)\right)^2}{2(x+y+z)} = \frac{1}{2}\left(\sum f(t)\right)^2 - \sum f^2(t).$$

Folosind **Lema** pentru $f(a) = m_a$ obținem:

$$\sum \frac{x}{y+z} m_a^2 \geq \frac{1}{2}\left(\sum m_a\right)^2 - \sum m_a^2 = \frac{1}{2}\left(\sum m_a^2 + 2\sum m_b m_c\right) - \sum m_a^2 =$$

$$= \sum m_b m_c - \frac{1}{2}\sum m_a^2 \geq p^2 - \frac{27R^2}{8} \overset{Euler}{\geq} \frac{16Fp-27R^3}{8R} = RHS.$$

Am folosit mai sus $\sum m_b m_c \geq p^2$ și $\sum m_a^2 \leq \frac{27R^2}{4}$.

Remarca.

Problema se poate întări.

If $x, y, z > 0$ then in ΔABC

$$\sum \frac{x}{y+z} m_a^2 \geq p^2 - \frac{27R^2}{8}.$$

Remarca.

If $x, y, z > 0$ then in ΔABC

1). $\sum \frac{x}{y+z} h_a^2 \ge \left(\frac{2r}{R} - \frac{1}{2}\right) p^2$.

2). $\sum \frac{x}{y+z} w_a^2 \ge 27r^2 - \frac{1}{2} p^2$.

3). $\sum \frac{x}{y+z} s_a^2 \ge 27r^2 - \frac{27R^2}{8}$.

4). $\sum \frac{x}{y+z} r_a^2 \ge 54r^2 - \frac{81}{8} R^2$.

Marin Chirciu

Aplicația81.

J.2608. In ΔABC

$$\sum \frac{m_a^5}{(h_b w_c)^3} \ge \frac{64r^5}{R^6}.$$

Zaza Mzhavanadze, Georgia, RMM-43, Winter-2024

Soluție.

$$LHS = \sum \frac{m_a^5}{(h_b w_c)^3} \overset{m_a \ge w_a \ge h_a}{\ge} \sum \frac{m_a^5}{(m_b m_c)^3} = \frac{\sum m_a^8}{(m_a m_b m_c)^3} \overset{m_a m_b m_c \le \frac{Rp^2}{2}}{\ge} \frac{\sum m_a^8}{\left(\frac{Rp^2}{2}\right)^3} \overset{CS}{\ge} \frac{\frac{\left(\sum m_a^2\right)^4}{27}}{\frac{R^3 p^6}{8}} \overset{\sum m_a^2 \ge p^2}{\ge}$$

$$\overset{\sum m_a^2 \ge p^2}{\ge} \frac{\frac{\left(p^2\right)^4}{27}}{\frac{R^3 p^6}{8}} = \frac{8p^2}{27R^3} \overset{p^2 \ge 27r^2}{\ge} \frac{8 \cdot 27r^2}{27R^3} = \frac{8r^2}{R^3} \overset{R \ge 2r}{\ge} \frac{64r^5}{R^6} = RHS.$$

Remarca.

In ΔABC

$$\sum \frac{m_a^{2k+3}}{(h_b w_c)^{2k+1}} \ge 27r^2 \left(\frac{2}{3R}\right)^{2k+1}, k \in \mathbf{N}.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{m_a^{2k+3}}{(h_bw_c)^{2k+1}}\overset{m_a\ge w_a\ge h_a}{\ge}\sum\frac{m_a^{2k+3}}{(m_bm_c)^{2k+1}}=\frac{\sum m_a^{4k+4}}{(m_am_bm_c)^{2k+1}}\overset{m_am_bm_c\le\frac{Rp^2}{2}}{\ge}\frac{\sum m_a^{4k+4}}{\left(\frac{Rp^2}{2}\right)^{2k+1}}\overset{CS}{\ge}$$

$$\overset{CS}{\ge}\frac{\frac{\left(\sum m_a^2\right)^{2k+2}}{3^{2k+1}}}{\frac{R^{2k+1}p^{4k+2}}{2^{2k+1}}}\overset{\sum m_a^2\ge p^2}{\ge}\overset{\sum m_a^2\ge p^2}{\ge}\frac{\frac{\left(p^2\right)^{2k+2}}{3^{2k+1}}}{\frac{R^{2k+1}p^{4k+2}}{2^{2k+1}}}=\frac{\frac{p^{4k+4}}{3^{2k+1}}}{\frac{R^{2k+1}p^{4k+2}}{2^{2k+1}}}=\left(\frac{2}{3R}\right)^{2k+1}p^2\overset{p^2\ge 27r^2}{\ge}\left(\frac{2}{3R}\right)^{2k+1}\cdot 27r^2=$$

$$=27r^2\left(\frac{2}{3R}\right)^{2k+1}=RHS.$$

Aplicația82.

J.2600. If $x,y,z>0$, In ΔABC

$$\sum\frac{x^2b\cdot m_a}{yz\sqrt{m_bm_c}}\ge 2\sqrt[4]{27}\sqrt{F}.$$

D.M.Bătinețu-Giurgiu, Romania, RMM-43, Winter-2024

Soluție.

$$LHS=\sum\frac{x^2b\cdot m_a}{yz\sqrt{m_bm_c}}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{x^2b\cdot m_a}{yz\sqrt{m_bm_c}}}=3\sqrt[3]{\prod\frac{x^2}{yz}\prod\frac{m_a}{\sqrt{m_bm_c}}\prod b}=3\sqrt[3]{1\cdot 1\cdot abc}=$$

$$=3(abc)^{\frac{1}{3}}\overset{Carlitz}{\ge}3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{1}{3}}=3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{1}{2}}=2\sqrt[4]{27}\sqrt{F}=RHS.$$

Am folosit mai sus inegalitatea lui Carlitz: $abc\overset{Carlitz}{\ge}\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}}$.

Remarca.

If $x,y,z>0$, In ΔABC

$$\sum\frac{x^2a^2\cdot m_a}{yz\sqrt{m_bm_c}}\ge 4\sqrt{3}F.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{x^2a^2\cdot m_a}{yz\sqrt{m_bm_c}}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^2a^2\cdot m_a}{yz\sqrt{m_bm_c}}}=3\sqrt[3]{\prod\frac{x^2}{yz}\prod\frac{m_a}{\sqrt{m_bm_c}}\prod a^2}=3\sqrt[3]{1\cdot1\cdot(abc)^2}=$$

$$=3(abc)^{\frac{2}{3}}\overset{Carlitz}{\geq}3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{2}{3}}=3\left(\frac{4F}{\sqrt{3}}\right)=4\sqrt{3}F=RHS.$$

Remarca.

If $x,y,z,k>0$, In ΔABC

$$\sum\frac{x^2a^k\cdot m_a}{yz\sqrt{m_bm_c}}\geq3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{k}{2}}.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{x^2a^k\cdot m_a}{yz\sqrt{m_bm_c}}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^2a^k\cdot m_a}{yz\sqrt{m_bm_c}}}=3\sqrt[3]{\prod\frac{x^2}{yz}\prod\frac{m_a}{\sqrt{m_bm_c}}\prod a^k}=3\sqrt[3]{1\cdot1\cdot(abc)^k}=$$

$$=3(abc)^{\frac{k}{3}}\overset{Carlitz}{\geq}3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{3}{2}\cdot\frac{k}{3}}=3\left(\frac{4F}{\sqrt{3}}\right)^{\frac{k}{2}}=RHS.$$

Remarca.

If $x,y,z,k>0$, In ΔABC

$$\sum\frac{x^2(b+c)^k\cdot m_a}{yz\sqrt{m_bm_c}}\geq\left(4\sqrt{3}r\right)^k.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{x^2(b+c)^k\cdot m_a}{yz\sqrt{m_bm_c}}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x^2(b+c)^k\cdot m_a}{yz\sqrt{m_bm_c}}}=3\sqrt[3]{\prod\frac{x^2}{yz}\prod\frac{m_a}{\sqrt{m_bm_c}}\prod(b+c)^k}=$$

$$=3\sqrt[3]{1\cdot1\cdot\prod(b+c)^k}=3\sqrt[3]{\prod(b+c)^k}\overset{(1)}{\geq}\left(4\sqrt{3}r\right)^k=RHS,$$

Am folosit mai sus unde (1) $\Leftrightarrow\sqrt[3]{\prod(b+c)}\geq4\sqrt{3}r$, vezi:

$$\prod(b+c)=2p\left(p^2+r^2+2Rr\right)\overset{Gerretsen}{\geq}2p\left(16Rr-5r^2+r^2+2Rr\right)=$$

$$=2p\left(18Rr-4r^2\right)\overset{Euler}{\geq}64r^2p\overset{Mitrinovic}{\geq}64r^2\cdot3\sqrt{3}r=\left(4\sqrt{3}r\right)^3.$$

Aplicația83.

J.2601. If $x, y, z > 0$, In ΔABC

$$\sum \frac{1}{w_a^2} \frac{y+z}{x} \geq \frac{18}{p^2}.$$

Mehmet Șahin, Turkey, RMM-43, Winter-2024

Soluție.

$$LHS = \sum \frac{1}{w_a^2} \frac{y+z}{x} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{1}{w_a^2} \frac{y+z}{x}} = 3\sqrt[3]{\prod \frac{y+z}{x} \prod \frac{1}{w_a^2}} \overset{Cesaro}{\geq} 3\sqrt[3]{8\prod \frac{1}{w_a^2}} =$$

$$= \frac{6}{\sqrt[3]{\prod w_a^2}} \overset{(1)}{\geq} \frac{6}{\frac{p^2}{3}} = \frac{18}{p^2} = RHS,$$

unde (1) $\Leftrightarrow \sqrt[3]{\prod w_a^2} \leq \frac{p^2}{3}$, vezi $\prod w_a^2 \leq \prod r_a^2 = \left(rp^2\right)^2 = r^2 p^4 \overset{Mitrinovic}{\leq} \frac{p^2}{27} \cdot p^4 = \frac{p^6}{27}$.

Remarca.

If $x, y, z > 0$, In ΔABC

1). $\sum \frac{1}{h_a^2} \frac{y+z}{x} \geq \frac{18}{p^2}$.

2). $\sum \frac{1}{m_a^2} \frac{y+z}{x} \geq \frac{8}{3R^2}$.

3). $\sum \frac{1}{s_a^2} \frac{y+z}{x} \geq \frac{8}{3R^2}$.

4). $\sum \frac{1}{r_a^2} \frac{y+z}{x} \geq \frac{18}{p^2}$.

Marin Chirciu

Aplicația84.

In ΔABC, I -incenter

$$3\sqrt{3} \leq \sum \frac{BC}{IA} \leq \frac{3R}{2r^2}\sqrt{R^2 - r^2}.$$

George Apostolopoulos, Greece, Mathematical Inequalities, 11/2023

Soluție.

Lema.

In ΔABC:

$$\sum \frac{BC}{IA} = \frac{1}{r}\sum a\sin\frac{A}{2}.$$

Inegalitatea din dreapta.

$$\sum \frac{BC}{IA} = \frac{1}{r}\sum a\sin\frac{A}{2} \overset{CBS}{\leq} \frac{1}{r}\sqrt{\sum a^2 \sum \sin^2\frac{A}{2}} \overset{Leibniz}{\leq} \frac{1}{r}\sqrt{9R^2\left(1-\frac{r}{2R}\right)} =$$

$$= \frac{3R}{r}\sqrt{1-\frac{r}{2R}} \overset{Euler}{\leq} \frac{3R}{2r^2}\sqrt{R^2-r^2}.$$

Inegalitatea din stânga.

$$\sum \frac{BC}{IA} = \frac{1}{r}\sum a\sin\frac{A}{2} \overset{AM-HM}{\geq} \frac{1}{r}\cdot 3\sqrt[3]{\prod a\sin\frac{A}{2}} = \frac{3}{r}\sqrt[3]{abc\prod \sin\frac{A}{2}} = \frac{3}{r}\sqrt[3]{4Rrp\frac{r}{4R}} =$$

$$= \frac{3}{r}\sqrt[3]{r^2 p} \overset{Mitrinovic}{\geq} \frac{3}{r}\sqrt[3]{r^2\cdot 3\sqrt{3}r} = \frac{3}{r}\sqrt[3]{\left(\sqrt{3}r\right)^3} = \frac{3}{r}\cdot\sqrt{3}r = 3\sqrt{3}.$$

Remarca.

In ΔABC, I -incenter

$$6\sqrt{3}\left(\frac{2r}{R}\right)^{\frac{1}{2}} \leq \sum \frac{b+c}{IA} \leq \frac{6R}{r}\sqrt{1-\frac{r}{2R}}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC:

$$\sum \frac{b+c}{IA} = \frac{1}{r}\sum (b+c)\sin\frac{A}{2}.$$

Inegalitatea din dreapta.

$$\sum \frac{b+c}{IA} = \frac{1}{r}\sum (b+c)\sin\frac{A}{2} \overset{CBS}{\leq} \frac{1}{r}\sqrt{\sum (b+c)^2 \sum \sin^2\frac{A}{2}} \overset{(1)}{\leq} \frac{1}{r}\sqrt{36R^2\left(1-\frac{r}{2R}\right)} =$$

$= \frac{6R}{r}\sqrt{1-\frac{r}{2R}}$, unde (1) $\Leftrightarrow$ $\sum (b+c)^2 \leq 36R^2$ și $\sum \sin\frac{A}{2} = 1-\frac{r}{2R}$, vezi:

$$\sum (b+c)^2 = 2\left(3p^2 - r^2 - 4Rr\right) \overset{Gerretsen}{\leq} 2\left(3\left(4R^2+4Rr+3r^2\right) - r^2 - 4Rr\right) =$$

$$= 8\left(3R^2+2Rr+2r^2\right) \overset{Euler}{\leq} 8\cdot\frac{9R^2}{2} = 36R^2.$$

Inegalitatea din stânga.

$$\sum\frac{b+c}{IA}=\frac{1}{r}\sum(b+c)\sin\frac{A}{2}\overset{AM-HM}{\geq}\frac{1}{r}\cdot 3\sqrt[3]{\prod(b+c)\sin\frac{A}{2}}=\frac{3}{r}\sqrt[3]{\prod(b+c)\prod\sin\frac{A}{2}}\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{3}{r}\sqrt[3]{\left(4\sqrt{3}r\right)^3\frac{r}{4R}}=\frac{3}{r}\cdot 4\sqrt{3}r\cdot\sqrt[3]{\frac{r}{4R}}=\frac{3}{r}\cdot 4\sqrt{3}r\cdot\frac{1}{2}\sqrt[3]{\frac{2r}{R}}=6\sqrt{3}\left(\frac{2r}{R}\right)^{\frac{1}{2}},$$

unde (2) $\Leftrightarrow \prod(b+c)\geq\left(4\sqrt{3}r\right)^3$, vezi:

$$\prod(b+c)=2p\left(p^2+r^2+2Rr\right)\overset{Gerretsen}{\geq}2p\left(16Rr-5r^2+r^2+2Rr\right)=$$

$$=2p\left(18Rr-4r^2\right)\overset{Euler}{\geq}64r^2p\overset{Mitrinovic}{\geq}64r^2\cdot 3\sqrt{3}r=\left(4\sqrt{3}r\right)^3.$$

Aplicația85.

In ΔABC

$$\sum\frac{a^3}{b^2+c^2-a^2+2bc}\geq 4\sqrt{3}\frac{r^2}{R}.$$

Konstantinos Geronikolas, Greece, RMM, 11/2023

Soluție.

Lema.

In ΔABC:

$$\sum\frac{a^3}{b^2+c^2-a^2+2bc}=\frac{p^2(2R-3r)+r^2(4R+r)}{2pr}.$$

$$RHS=\sum\frac{a^3}{b^2+c^2-a^2+2bc}\overset{Lema}{=}\frac{p^2(2R-3r)+r^2(4R+r)}{2pr}\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{\left(16Rr-5r^2\right)(2R-3r)+r^2(4R+r)}{2pr}=\frac{16R^2-27Rr+8r^2}{p}\overset{Euler}{\geq}\frac{9R^2}{2p}\overset{Mitrinovic}{\geq}\frac{9R^2}{2\cdot\frac{3\sqrt{3}R}{2}}=$$

$$=\sqrt{3}R\overset{Euler}{\geq}4\sqrt{3}\frac{r^2}{R}=LHS.$$

Remarca.

In ΔABC

$$\sqrt{3}R\leq\sum\frac{a^3}{(b+c)^2-a^2}\leq\frac{2\sqrt{3}\left(2R^3-7r^3\right)}{9r^2}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC:

$$\sum \frac{a^3}{(b+c)^2-a^2}=\frac{p^2(2R-3r)+r^2(4R+r)}{2pr}.$$

Inegalitatea din dreapta.

$$\sum \frac{a^3}{(b+c)^2-a^2}\overset{Lema}{=}\frac{p^2(2R-3r)+r^2(4R+r)}{2pr}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)(2R-3r)+r^2(4R+r)}{2pr}=\frac{4R^3-2R^2r-Rr^2-4r^3}{pr}\overset{Euler}{\leq}$$

$$\overset{Euler}{\leq}\frac{2(2R^3-7r^2)}{pr}\overset{Mitrinovic}{\leq}\frac{2(2R^3-7r^2)}{r\cdot 3\sqrt{3}r}=\frac{2\sqrt{3}(2R^3-7r^3)}{9r^2}.$$

Inegalitatea din stânga.

$$\sum \frac{a^3}{(b+c)^2-a^2}\overset{Lema}{=}\frac{p^2(2R-3r)+r^2(4R+r)}{2pr}\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(2R-3r)+r^2(4R+r)}{2pr}=$$

$$=\frac{16R^2-27Rr+8r^2}{p}\overset{Euler}{\geq}\frac{9R^2}{2p}\overset{Mitrinovic}{\geq}\frac{9R^2}{2\cdot\frac{3\sqrt{3}R}{2}}=\sqrt{3}R.$$

Aplicația86.

J.2592. In ΔABC

$$\sum(2b+a)(2c+a)\leq 81R^2$$

Daniel Sitaru, Romania, RMM-43, Winter-2024

Soluție.

Lema.

In ΔABC

$$\sum(2b+a)(2c+a)=2(5p^2+3r^2+12Rr).$$

$$LHS=\sum(2b+a)(2c+a)=2(5p^2+3r^2+12Rr)\overset{Gerretsen}{\leq}2(5(4R^2+4Rr+3r^2)+3r^2+12Rr)=$$

$$=4\left(10R^2+16Rr+9r^2\right)\overset{Euler}{\leq}81R^2=RHS.$$

Remarca.

In ΔABC

$$\sum(2b+a)(2c+a)\geq 324r^2$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum(2b+a)(2c+a)=2\left(5p^2+3r^2+12Rr\right).$$

$$LHS=\sum(2b+a)(2c+a)=2\left(5p^2+3r^2+12Rr\right)\overset{Gerretsen}{\geq}2\left(5\left(16Rr-5r^2\right)+3r^2+12Rr\right)=$$

$$=4\left(46Rr-11r^2\right)\overset{Euler}{\geq}324r^2=RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$324r^2\leq\sum(2b+a)(2c+a)\leq 81R^2$$

Remarca.

If $\lambda\geq 0$, In ΔABC

$$36(\lambda+1)^2r^2\leq\sum(\lambda b+a)(\lambda c+a)\leq 9(\lambda+1)^2R^2.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum(\lambda b+a)(\lambda c+a)=\left(\lambda^2+2\lambda+2\right)p^2+\left(\lambda^2+2\lambda-2\right)r^2+\left(4\lambda^2+8\lambda-8\right).$$

Inegalitatea din dreapta.

$$\sum(\lambda b+a)(\lambda c+a)=\left(\lambda^2+2\lambda+2\right)p^2+\left(\lambda^2+2\lambda-2\right)r^2+\left(4\lambda^2+8\lambda-8\right)\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\left(\lambda^2+2\lambda+2\right)\left(4R^2+4Rr+3r^2\right)+\left(\lambda^2+2\lambda-2\right)r^2+\left(4\lambda^2+8\lambda-8\right)=$$

$$=4\left(\left(2\lambda^2+2\lambda+2\right)R^2+\left(8\lambda^2+16\lambda\right)Rr+\left(4\lambda^2+8\lambda+4\right)r^2\right)\overset{Euler}{\leq}9\left(\lambda+1\right)^2R^2.$$

<u>Inegalitatea din stânga.</u>

$$\sum(\lambda b+a)(\lambda c+a)=\left(\lambda^2+2\lambda+2\right)p^2+\left(\lambda^2+2\lambda-2\right)r^2+\left(4\lambda^2+8\lambda-8\right)\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\left(\lambda^2+2\lambda+2\right)\left(16Rr-5r^2\right)+\left(\lambda^2+2\lambda-2\right)r^2+\left(4\lambda^2+8\lambda-8\right)=$$

$$=\left(20\lambda^2+40\lambda+24\right)Rr-\left(4\lambda^2+8\lambda+12\right)r^2\overset{Euler}{\geq}36\left(\lambda+1\right)^2r^2.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

<u>Aplicația87.</u>

In ΔABC:

$$2Rr\leq\sum\left(\frac{m_a}{3\cos\frac{A}{2}}\right)^2.$$

Neculai Stanciu,RMM, 11/2023

<u>Soluție.</u>

<u>Lema.</u>

In ΔABC:

$$\frac{m_a}{\cos\frac{A}{2}}\geq\sqrt{bc}.$$

$$RHS=\sum\left(\frac{m_a}{3\cos\frac{A}{2}}\right)^2\overset{Lema}{\geq}\sum\left(\frac{\sqrt{bc}}{3}\right)^2=\frac{1}{9}\sum bc\overset{(1)}{\geq}\frac{1}{9}\cdot18Rr=2Rr=LHS\text{ ,(1)}\Leftrightarrow\sum bc\geq18Rr.$$

<u>Remarca.</u>

In ΔABC:

$$\sum\left(\frac{m_a}{\lambda\cos\frac{A}{2}}\right)^2\geq\frac{18}{\lambda^2}Rr.$$

Marin Chirciu

<u>Aplicația88.</u>

In acute ΔABC :

$$\sum h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)\geq 6.$$

Lan Tran, Vietnam, Mathematical inequalities, 11/2023

Soluție.

Lema.

In acute ΔABC :

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2};$$

$$\frac{h_b^2+h_c^2}{m_a^2}\geq 8\sin^2\frac{A}{2}.$$

Demonstrație.

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}\Leftrightarrow m_a^2\leq\frac{b^2+c^2}{2}\cos^2\frac{A}{2}\Leftrightarrow\frac{2b^2+2c^2-a^2}{4}\leq\frac{b^2+c^2}{2}\cdot\frac{p(p-a)}{bc}\Leftrightarrow$$

$\Leftrightarrow\dfrac{2b^2+2c^2-a^2}{4}\leq\dfrac{b^2+c^2}{2}\cdot\dfrac{(a+b+c)(b+c-a)}{4bc}\Leftrightarrow(b^2+c^2-a^2)(b-c)^2\geq 0$, care rezultă din $(b^2+c^2-a^2)>0$, in acute triangle și $(b-c)^2\geq 0$, cu egalitate pentru $b=c$.

Din $h_a=\dfrac{bc}{2R}\Rightarrow h_b^2+h_c^2=\dfrac{a^2}{4R^2}(b^2+c^2)\Rightarrow b^2+c^2=\dfrac{4R^2}{a^2}(h_b^2+h_c^2)$.

Folosind **Lema** obținem:

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}=\sqrt{\frac{b^2+c^2}{2}}\cdot\frac{a}{4R\sin\frac{A}{2}}=\sqrt{\frac{\frac{4R^2}{a^2}(h_b^2+h_c^2)}{2}}\cdot\frac{a}{4R\sin\frac{A}{2}}=$$

$$=\sqrt{\frac{2R^2(h_b^2+h_c^2)}{a^2}}\cdot\frac{a}{4R\sin\frac{A}{2}}=\sqrt{2(h_b^2+h_c^2)}\cdot\frac{1}{4\sin\frac{A}{2}}.$$

Din $m_a\leq\sqrt{2(h_b^2+h_c^2)}\cdot\dfrac{1}{4\sin\frac{A}{2}}\Rightarrow m_a^2\leq 2(h_b^2+h_c^2)\cdot\dfrac{1}{16\sin^2\frac{A}{2}}\Rightarrow m_a^2\leq\dfrac{h_b^2+h_c^2}{8\sin^2\frac{A}{2}}\Rightarrow$

$$\Rightarrow h_b^2+h_c^2\geq 8m_a^2\sin^2\frac{A}{2}\Rightarrow\frac{h_b^2+h_c^2}{m_a^2}\geq 8\sin^2\frac{A}{2}.$$

$$\sum h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)=\sum\frac{h_b^2+h_c^2}{m_a^2}\geq\sum 8\sin^2\frac{A}{2}=8\sum\sin^2\frac{A}{2}=8\cdot\left(1-\frac{r}{2R}\right)\overset{Euler}{\geq}8\cdot\frac{3}{4}=6.$$

Remarca.

In acute ΔABC:

$$\sum\frac{1}{m_a^2}\left(h_b^2+h_c^2\right)\geq 8-\frac{4r}{R}.$$

Marin Chirciu

Soluție.

Lema.

In acute ΔABC:

$$\frac{h_b^2+h_c^2}{m_a^2}\geq 8\sin^2\frac{A}{2}.$$

Demonstrație.

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}\Leftrightarrow m_a^2\leq\frac{b^2+c^2}{2}\cos^2\frac{A}{2}\Leftrightarrow\frac{2b^2+2c^2-a^2}{4}\leq\frac{b^2+c^2}{2}\cdot\frac{p(p-a)}{bc}\Leftrightarrow$$

$\Leftrightarrow\dfrac{2b^2+2c^2-a^2}{4}\leq\dfrac{b^2+c^2}{2}\cdot\dfrac{(a+b+c)(b+c-a)}{4bc}\Leftrightarrow\left(b^2+c^2-a^2\right)(b-c)^2\geq 0$, care rezultă din $\left(b^2+c^2-a^2\right)>0$, in acute triangle și $(b-c)^2\geq 0$, cu egalitate pentru $b=c$.

Remarca.

In acute ΔABC:

$$\sum m_a\sec\frac{A}{2}\leq 3\sqrt{3}R.$$

Marin Chirciu

Soluție.

Lema.

In acute ΔABC:

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2};$$

Demonstrație.

$$m_a\leq\sqrt{\frac{b^2+c^2}{2}}\cos\frac{A}{2}\Leftrightarrow m_a^2\leq\frac{b^2+c^2}{2}\cos^2\frac{A}{2}\Leftrightarrow\frac{2b^2+2c^2-a^2}{4}\leq\frac{b^2+c^2}{2}\cdot\frac{p(p-a)}{bc}\Leftrightarrow$$

$\Leftrightarrow \frac{2b^2+2c^2-a^2}{4} \le \frac{b^2+c^2}{2} \cdot \frac{(a+b+c)(b+c-a)}{4bc} \Leftrightarrow (b^2+c^2-a^2)(b-c)^2 \ge 0$, care rezultă din $(b^2+c^2-a^2) > 0$, in acute triangle și $(b-c)^2 \ge 0$, cu egalitate pentru $b=c$.

Aplicația89.

J.2590. In ΔABC

$$\sqrt[3]{(a+b)(b+c)(c+a)} \ge 4\sqrt{3}r$$

Daniel Sitaru, Dan Nănuți, Romania, RMM-43, Winter-2024

Soluție.

Lema.

In ΔABC

$$(a+b)(b+c)(c+a) = 2p\left(p^2+r^2+2Rr\right).$$

Inegalitatea se scrie:

$2p\left(p^2+r^2+2Rr\right) \ge \left(4\sqrt{3}r\right)^3 \Leftrightarrow 2p\left(p^2+r^2+2Rr\right) \ge 64 \cdot 3\sqrt{3}r^3$,vezi $p \ge 3\sqrt{3}r$.

Rămâne să arătăm că:

$2 \cdot 3\sqrt{3}r\left(p^2+r^2+2Rr\right) \ge 64 \cdot 3\sqrt{3}r^3 \Leftrightarrow p^2+r^2+2Rr \ge 32r^2$, vezi $p^2 \ge 16Rr-5r^2$.

Remarca.

In ΔABC

$$\sqrt[3]{(a+b)(b+c)(c+a)} \le 2\sqrt{3}R$$

Marin Chirciu

Inegalitatea se scrie:

$2p\left(p^2+r^2+2Rr\right) \ge \left(2\sqrt{3}R\right)^3 \Leftrightarrow 2p\left(p^2+r^2+2Rr\right) \ge 8 \cdot 3\sqrt{3}R^3$,vezi $p \le \frac{3\sqrt{3}R}{2}$.

Remarca.

In ΔABC

$$4\sqrt{3}r \le \sqrt[3]{(a+b)(b+c)(c+a)} \le 2\sqrt{3}R .$$

In ΔABC

$$5\sqrt{3}r \le \sqrt[3]{(p+a)(p+b)(p+c)} \le \frac{5\sqrt{3}R}{2}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod(p+a) = p\left(4p^2 + r^2 + 8Rr\right).$$

Remarca.

In ΔABC

$$8\sqrt{3}r \le \sqrt[3]{(2p+a)(2p+b)(2p+c)} \le 4\sqrt{3}R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod(2p+a) = 2p\left(9p^2 + r^2 + 6Rr\right).$$

Remarca.

In ΔABC

$$4r \le \sqrt[3]{(r+r_a)(r+r_b)(r+r_c)} \le 2R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod(r+r_a) = 2r\left(p^2 + r^2 + 2Rr\right).$$

Remarca.

In ΔABC

$$2r \le \sqrt[3]{(r_a-r)(r_b-r)(r_c-r)} \le R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod(r_a - r) = 4Rr^2.$$

Remarca.

In ΔABC

$$9r\left(\frac{2r}{R}\right)^{\frac{1}{3}} \le \sqrt[3]{(h_a + 2r_a)(h_b + 2r_b)(h_c + 2r_c)} \le \frac{9R}{2}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod(h_a + 2r_a) = \frac{2p^4}{R}.$$

Remarca.

In ΔABC

$$6r \le \sqrt[3]{(r_a + r_b)(r_b + r_c)(r_c + r_a)} \le 3R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod(r_b + r_c) = 4Rp^2.$$

Remarca.

In ΔABC

$$6r\left(\frac{2r}{R}\right)^{\frac{2}{3}} \le \sqrt[3]{(h_a + h_b)(h_b + h_c)(h_c + h_a)} \le 3R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod\left(h_b+h_c\right)=\frac{rp^2\left(p^2+r^2+2Rr\right)}{R^2}.$$

Remarca.

In ΔABC

$$6r\left(\frac{2r}{R}\right)^{\frac{1}{3}}\leq\sqrt[3]{\left(h_a+r_a\right)\left(h_b+r_b\right)\left(h_c+r_c\right)}\leq 3R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod\left(h_a+r_a\right)=\frac{p^2\left(p^2+r^2+2Rr\right)}{2R}.$$

Remarca.

In ΔABC

$$2r\left(\frac{2r}{R}\right)^{\frac{1}{3}}\leq\sqrt[3]{\left(h_a-r\right)\left(h_b-r\right)\left(h_c-r\right)}\leq R.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod\left(h_a-r\right)=\frac{r^2}{2R}\left(p^2+r^2+2Rr\right).$$

Remarca.

În aceeași clasă de probleme.

In ΔABC

$$r\left(\frac{2r}{R}\right)^{\frac{1}{3}}\leq\sqrt[3]{\left(h_a-2r\right)\left(h_b-2r\right)\left(h_c-2r\right)}\leq\frac{R}{2}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod (h_a - 2r) = \frac{2r^4}{R}.$$

Remarca.

În aceeași clasă de probleme.

In ΔABC

$$\sqrt[3]{(r_a - 2r)(r_b - 2r)(r_c - 2r)} \le r.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod (r_a - 2r) = r\left(16Rr - 4r^2 - p^2\right).$$

Remarca.

In acute ΔABC

$$\sqrt[3]{\left(b^2 + c^2 - a^2\right)\left(c^2 + a^2 - b^2\right)\left(a^2 + b^2 - c^2\right)} \le 3R^2.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\prod \left(b^2 + c^2 - a^2\right) = 32r^2 p^2 \left(p^2 - (2R + r)^2\right).$$

Aplicația90.

If $x, y, z > 0$, in ΔABC

$$\sum \left(\frac{r_a}{h_a}\right)^2 \frac{y+z}{x} \ge 6.$$

Mehmet Şahin, Turkey,RMM 4/2023

Soluție.

Lemă.

Fie $x,y,z>0$ și $f:\mathrm{D}\to \boldsymbol{R}$ o funcție . Are loc relația

$$\sum \frac{y+z}{x} f^2(a) \geq 2\sum f(b)f(c).$$

Demonstrație.

Avem $\sum \frac{y+z}{x} f^2(a) = \sum\left(\frac{y+z}{x}+1-1\right) f^2(a) = \sum \frac{x+y+z}{x} f^2(a) - \sum f^2(a) \overset{CS}{\geq}$

$$\overset{CS}{\geq}(x+y+z)\frac{\left(\sum f(a)\right)^2}{x+y+z} - \sum f^2(a) = (x+y+z)\frac{\left(\sum f(a)\right)^2}{(x+y+z)} - \sum f^2(a) =$$

$$=\left(\sum f(a)\right)^2 - \sum f^2(a) = \sum f^2(a) + 2\sum f(b)f(c) - \sum f^2(a) = 2\sum f(b)f(c).$$

SFolosind **Lema** pentru $f(a)=\frac{r_a}{h_a}$ obţinem:

$$LHS = \sum\left(\frac{r_a}{h_a}\right)^2 \frac{y+z}{x} \overset{Lema}{\geq} 2\sum f(b)f(c) = 2\sum \frac{r_b}{h_b}\cdot\frac{r_c}{h_c} = 2\sum \frac{r_b r_c}{h_b h_c} = 2\cdot\frac{p^2+r^2-8Rr}{4r^2} \overset{Gerretsen}{\geq}$$

$$\overset{Lema}{\geq} \frac{16Rr-5r^2+r^2-8Rr}{2r^2} = \frac{8Rr-4r^2}{2r^2} = \frac{2(2R-r)}{r} \overset{Euler}{\geq} 6.$$

Am folosit mai sus: $\sum \frac{r_b r_c}{h_b h_c} = \frac{p^2+r^2-8Rr}{4r^2}$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

Problema se poate întări.

If $x,y,z>0$, in ΔABC

$$\sum\left(\frac{r_a}{h_a}\right)^2 \frac{y+z}{x} \geq \frac{4R}{r} - 2.$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a)=\frac{r_a}{h_a}$ obţinem:

$$LHS = \sum\left(\frac{r_a}{h_a}\right)^2 \frac{y+z}{x} \overset{Lema}{\geq} 2\sum f(b)f(c) = 2\sum \frac{r_b}{h_b}\cdot\frac{r_c}{h_c} = 2\sum \frac{r_b r_c}{h_b h_c} = 2\cdot\frac{p^2+r^2-8Rr}{4r^2} \overset{Gerretsen}{\geq}$$

$$\overset{Lema}{\geq}\frac{16Rr-5r^2+r^2-8Rr}{2r^2}=\frac{8Rr-4r^2}{2r^2}=\frac{2(2R-r)}{r}=\frac{4R}{r}-2\overset{Euler}{\geq}6.$$

Am folosit mai sus: $\sum\frac{r_br_c}{h_bh_c}=\frac{p^2+r^2-8Rr}{4r^2}$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

If $x,y,z>0$, in ΔABC

$$\sum\left(\frac{h_a}{r_a}\right)^2\frac{y+z}{x}\geq 4\left(2-\frac{r}{R}\right).$$

Marin Chirciu

Soluție.

Folosind **Lema** pentru $f(a)=\frac{h_a}{r_a}$ obţinem:

$$LHS=\sum\left(\frac{h_a}{r_a}\right)^2\frac{y+z}{x}\overset{Lema}{\geq}2\sum f(b)f(c)=2\sum\frac{h_b}{r_b}\cdot\frac{h_c}{r_c}=2\sum\frac{h_bh_c}{r_br_c}=2\cdot 2\left(2-\frac{r}{R}\right)=$$

$=4\left(2-\frac{r}{R}\right)$. Am folosit mai sus: $\sum\frac{h_bh_c}{r_br_c}=2\left(2-\frac{r}{R}\right)$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\frac{r_br_c}{h_bh_c}\geq\sum\frac{h_bh_c}{r_br_c}.$$

Marin Chirciu

Demonstrație.

Folosind sumele $\sum\frac{r_br_c}{h_bh_c}=\frac{p^2+r^2-8Rr}{4r^2}$ şi $\sum\frac{h_bh_c}{r_br_c}=2\left(2-\frac{r}{R}\right)$ inegalitatea se scrie:

$\frac{p^2+r^2-8Rr}{4r^2}\geq 2\left(2-\frac{r}{R}\right)\Leftrightarrow\ Rp^2\geq r\left(8R^2+15Rr-8r^2\right)$, care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$. Rămâne să arătăm că:

$$R\left(16Rr-5r^2\right)\geq r\left(8R^2+15Rr-8r^2\right)\Leftrightarrow 2R^2-5Rr+2r^2\geq 0\Leftrightarrow(R-2r)(2R-r)\geq 0.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Aplicația91.

In ΔABC

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} \geq (a+b+c)^2 .$$

Marian Cucoaneş, Mărăşeşti, Mathematical Inequalities,4/2017

Soluție.

$$LHS = \sum \frac{r_a^2}{\sin^2 \frac{A}{2}} \overset{CS}{\geq} \frac{\left(\sum r_a\right)^2}{\sum \sin^2 \frac{A}{2}} = \frac{(4R+r)^2}{1-\frac{r}{2R}} \overset{Bergstrom}{\geq} 4p^2 = (a+b+c)^2 = RHS .$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca

In ΔABC

$$(4R+r)^2 \cdot \frac{R+2r}{R+r} \leq \sum \frac{r_a^2}{\sin^2 \frac{A}{2}} \leq \frac{4}{3}(4R+r)^2 .$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} = p^2 + (4R+r)^2 .$$

Inegalitatea din dreapta.

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} = p^2 + (4R+r)^2 \overset{Gerretsen}{\leq} \frac{R(4R+r)^2}{2(2R-r)} + (4R+r)^2 = (4R+r)^2 \left[\frac{R}{2(2R-r)} + 1\right] =$$

$$= (4R+r)^2 \cdot \frac{5R-2r}{2(2R-r)} \overset{Euler}{\leq} (4R+r)^2 \cdot \frac{4}{3} = \frac{4}{3}(4R+r)^2 .$$

Inegalitatea din stânga.

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} = p^2 + (4R+r)^2 \overset{Gerretsen}{\geq} \frac{r(4R+r)^2}{R+r} + (4R+r)^2 = (4R+r)^2 \left[\frac{r}{R+r} + 1\right] =$$

$$=(4R+r)^2\cdot\frac{R+2r}{R+r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca

In ΔABC

$$2R(4R+r)\le\sum\frac{r_a^2}{\cos^2\frac{A}{2}}\le 4\left(\frac{4R}{r}+1\right)\left(R^2-3r^2\right).$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\sum\frac{r_a^2}{\cos^2\frac{A}{2}}=(4R+r)\left[r-12R+\frac{(4R+r)^3}{p^2}\right].$$

Inegalitatea din dreapta.

$$\sum\frac{r_a^2}{\cos^2\frac{A}{2}}=(4R+r)\left[r-12R+\frac{(4R+r)^3}{p^2}\right]\overset{Gerretsen}{\le}(4R+r)\left[r-12R+\frac{(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=(4R+r)\left[r-12R+\frac{(4R+r)(R+r)}{r}\right]=(4R+r)\cdot\frac{4R^2-7Rr+2r^2}{r}\overset{Euler}{\le}(4R+r)\cdot\frac{4R^2-12r^2}{r}=$$

$$=4\left(\frac{4R}{r}+1\right)\left(R^2-3r^2\right).$$

Inegalitatea din stânga.

$$\sum\frac{r_a^2}{\cos^2\frac{A}{2}}=(4R+r)\left[r-12R+\frac{(4R+r)^3}{p^2}\right]\overset{Gerretsen}{\ge}(4R+r)\left[r-12R+\frac{(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=(4R+r)\left[r-12R+\frac{2(4R+r)(2R-r)}{R}\right]=(4R+r)\cdot\frac{4R^2-3Rr-2r^2}{R}\overset{Euler}{\ge}$$

$$\overset{Euler}{\ge}(4R+r)\cdot\frac{2R^2}{R}=2R(4R+r)$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} \le 3\sum \frac{r_a^2}{\cos^2 \frac{A}{2}}.$$

Marin Chirciu

Soluţie

Lema1

In ΔABC

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} = p^2 + (4R+r)^2.$$

Lema2

In ΔABC

$$\sum \frac{r_a^2}{\cos^2 \frac{A}{2}} = (4R+r)\left[r - 12R + \frac{(4R+r)^3}{p^2}\right].$$

Inegalitatea se scrie:

$$p^2 + (4R+r)^2 \le 3(4R+r)\left[r - 12R + \frac{(4R+r)^3}{p^2}\right] \Leftrightarrow$$

$\Leftrightarrow p^2\left(p^2 + 160R^2 + 32Rr - 2r^2\right) \le 3(4R+r)^4$, care rezultă din inegalitatea lui Gerretsen

$p^2 \le \frac{R(4R+r)^2}{2(2R-r)} \le 4R^2 + 4Rr + 3r^2$. Rămâne să arătăm că:

$$\frac{R(4R+r)^2}{2(2R-r)}\left(4R^2 + 4Rr + 3r^2 + 160R^2 + 32Rr - 2r^2\right) \le 3(4R+r)^4 \Leftrightarrow$$

$\Leftrightarrow 28R^3 - 36R^2r - 37Rr^2 - 6r^3 \ge 0 \Leftrightarrow (R-2r)\left(28R^2 + 20Rr + 3r^2\right) \ge 0$, vezi $R \ge 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca

In ΔABC

$$(4R+r)^2\cdot\frac{R+2r}{R+r}\le\sum\frac{h_a^2}{\sin^2\frac{A}{2}}\le\frac{4}{3}(4R+r)^2.$$

Marin Chirciu

<u>Soluție.</u>

<u>Lema</u>

In ΔABC

$$\sum\frac{h_a^2}{\sin^2\frac{A}{2}}=\frac{p^6+p^4\left(3r^2-16Rr\right)+p^2r^2\left(32R^2-8Rr+3r^2\right)+r^4\left(4R+r\right)^2}{4R^2r^2}.$$

<u>Inegalitatea din dreapta.</u>

$$\sum\frac{h_a^2}{\sin^2\frac{A}{2}}=\frac{p^6+p^4\left(3r^2-16Rr\right)+p^2r^2\left(32R^2-8Rr+3r^2\right)+r^4\left(4R+r\right)^2}{4R^2r^2}=$$

$$=\frac{p^2\left[p^2\left(p^2+3r^2-16Rr\right)+r^2\left(32R^2-8Rr+3r^2\right)\right]+r^4\left(4R+r\right)^2}{4R^2r^2}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{p^2\left[p^2\left(4R^2+4Rr+3r^2+3r^2-16Rr\right)+r^2\left(32R^2-8Rr+3r^2\right)\right]+r^4\left(4R+r\right)^2}{4R^2r^2}=$$

$$=\frac{p^2\left[p^2\left(4R^2-12Rr+6r^2\right)+r^2\left(32R^2-8Rr+3r^2\right)\right]+r^4\left(4R+r\right)^2}{4R^2r^2}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{p^2\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2-12Rr+6r^2\right)+r^2\left(32R^2-8Rr+3r^2\right)\right]+r^4\left(4R+r\right)^2}{4R^2r^2}=$$

$$=\frac{p^2\left(16R^4-32R^3r+20R^2r^2-20Rr^3+21r^4\right)+r^4\left(4R+r\right)^2}{4R^2r^2}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{\left(4R^2+4Rr+3r^2\right)\left(16R^4-32R^3r+20R^2r^2-20Rr^3+21r^4\right)+r^4\left(4R+r\right)^2}{4R^2r^2}=$$

$$=\frac{64R^6-64R^5r+64R^3r^3+240R^2r^4+12Rr^5+64r^6}{4R^2r^2}=$$

$$=\frac{16R^6-16R^5r+16R^3r^3+60R^2r^4+3Rr^5+16r^6}{R^2r^2}=$$

<u>Inegalitatea din stânga.</u>

$$\sum \frac{r_a^2}{\sin^2 \frac{A}{2}} = p^2 + (4R+r)^2 \overset{Gerretsen}{\geq} \frac{r(4R+r)^2}{R+r} + (4R+r)^2 = (4R+r)^2 \left[\frac{r}{R+r}+1\right] =$$

$$= (4R+r)^2 \cdot \frac{R+2r}{R+r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca

In ΔABC

$$2R(4R+r) \leq \sum \frac{h_a^2}{\cos^2 \frac{A}{2}} \leq 4\left(\frac{4R}{r}+1\right)\left(R^2-3r^2\right).$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\sum \frac{h_a^2}{\cos^2 \frac{A}{2}} = \frac{p^6 + p^4\left(3r^2-8Rr\right) + p^2r^2\left(-16R^2+8Rr+3r^2\right) + r^2(4R+r)^4}{4p^2R^2}.$$

Inegalitatea din dreapta.

$$\sum \frac{r_a^2}{\cos^2 \frac{A}{2}} = (4R+r)\left[r-12R+\frac{(4R+r)^3}{p^2}\right] \overset{Gerretsen}{\leq} (4R+r)\left[r-12R+\frac{(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right] =$$

$$= (4R+r)\left[r-12R+\frac{(4R+r)(R+r)}{r}\right] = (4R+r)\cdot\frac{4R^2-7Rr+2r^2}{r} \overset{Euler}{\leq} (4R+r)\cdot\frac{4R^2-12r^2}{r} =$$

$$= 4\left(\frac{4R}{r}+1\right)\left(R^2-3r^2\right).$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral

Inegalitatea din stânga.

$$\sum \frac{r_a^2}{\cos^2 \frac{A}{2}} = (4R+r)\left[r-12R+\frac{(4R+r)^3}{p^2}\right] \overset{Gerretsen}{\geq} (4R+r)\left[r-12R+\frac{(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$=(4R+r)\left[r-12R+\frac{2(4R+r)(2R-r)}{R}\right]=(4R+r)\cdot\frac{4R^2-3Rr-2r^2}{R}\overset{Euler}{\geq}$$

$$\overset{Euler}{\geq}(4R+r)\cdot\frac{2R^2}{R}=2R(4R+r).$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\frac{h_a^2}{\sin^2\frac{A}{2}}\geq 3\sum\frac{h_a^2}{\cos^2\frac{A}{2}}.$$

Marin Chirciu

Soluție

$$\frac{p^6+p^4(3r^2-16Rr)+p^2r^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{4R^2r^2}\geq$$

$$\geq 3\cdot\frac{p^6+p^4(3r^2-8Rr)+p^2r^2(-16R^2+8Rr+3r^2)+r^2(4R+r)^4}{4p^2R^2}\Leftrightarrow$$

$$\Leftrightarrow p^2\left[p^6+p^4(3r^2-16Rr)+p^2r^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2\right]\geq$$

$$\geq 3r^2\left[p^6+p^4(3r^2-8Rr)+p^2r^2(-16R^2+8Rr+3r^2)+r^2(4R+r)^4\right]\Leftrightarrow$$

$$\Leftrightarrow p^8-16p^6Rr+p^4r^2(32R^2+16Rr-6r^2)+p^2r^4(64R^2-16Rr-8r^2)\geq 3r^4(4R+r)^4\Leftrightarrow$$

$$\Leftrightarrow p^2\left[p^6-16p^4Rr+p^2r^2(32R^2+16Rr-6r^2)+r^4(64R^2-16Rr-8r^2)\right]\geq 3r^4(4R+r)^4$$

$p^2\left[p^2\left(p^2\left(p^2-16Rr\right)+r^2(32R^2+16Rr-6r^2)\right)+r^4(64R^2-16Rr-8r^2)\right]\geq 3r^4(4R+r)^4$, care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2\geq\dfrac{r(4R+r)^2}{R+r}$.

Rămâne să arătăm că:

$$\frac{r(4R+r)^2}{R+r}\left[p^2\left(p^2\left(p^2-16Rr\right)+r^2(32R^2+16Rr-6r^2)\right)+r^4(64R^2-16Rr-8r^2)\right]\geq 3r^4(4R+r)^4$$

$$(16Rr-5r^2)\left((16Rr-5r^2)(16Rr-5r^2-16Rr)+r^2(32R^2+16Rr-6r^2)\right)+r^4(64R^2-16Rr-8r^2)\geq$$

$$\geq 3r^3(4R+r)^2(R+r)\Leftrightarrow$$

$(16R-5r)\left((16R-5r)(-5r)+\left(32R^2+16Rr-6r^2\right)\right)+r\left(64R^2-16Rr-8r^2\right)\geq 3(4R+r)^2(R+r)$

$464R^3-1192R^2r+581Rr^2-106r^3\geq 0 \Leftrightarrow (R-2r)\left(464R^2-264Rr+53r^2\right)\geq 0$, vezu $R\geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\frac{h_a^2}{\sin^2\frac{A}{2}}\geq\frac{2r}{R}\sum\frac{r_a^2}{\sin^2\frac{A}{2}}.$$

Marin Chirciu

Soluție.

Folosind sumele:

$\sum\frac{h_a^2}{\sin^2\frac{A}{2}}=\frac{p^6+p^4\left(3r^2-16Rr\right)+p^2r^2\left(32R^2-8Rr+3r^2\right)+r^4(4R+r)^2}{4R^2r^2}$ și

$\sum\frac{r_a^2}{\sin^2\frac{A}{2}}=p^2+(4R+r)^2$ inegalitatea se scrie:

$\frac{p^6+p^4\left(3r^2-16Rr\right)+p^2r^2\left(32R^2-8Rr+3r^2\right)+r^4(4R+r)^2}{4R^2r^2}\geq\frac{2r}{R}\left[p^2+(4R+r)^2\right]\Leftrightarrow$

$\Leftrightarrow p^2\left[p^2\left(p^2+3r^2-16Rr\right)+r^2\left(32R^2-16Rr+3r^2\right)\right]\geq r^3(4R+r)^2(8R-r)$,

care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}$.

Rămâne să arătăm că:

$\frac{r(4R+r)^2}{R+r}\left[\left(16Rr-5r^2\right)\left(16Rr-5r^2+3r^2-16Rr\right)+r^2\left(32R^2-16Rr+3r^2\right)\right]\geq r^3(4R+r)^2(8R-r)$

$24R^2-55Rr+14r^2\geq 0\Leftrightarrow (R-2r)(24R-7r)\geq 0$, evident din inegalitatea lui Euler $R\geq 2r$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Aplicația92.

In ΔABC

$$\sum \frac{r_a}{4r_b^2 + r_b r_c + 4r_c^2} \ge \frac{2}{9R}.$$

Zaza Mzhavanadze, Georgia, RMM 3/2023

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum \frac{x}{4y^2 + yz + 4z^2} \ge \frac{\left(\sum x\right)^2}{4\sum x \sum yz - 9xyz}.$$

Demonstraţie.

$$\sum \frac{x}{4y^2 + yz + 4z^2} = \sum \frac{x^2}{4xy^2 + xyz + 4xz^2} \overset{CS}{\ge} \frac{\left(\sum x\right)^2}{\sum\left(4xy^2 + xyz + 4xz^2\right)} = \frac{\left(\sum x\right)^2}{4\sum x \sum yz - 9xyz}.$$

Folosind **Lema** pentru $(x, y, z) = (r_a, r_b, r_c)$ obţinem

$$\sum \frac{r_a}{4r_b^2 + r_b r_c + 4r_c^2} \overset{Lema}{\ge} \frac{\left(\sum r_a\right)^2}{4\sum r_a \sum r_b r_c - 9r_a r_b r_c} = \frac{(4R+r)^2}{4(4R+r)\cdot p^2 - 9rp^2} = \frac{(4R+r)^2}{p^2(16R-5r)} \overset{(1)}{\ge} \frac{2}{9R},$$

unde $\frac{(4R+r)^2}{p^2(16R-5r)} \overset{(1)}{\ge} \frac{2}{9R}$, rezultă din inegalitatea Blundon- Gerretsen: $p^2 \le \frac{R(4R+r)^2}{2(2R-r)}$.

Rămâne să arătăm că:

$$\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}(16R-5r)} \ge \frac{2}{9R} \Leftrightarrow 9(2R-r) \ge 16R-5r \Leftrightarrow R \ge 2r, (Euler).$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \frac{h_a}{4h_b^2 + h_b h_c + 4h_c^2} \ge \left(\frac{2}{3}\right)^6 \frac{r}{R^2}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum \frac{x}{4y^2+yz+4z^2} \geq \frac{\left(\sum x\right)^2}{4\sum x \sum yz - 9xyz}.$$

Folosind **Lema** pentru $(x,y,z)=(h_a,h_b,h_c)$ obţinem

$$\sum \frac{h_a}{4h_b^2+h_bh_c+4h_c^2} \overset{Lema}{\geq} \frac{\left(\sum h_a\right)^2}{4\sum h_a \sum h_bh_c - 9h_ah_bh_c} = \frac{\left(\dfrac{p^2+r^2+4Rr}{2R}\right)^2}{4\cdot\dfrac{p^2+r^2+4Rr}{2R}\cdot\dfrac{2rp^2}{R}-9\cdot\dfrac{2r^2p^2}{R}} =$$

$$= \frac{\left(p^2+r^2+4Rr\right)^2}{8rp^2\left(2p^2+2r^2-Rr\right)} \overset{Gerretsen}{\geq} \frac{\left(16Rr-5r^2+r^2+4Rr\right)^2}{8r\cdot\dfrac{27R^2}{4}\left(2\left(4R^2+4Rr+3r^2\right)+2r^2-Rr\right)} =$$

$$= \frac{8r\left(5R-2r\right)^2}{27R^2\left(8R^2+7Rr+8r^2\right)} \overset{Euler}{\geq} \frac{8r}{27R^2}\cdot\frac{32}{27} = \frac{256r}{729R^2} = \left(\frac{2}{3}\right)^6 \frac{r}{R^2}.$$

Egalitatea are loc dacă triunghiul este echilateral.

Aplicaţia93.

In ΔABC

$$\sum a^2\sqrt[3]{bc} \leq R^2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^2.$$

Daniel Sitaru, RMM 3/2023

Soluţie.

Tripletele $\left(a^2,b^2,c^2\right)$ şi $\left(\sqrt[3]{bc},\sqrt[3]{ca},\sqrt[3]{ab}\right)$ sunt invers ordonate.

Cu inegalitatea lui Chebyshev obţinem:

$$\sum a^2\sqrt[3]{bc} \overset{Chebyshev}{\leq} \frac{1}{3}\sum a^2 \sum \sqrt[3]{bc} \overset{Leibniz}{\leq} \frac{1}{3}\cdot 9R^2 \sum \sqrt[3]{bc} \overset{SOS}{\leq} 3R^2\cdot\frac{1}{3}\left(\sum \sqrt[3]{a}\right)^2 = R^2\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^2.$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum a^4\sqrt[3]{bc} \leq \frac{27R^5}{2r}\sqrt[3]{3R^2}.$$

Marin Chirciu

Soluţie.

Tripletele $\left(a^4,b^4,c^4\right)$ şi $\left(\sqrt[3]{bc},\sqrt[3]{ca},\sqrt[3]{ab}\right)$ sunt invers ordonate.

Cu inegalitatea lui Chebyshev obţinem:

$$\sum a^4\sqrt[3]{bc}\overset{Chebyshev}{\leq}\frac{1}{3}\sum a^4\sum\sqrt[3]{bc}\overset{Goldner}{\leq}\frac{1}{3}\cdot\frac{27R^5}{2r}\sum\sqrt[3]{bc}\overset{SOS}{\leq}\frac{9R^5}{2r}\cdot\frac{1}{3}\left(\sum\sqrt[3]{a}\right)^2=$$

$$=\frac{3R^5}{2r}\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^2\overset{Holder}{\leq}\frac{3R^5}{2r}\left(\sqrt[3]{9(a+b+c)}\right)^2=\frac{3R^5}{2r}\left(\sqrt[3]{9\cdot 2p}\right)^2\overset{Mitinovic}{\leq}\frac{3R^5}{2r}\left(\sqrt[3]{18\cdot\frac{3\sqrt{3}R}{2}}\right)^2=$$

$$=\frac{3R^5}{2r}\left(\sqrt[3]{9\cdot 3\sqrt{3}R}\right)^2=\frac{3R^5}{2r}\left(3\sqrt[3]{\sqrt{3}R}\right)^2=\frac{27R^5}{2r}\left(\sqrt[3]{\sqrt{3}R}\right)^2=\frac{27R^5}{2r}\sqrt[3]{3R^2}.$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum m_a^2\sqrt[3]{m_bm_c}\leq\frac{27R^2}{4}\sqrt[3]{\frac{9R^2}{4}}.$$

Marin Chirciu

Soluție.

Lema

If $x,y,z>0$ then

$$\sum x^2\sqrt[3]{yz}\leq\sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Tripletele $\left(x^2,y^2,z^2\right)$ şi $\left(\sqrt[3]{yz},\sqrt[3]{zx},\sqrt[3]{xy}\right)$ sunt invers ordonate.

Cu inegalitatea lui Chebyshev obţinem:

$$\sum x^2\sqrt[3]{yz}\overset{Chebyshev}{\leq}\frac{1}{3}\sum x^2\sum\sqrt[3]{yz}\overset{SOS}{\leq}\frac{1}{3}\sum x^2\cdot\frac{1}{3}\left(\sum\sqrt[3]{x}\right)^2=$$

$$=\frac{1}{9}\sum x^2\left(\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}\right)^2\overset{Holder}{\leq}\frac{1}{9}\sum x^2\left(\sqrt[3]{9(x+y+z)}\right)^2=\frac{1}{9}\sum x^2\left(\sqrt[3]{9\sum x}\right)^2=$$

$$=\frac{1}{3}\sum x^2\sqrt[3]{3\left(\sum x\right)^2}\overset{SOS}{\leq}\frac{1}{3}\sum x^2\sqrt[3]{9\sum x^2}=\frac{\sqrt[3]{9}}{3}\left(\sum x^2\right)^{\frac{4}{3}}=\frac{1}{\sqrt[3]{3}}\left(\sum x^2\right)^{\frac{4}{3}}=\sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x,y,z)=\left(m_a,m_b,m_c\right)$ obţinem:

$$\sum m_a^2\sqrt[3]{m_bm_c}\overset{Lema}{\leq}\sqrt[3]{\frac{1}{3}\left(\sum m_a^2\right)^4}\leq\sqrt[3]{\frac{1}{3}\left(\frac{27R^2}{4}\right)^4}=\frac{27R^2}{4}\sqrt[3]{\frac{9R^2}{4}}.$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum h_a^2 \sqrt[3]{h_b h_c} \le p^2 \sqrt[3]{\frac{1}{3} p^2} .$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4} .$$

Folosind **Lema** pentru $(x, y, z) = (h_a, h_b, h_c)$ obţinem:

$$\sum h_a^2 \sqrt[3]{h_b h_c} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum h_a^2\right)^4} \le \sqrt[3]{\frac{1}{3}\left(p^2\right)^4} = p^2 \sqrt[3]{\frac{1}{3} p^2} .$$

Remarca.

In ΔABC

$$\sum w_a^2 \sqrt[3]{w_b w_c} \le p^2 \sqrt[3]{\frac{1}{3} p^2} .$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4} .$$

Folosind **Lema** pentru $(x, y, z) = (w_a, w_b, w_c)$ obţinem:

$$\sum w_a^2 \sqrt[3]{w_b w_c} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum w_a^2\right)^4} \le \sqrt[3]{\frac{1}{3}\left(p^2\right)^4} = p^2 \sqrt[3]{\frac{1}{3} p^2} .$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum s_a^2 \sqrt[3]{s_b s_c} \le \frac{27R^2}{4}\sqrt[3]{\frac{9R^2}{4}}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x, y, z) = (s_a, s_b, s_c)$ obţinem:

$$\sum s_a^2 \sqrt[3]{s_b s_c} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum s_a^2\right)^4} \le \sqrt[3]{\frac{1}{3}\left(\frac{27R^2}{4}\right)^4} = \frac{27R^2}{4}\sqrt[3]{\frac{9R^2}{4}}.$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum r_a^2 \sqrt[3]{r_b r_c} \le \frac{81}{8}\sqrt[3]{\frac{2}{3}\left(3R^2 - 8r^2\right)^4}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x, y, z) = (r_a, r_b, r_c)$ obţinem:

$$\sum r_a^2 \sqrt[3]{r_b r_c} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum r_a^2\right)^4} \le \sqrt[3]{\frac{1}{3}\left(\frac{27}{4}\left(3R^2 - 8r^2\right)\right)^4} = \frac{81}{8}\sqrt[3]{\frac{2}{3}\left(3R^2 - 8r^2\right)^4}.$$

Am folosit mai sus

$$\sum r_a^2 = (4R + r)^2 - 2p^2 \overset{Gerretsen}{\le} (4R + r)^2 - 2\left(16Rr - 5r^2\right) = 16R^2 - 24Rr + 11r^2 \overset{Euler}{\le}$$
$$\overset{Euler}{\le} \frac{27}{4}\left(3R^2 - 8r^2\right).$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

În aceeași clasă de probleme.

In ΔABC

$$\sum \sin^2 \frac{A}{2} \sqrt[3]{\sin \frac{B}{2} \sin \frac{C}{2}} \le \sqrt[3]{\frac{1}{3}\left(1-\frac{r}{2R}\right)^4}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x, y, z) = \left(\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2}\right)$ obţinem:

$$\sum \sin^2 \frac{A}{2} \sqrt[3]{\sin \frac{B}{2} \sin \frac{C}{2}} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum \sin^2 \frac{A}{2}\right)^4} \le \sqrt[3]{\frac{1}{3}\left(1-\frac{r}{2R}\right)^4}.$$

Am folosit mai sus: $\sum \sin^2 \frac{A}{2} = 1 - \frac{r}{2R}$.

Remarca.

In ΔABC

$$\sum \cos^2 \frac{A}{2} \sqrt[3]{\cos \frac{B}{2} \cos \frac{C}{2}} \le \frac{9}{4}\sqrt[3]{\frac{3}{4}}.$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x, y, z) = \left(\cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2}\right)$ obţinem:

$$\sum \cos^2 \frac{A}{2} \sqrt[3]{\cos \frac{B}{2} \cos \frac{C}{2}} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum \cos^2 \frac{A}{2}\right)^4} \le \sqrt[3]{\frac{1}{3}\left(2+\frac{r}{2R}\right)^4} \le \sqrt[3]{\frac{1}{3}\left(\frac{9}{4}\right)^4} = \frac{9}{4}\sqrt[3]{\frac{1}{3}\cdot\frac{9}{4}} = \frac{9}{4}\sqrt[3]{\frac{3}{4}}.$$

Am folosit mai sus: $\sum \cos^2 \frac{A}{2} = 2 + \frac{r}{2R} \le \frac{9}{4}$.

Remarca.

In ΔABC

$$\sum \tan^2 \frac{A}{2} \sqrt[3]{\tan \frac{B}{2} \tan \frac{C}{2}} \le \frac{R}{r} \sqrt[3]{\frac{R}{3r}} .$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3} \left(\sum x^2\right)^4} .$$

Folosind **Lema** pentru $(x, y, z) = \left(\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2} \right)$ obţinem:

$$\sum \tan^2 \frac{A}{2} \sqrt[3]{\tan \frac{B}{2} \tan \frac{C}{2}} \overset{Lema}{\le} \sqrt[3]{\frac{1}{3}\left(\sum \tan^2 \frac{A}{2}\right)^4} \le \sqrt[3]{\frac{1}{3}\left(\frac{(4R+r)^2 - p^2}{p^2}\right)^4} \le \sqrt[3]{\frac{1}{3}\left(\frac{R}{r}\right)^4} = \frac{R}{r} \sqrt[3]{\frac{R}{3r}} .$$

Am folosit mai sus:

$$\sum \tan^2 \frac{A}{2} = \frac{(4R+r)^2 - p^2}{p^2} = \frac{(4R+r)^2}{p^2} - 1 \overset{Gerretsen}{\le} \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} - 1 = \frac{R+r}{r} - 1 = \frac{R}{r} .$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \cot^2 \frac{A}{2} \sqrt[3]{\cot \frac{B}{2} \cot \frac{C}{2}} \le \sqrt[3]{\frac{1}{3}\left(\frac{2R}{r} - 1\right)^8} .$$

Marin Chirciu

Soluție.

Lema

If $x, y, z > 0$ then

$$\sum x^2 \sqrt[3]{yz} \le \sqrt[3]{\frac{1}{3} \left(\sum x^2\right)^4} .$$

Folosind **Lema** pentru $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ obţinem:

$$\sum\cot^2\frac{A}{2}\sqrt[3]{\cot\frac{B}{2}\cot\frac{C}{2}}\overset{Lema}{\le}\sqrt[3]{\frac{1}{3}\left(\sum\cot^2\frac{A}{2}\right)^4}\le\sqrt[3]{\frac{1}{3}\left(\frac{p^2-2r^2-8Rr}{r^2}\right)^4}\le\sqrt[3]{\frac{1}{3}\left(\left(\frac{2R}{r}-1\right)^2\right)^4}=$$

$$=\sqrt[3]{\frac{1}{3}\left(\frac{2R}{r}-1\right)^8}.$$

Am folosit mai sus:

$$\sum\tan^2\frac{A}{2}=\frac{p^2-2r^2-8Rr}{r^2}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-2r^2-8Rr}{r^2}=\frac{4R^2-4Rr+r^2}{r^2}=$$

$$=\frac{(2R-r)^2}{r^2}=\left(\frac{2R}{r}-1\right)^2$$

Egalitatea are loc dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\sec^2\frac{A}{2}\sqrt[3]{\sec\frac{B}{2}\sec\frac{C}{2}}\le\sqrt[3]{\frac{1}{3}\left(2+\frac{R}{r}\right)^8}.$$

Marin Chirciu

Soluție.

Lema

If $x,y,z>0$ then

$$\sum x^2\sqrt[3]{yz}\le\sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x,y,z)=\left(\sec\frac{A}{2},\sec\frac{B}{2},\sec\frac{C}{2}\right)$ obţinem:

$$\sum\sec^2\frac{A}{2}\sqrt[3]{\sec\frac{B}{2}\sec\frac{C}{2}}\overset{Lema}{\le}\sqrt[3]{\frac{1}{3}\left(\sum\sec^2\frac{A}{2}\right)^4}\le\sqrt[3]{\frac{1}{3}\left(\frac{p^2+(4R+r)^2}{p^2}\right)^4}\le\sqrt[3]{\frac{1}{3}\left(\left(2+\frac{R}{r}\right)^2\right)^4}=$$

$$=\sqrt[3]{\frac{1}{3}\left(2+\frac{R}{r}\right)^8}.$$

Am folosit mai sus:

$$\sum \sec^2\frac{A}{2}=\sum\frac{1}{\cos^2\frac{A}{2}}=\frac{p^2+(4R+r)^2}{p^2}=1+\frac{(4R+r)^2}{p^2}\overset{Gerretsen}{\leq}1+\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}=2+\frac{R}{r}.$$

Remarca.

In ΔABC

$$\sum \csc^2\frac{A}{2}\sqrt[3]{\csc\frac{B}{2}\csc\frac{C}{2}}\leq 4\sqrt[3]{\frac{4}{3}\left(\frac{R^2}{r^2}-1\right)^4}.$$

Marin Chirciu

Soluție.

Lema

If $x,y,z>0$ then

$$\sum x^2\sqrt[3]{yz}\leq\sqrt[3]{\frac{1}{3}\left(\sum x^2\right)^4}.$$

Folosind **Lema** pentru $(x,y,z)=\left(\csc\frac{A}{2},\csc\frac{B}{2},\csc\frac{C}{2}\right)$ obţinem:

$$\sum \csc^2\frac{A}{2}\sqrt[3]{\csc\frac{B}{2}\csc\frac{C}{2}}\overset{Lema}{\leq}\sqrt[3]{\frac{1}{3}\left(\sum\csc^2\frac{A}{2}\right)^4}\leq\sqrt[3]{\frac{1}{3}\left(\frac{p^2+r^2-8Rr}{r^2}\right)^4}\leq\sqrt[3]{\frac{1}{3}\left(4\left(\frac{R^2}{r^2}-1\right)\right)^4}=$$

$$=\sqrt[3]{\frac{4^4}{3}\left(\frac{R^2}{r^2}-1\right)^4}=4\sqrt[3]{\frac{4}{3}\left(\frac{R^2}{r^2}-1\right)^4}.$$

Am folosit mai sus:

$$\sum\csc^2\frac{A}{2}=\sum\frac{1}{\sin^2\frac{A}{2}}=\frac{p^2+r^2-8Rr}{r^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{r^2}=\frac{4R^2-4Rr+4r^2}{r^2}=$$

$$=\frac{4(R^2-Rr+r^2)}{r^2}\overset{Euler}{\leq}\frac{4(R^2-r^2)}{r^2}=4\left(\frac{R^2}{r^2}-1\right).$$

Aplicația94.

G441. In ΔABC

$$\sum a\sum a^2\geq 2\sum(b+c)h_a^2.$$

Nguyen Viet Hung, Vietnam, Recreații Matematice 1/2023

Remarca.

In ΔABC

$$36r^2p \le \sum(b+c)h_a^2 \le 9R^2p.$$

Soluție.

Lema

In ΔABC

$$\sum(b+c)h_a^2 = \frac{p\left[p^2\left(p^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2}.$$

Inegalitatea din dreapta.

$$\sum(b+c)h_a^2 = \frac{p\left[p^2\left(p^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2} \overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le} \frac{p\left[\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2} =$$

$$= \frac{p\left(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4\right)}{2R^2} = \frac{4p\left(2R^4-R^3r+2R^2r^2+Rr^3+2r^4\right)}{R^2} \overset{Euler}{\le}$$

$$\overset{Euler}{\le} \frac{4p\cdot\frac{9R^4}{4}}{R^2} = 9R^2p.$$

Am folosit mai sus:

$$2R^4-R^3r+2R^2r^2+Rr^3+2r^4 \le \frac{9R^4}{4} \Leftrightarrow R^4+4R^3r-8R^2r^2-4Rr^3-8r^4 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)\left(R^3+6R^2r+4Rr^2+4r^3\right) \ge 0$, evident din inegalitatea lui Euler $R \ge 2r$.

Inegalitatea din stânga.

$$\sum(b+c)h_a^2 = \frac{p\left[p^2\left(p^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2} \overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge} \frac{p\left[\left(16Rr-5r^2\right)\left(16Rr-5r^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2} =$$

$$= \frac{pr^2\left(104R^2-72Rr+16r^2\right)}{2R^2} = \frac{4pr^2\left(13R^2-9Rr+2r^2\right)}{R^2} \overset{Euler}{\ge} \frac{4pr^2\cdot 9R^2}{R^2} = 36pr^2$$

Am folosit mai sus

$13R^2-9Rr+2r^2 \ge 9R^2 \Leftrightarrow 4R^2-9Rr+2r^2 \ge 0 \Leftrightarrow (R-2r)(4R-r) \ge 0$,vezi $R \ge 2r$.

Remarca.

In ΔABC

$$18Rrp \le \sum(b+c)r_a^2 \le \frac{9R^3p}{2r}.$$

Soluție.

Lema

In ΔABC

$$\sum(b+c)r_a^2 = 2p\left(8R^2+6Rr+r^2-p^2\right).$$

Inegalitatea din dreapta.

$$\sum(b+c)r_a^2 = 2p\left(8R^2+6Rr+r^2-p^2\right) \overset{Gerretsen}{\le} 2p\left(8R^2+6Rr+r^2-16Rr+5r^2\right) =$$

$$= 4p\left(4R^2-5Rr+3r^2\right) \overset{Euler}{\le} 4p\cdot\frac{9R^3}{8r} = \frac{9R^3p}{2r}.$$

Am folosit mai sus:

$$4R^2-5Rr+3r^2 \le \frac{9R^3}{8r} \Leftrightarrow 9R^3-32R^2r+40Rr^2-24r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)\left(9R^2-14Rr+12r^2\right) \ge 0$, evident din inegalitatea lui Euler $R \ge 2r$.

Inegalitatea din stânga.

$$\sum(b+c)r_a^2 = 2p\left(8R^2+6Rr+r^2-p^2\right) \overset{Gerretsen}{\ge} 2p\left(8R^2+6Rr+r^2-4R^2-4Rr-3r^2\right) =$$

$$= 4p\left(2R^2+Rr-r^2\right) \overset{Euler}{\ge} 4p\cdot\frac{9Rr}{2} = 18Rrp.$$

Am folosit mai sus:

$2R^2+Rr-r^2 \ge \frac{9Rr}{2} \Leftrightarrow 4R^2-7Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(4R+r) \ge 0$,vezi $R \ge 2r$.

Remarca.

In ΔABC

$$\sum(b+c)h_a^2 \le \sum(b+c)r_a^2.$$

Marin Chirciu

Soluție.

Folosind **Lemele** de mai sus avem sumele

$\sum(b+c)h_a^2 = \dfrac{p\left[p^2\left(p^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2}$ și

$\sum(b+c)r_a^2 = 2p\left(8R^2+6Rr+r^2-p^2\right)$.

Inegalitatea se scrie:

$$\frac{p\left[p^2\left(p^2+2r^2-10Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right]}{2R^2} \le 2p\left(8R^2+6Rr+r^2-p^2\right) \Leftrightarrow$$

$$\Leftrightarrow p^2\left(p^2+2r^2-10Rr+4R^2\right) \le 32R^4+24R^3r-4R^2r^2-6Rr^3-r^4,$$

care rezultă din inegalitatea lui Gerretsen $p^2 \le 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$$\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2+2r^2-10Rr+4R^2\right) \le 32R^4+24R^3r-4R^2r^2-6Rr^3-r^4 \Leftrightarrow$$

$2R^3-3R^2r-Rr^2-2r^3 \ge 0 \Leftrightarrow (R-2r)\left(2R^2+Rr+r^2\right) \ge 0$,vezi $R \ge 2r$.

Remarca.

In ΔABC

$$36r^2p \le \sum(b+c)h_a^2 \le \sum(b+c)r_a^2 \le \frac{9R^3p}{2r}.$$

Marin Chirciu

Remarca.

In ΔABC

$$36r^2p \le \sum(b+c)m_a^2 \le \frac{9R^3p}{2r}.$$

Soluție.

Lema

In ΔABC

$$\sum(b+c)m_a^2 = \frac{1}{2}p\left(5p^2-11r^2-26Rr\right).$$

Demonstrație.

Inegalitatea din dreapta.

$$\sum(b+c)m_a^2 = \frac{1}{2}p\left(5p^2-11r^2-26Rr\right) \overset{Gerretsen}{\le} \frac{1}{2}p\left(5\left(4R^2+4Rr+3r^2\right)-11r^2-26Rr\right) =$$

$$= p\left(10R^2 - 3Rr + 2r^2\right) \overset{Euler}{\leq} p \cdot \frac{9R^3}{2r} = \frac{9R^3 p}{2r}.$$

Am folosit mai sus:

$$10R^2 - 3Rr + 2r^2 \leq \frac{9R^3}{2r} \Leftrightarrow 9R^3 - 20R^2 r + 6Rr^2 - 4r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)\left(9R^2 - 2Rr + 2r^2\right) \geq 0$, evident din inegalitatea lui Euler $R \geq 2r$.

Inegalitatea din stânga.

$$\sum (b+c) m_a^2 = \frac{1}{2} p\left(5p^2 - 11r^2 - 26Rr\right) \overset{Gerretsen}{\geq} \frac{1}{2} p\left(5\left(16Rr - 5r^2\right) - 11r^2 - 26Rr\right) =$$

$$= pr\left(27R - 18r\right) \overset{Euler}{\geq} pr \cdot 36r = 36r^2 p.$$

Remarca.

In ΔABC

$$36r^2 p \leq \sum (b+c) l_a^2 \leq \frac{9R^3 p}{2r}.$$

Marin Chirciu

Soluție.

Lema

In ΔABC

$$\sum (b+c) l_a^2 = 2p \cdot \frac{p^2\left(p^2 + 2r^2 - 4Rr\right) + r^3\left(4R + r\right)}{p^2 + r^2 + 2Rr}.$$

Inegalitatea din dreapta.

$$\sum (b+c) l_a^2 = 2p \cdot \frac{p^2\left(p^2 + 2r^2 - 4Rr\right) + r^3\left(4R + r\right)}{p^2 + r^2 + 2Rr} \overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq} 2p \cdot \frac{\left(4R^2 + 4Rr + 3r^2\right)\left(4R^2 + 4Rr + 3r^2 + 2r^2 - 4Rr\right) + r^3\left(4R + r\right)}{16Rr - 5r^2 + r^2 + 2Rr} =$$

$$= 2p \cdot \frac{8\left(2R^4 + 2R^3 r + 4R^2 r^2 + 3Rr^3 + 2r^4\right)}{2r\left(9R - 2r\right)} \overset{Euler}{\leq} 16p \cdot \frac{1}{2r} \cdot \frac{9R^3}{16} = \frac{9R^3 p}{2r}.$$

Am folosit mai sus:

$$\frac{2R^4+2R^3r+4R^2r^2+3Rr^3+2r^4}{9R-2r}\le\frac{9R^3}{16}\Leftrightarrow 49R^4-50R^3r-64R^2r^2-48Rr^3-32r^4\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(49R^3+48R^2r+32Rr^2+16r^3)\ge 0$, evident din inegalitatea lui Euler $R\ge 2r$.

Inegalitatea din stânga.

$$\sum(b+c)l_a^2=2p\cdot\frac{p^2(p^2+2r^2-4Rr)+r^3(4R+r)}{p^2+r^2+2Rr}\overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge} 2p\cdot\frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-4Rr)+r^3(4R+r)}{4R^2+4Rr+3r^2+r^2+2Rr}=8pr^2\cdot\frac{24R^2-13Rr+2r^2}{2R^2+3Rr+2r^2}\overset{Euler}{\ge}$$

$$\overset{Euler}{\ge} 8pr^2\cdot\frac{9}{2}=36r^2p.$$

$\dfrac{24R^2-13Rr+2r^2}{2R^2+3Rr+2r^2}\ge\dfrac{9}{2}\Leftrightarrow 30R^2-53Rr-14r^2\ge 0\Leftrightarrow (R-2r)(30R+7r)\ge 0$, vezu $R\ge 2r$.

Remarca.

In ΔABC

$$36r^2p\le\sum(b+c)s_a^2\le\frac{9R^3p}{2r}.$$

Marin Chirciu

Soluție.

Folosind $h_a\le s_a\le m_a$ și inegalitățile de mai sus $\sum(b+c)m_a^2\le\dfrac{9R^3p}{2r}$, $36r^2p\le\sum(b+c)h_a^2$,

rezultă concluzia $36r^2p\le\sum(b+c)s_a^2\le\dfrac{9R^3p}{2r}$.

Aplicația95.

If $a,b,c>0$, $ab+bc+ca=3$ then

$$\sum\frac{2+b}{1+a^2}\ge\frac{9}{2}.$$

Imad Zak, Lebanon, Mathematical Inequalities 2/2023

Soluție

Lema

If $a,b,c>0$ then

$$\frac{2+b}{1+a^2} \geq 2-a+b-\frac{1}{2}ab.$$

Demonstrație

$$\frac{1}{1+a^2}=1-\frac{a^2}{1+a^2} \geq 1-\frac{a^2}{2a}=1-\frac{a}{2} \Rightarrow \frac{2+b}{1+a^2} \geq (2+b)\left(1-\frac{a}{2}\right)=2-a+b-\frac{1}{2}ab.$$

$$LHS=\sum\frac{2+b}{1+a^2} \overset{Lema}{\geq} \sum\left(2-a+b-\frac{1}{2}ab\right)=6-\frac{1}{2}\sum bc=6-\frac{1}{2}\cdot 3=\frac{9}{2}=RHS.$$

Remarca.

In ΔABC

$$\sum\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq \frac{9}{2}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z>0$, $xy+yz+zx=3$ then

$$\sum\frac{2+y}{1+x^2} \geq \frac{9}{2}.$$

Demonstrație

$$\frac{1}{1+x^2}=1-\frac{x^2}{1+x^2} \geq 1-\frac{x^2}{2x}=1-\frac{x}{2} \Rightarrow \frac{2+y}{1+x^2} \geq (2+y)\left(1-\frac{x}{2}\right)=2-x+y-\frac{1}{2}xy.$$

$$LHS=\sum\frac{2+y}{1+x^2} \geq \sum\left(2-x+y-\frac{1}{2}xy\right)=6-\frac{1}{2}\sum yz=6-\frac{1}{2}\cdot 3=\frac{9}{2}=RHS.$$

Se cunoaște identitatea în triunghi $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1 \Leftrightarrow \sum\sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2}=3$.

Facem substituția $\left(\sqrt{3}\tan\frac{A}{2}, \sqrt{3}\tan\frac{B}{2}, \sqrt{3}\tan\frac{C}{2}\right)=(x, y, z)$ în **Lema** de mai sus.

Rezultă: $\sum\frac{2+\sqrt{3}\tan\frac{B}{2}}{1+3\tan^2\frac{A}{2}} \geq \frac{9}{2}$.

Remarca.

If $a, b, c>0$, $a+b+c=3$ then

$$\sum \frac{2+b}{1+a^2} \geq \frac{9}{2}.$$

Marin Chirciu

Soluție

Lema

If $a,b,c>0$ then

$$\frac{2+b}{1+a^2} \geq 2-a+b-\frac{1}{2}ab.$$

Demonstrație

$$\frac{1}{1+a^2}=1-\frac{a^2}{1+a^2} \geq 1-\frac{a^2}{2a}=1-\frac{a}{2} \Rightarrow \frac{2+b}{1+a^2} \geq (2+b)\left(1-\frac{a}{2}\right)=2-a+b-\frac{1}{2}ab.$$

$$\sum \frac{2+b}{1+a^2} \overset{Lema}{\geq} \sum\left(2-a+b-\frac{1}{2}ab\right)=6-\frac{1}{2}\sum bc=6-\frac{1}{2}\cdot\frac{1}{3}\left(\sum a\right)^2=6-\frac{1}{2}\cdot\frac{1}{3}\cdot 3^2=\frac{9}{2}.$$

Remarca.

In ΔABC

$$\sum \frac{2+\frac{3r}{r_b}}{1+\left(\frac{3r}{r_a}\right)^2} \geq \frac{9}{2}.$$

Marin Chirciu

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$\sum \frac{2+y}{1+x^2} \geq \frac{9}{2}.$$

Demonstrație

$$\frac{1}{1+x^2}=1-\frac{x^2}{1+x^2} \geq 1-\frac{x^2}{2x}=1-\frac{x}{2} \Rightarrow \frac{2+y}{1+x^2} \geq (2+y)\left(1-\frac{x}{2}\right)=2-x+y-\frac{1}{2}xy.$$

Obținem:

$$\sum\frac{2+y}{1+x^2}\geq\sum\left(2-x+y-\frac{1}{2}xy\right)=6-\frac{1}{2}\sum yz=6-\frac{1}{2}\cdot\frac{1}{3}\left(\sum x\right)^2=6-\frac{1}{2}\cdot\frac{1}{3}\cdot 3^2=\frac{9}{2}$$

Cu identitatea în triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$ facem substituția

$\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)=(x,y,z)$ în **Lema** de mai sus. Rezultă: $\sum\frac{2+\frac{3r}{r_b}}{1+\left(\frac{3r}{r_a}\right)^2}\geq\frac{9}{2}$.

Remarca.

In ΔABC

$$\sum\frac{2+\frac{3r}{h_b}}{1+\left(\frac{3r}{h_a}\right)^2}\geq\frac{9}{2}.$$

Marin Chirciu

Soluție

Lema

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{2+y}{1+x^2}\geq\frac{9}{2}.$$

Demonstrație

$$\frac{1}{1+x^2}=1-\frac{x^2}{1+x^2}\geq 1-\frac{x^2}{2x}=1-\frac{x}{2}\Rightarrow\frac{2+y}{1+x^2}\geq(2+y)\left(1-\frac{x}{2}\right)=2-x+y-\frac{1}{2}xy.$$

$$\sum\frac{2+y}{1+x^2}\geq\sum\left(2-x+y-\frac{1}{2}xy\right)=6-\frac{1}{2}\sum yz=6-\frac{1}{2}\cdot\frac{1}{3}\left(\sum x\right)^2=6-\frac{1}{2}\cdot\frac{1}{3}\cdot 3^2=\frac{9}{2}$$

Cu identitatea în triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Leftrightarrow\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1\Leftrightarrow\frac{3r}{h_a}+\frac{3r}{h_b}+\frac{3r}{h_c}=3$ facem substituția

$\left(\frac{3r}{h_a},\frac{3r}{h_b},\frac{3r}{h_c}\right)=(x,y,z)$ în **Lema** de mai sus. Rezultă: $\sum\frac{2+\frac{3r}{h_b}}{1+\left(\frac{3r}{h_a}\right)^2}\geq\frac{9}{2}$.

Aplicația96.

In acute ΔABC

$$\sum m_a\sqrt{\cot A} > 6r.$$

Vasile Mircea Popa, RMM 2/2023

Soluție

Tripletele (m_a, m_b, m_c) și $(\sqrt{\cot A}, \sqrt{\cot B}, \sqrt{\cot C})$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum m_a\sqrt{\cot A} \overset{Chebyshev}{\geq} \frac{1}{3}\sum m_a \sum\sqrt{\cot A} > \frac{1}{3}\cdot 9r\cdot 2 = 6r = RHS.$$

Am folosit mai sus $\sum m_a \geq 9r$ și $\sum\sqrt{\cot A} > 2$, care rezultă din:

Cu substituția $(x, y, z) = (\cot A, \cot B, \cot C)$, $xy + yz + zx = 1$ avem:

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \overset{AM-GM}{\geq} 2\sqrt{\sqrt{x}\left(\sqrt{y} + \sqrt{z}\right)} \Rightarrow$$

$$\left(\frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{2}\right)^4 \geq x\left(\sqrt{y} + \sqrt{z}\right)^2 = x\left(y + z + 2\sqrt{yz}\right) > xy + yz + zx = 1.$$

Remarca.

In acute ΔABC

$$\sum m_a \cot A \geq 3\sqrt{3}r.$$

Marin Chirciu

Soluție

Tripletele (m_a, m_b, m_c) și $(\cot A, \cot B, \cot C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum m_a \cot A \overset{Chebyshev}{\geq} \frac{1}{3}\sum m_a \sum \cot A \geq \frac{1}{3}\cdot 9r\cdot\sqrt{3} = 3\sqrt{3}r = RHS.$$

Am folosit $\sum m_a \geq 9r$ și $\sum \cot A \geq 3\sqrt{3}$, care rezultă din $\sum \cot A = \frac{p^2 - r^2 - 4Rr}{2pr} \geq 3\sqrt{3}$.

Remarca.

In acute ΔABC

$$\sum h_a \cot A \geq 3\sqrt{3}r.$$

Marin Chirciu

Soluție

Tripletele (h_a, h_b, h_c) și $(\cot A, \cot B, \cot C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum h_a \cot A \overset{Chebyshev}{\geq} \frac{1}{3}\sum h_a \sum \cot A \geq \frac{1}{3}\cdot 9r \cdot \sqrt{3} = 3\sqrt{3}r = RHS .$$

Am folosit $\sum h_a \geq 9r$ și $\sum \cot A \geq 3\sqrt{3}$, care rezultă din $\sum \cot A = \frac{p^2 - r^2 - 4Rr}{2pr} \geq 3\sqrt{3}$.

Remarca.

In acute ΔABC

$$\sum w_a \cot A \geq 3\sqrt{3}r .$$

Marin Chirciu

Soluție

Tripletele (w_a, w_b, w_c) și $(\cot A, \cot B, \cot C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum w_a \cot A \overset{Chebyshev}{\geq} \frac{1}{3}\sum w_a \sum \cot A \geq \frac{1}{3}\cdot 9r \cdot \sqrt{3} = 3\sqrt{3}r = RHS .$$

Am folosit $\sum w_a \geq 9r$ și $\sum \cot A \geq 3\sqrt{3}$, care rezultă din $\sum \cot A = \frac{p^2 - r^2 - 4Rr}{2pr} \geq 3\sqrt{3}$.

Remarca.

In acute ΔABC

$$\sum s_a \cot A \geq 3\sqrt{3}r .$$

Marin Chirciu

Soluție

Tripletele (s_a, s_b, s_c) și $(\cot A, \cot B, \cot C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum s_a \cot A \overset{Chebyshev}{\geq} \frac{1}{3}\sum s_a \sum \cot A \geq \frac{1}{3}\cdot 9r \cdot \sqrt{3} = 3\sqrt{3}r = RHS .$$

Am folosit $\sum w_a \geq 9r$ și $\sum \cot A \geq 3\sqrt{3}$, care rezultă din $\sum \cot A = \frac{p^2 - r^2 - 4Rr}{2pr} \geq 3\sqrt{3}$.

Remarca

In acute ΔABC

$$\sum r_a \cot A \geq 3\sqrt{3}r .$$

Marin Chirciu

Soluție

Tripletele (r_a, r_b, r_c) și $(\cot A, \cot B, \cot C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum r_a \cot A \overset{Chebyshev}{\geq} \frac{1}{3}\sum r_a \sum \cot A \geq \frac{1}{3} \cdot 9r \cdot \sqrt{3} = 3\sqrt{3}r = RHS .$$

Am folosit $\sum r_a \geq 9r$ și $\sum \cot A \geq 3\sqrt{3}$, care rezultă din $\sum \cot A = \dfrac{p^2 - r^2 - 4Rr}{2pr} \geq 3\sqrt{3}$.

Remarca.

In acute ΔABC

$$\sum m_a \cos A \geq 3r\left(1 + \frac{r}{R}\right).$$

Marin Chirciu

Soluție

Tripletele (m_a, m_b, m_c) și $(\cos A, \cos B, \cos C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum m_a \cos A \overset{Chebyshev}{\geq} \frac{1}{3}\sum m_a \sum \cos A \geq \frac{1}{3} \cdot 9r \cdot \left(1 + \frac{r}{R}\right) = 3r\left(1 + \frac{r}{R}\right) = RHS .$$

Am folosit mai sus $\sum m_a \geq 9r$ și $\sum \cos A = 1 + \dfrac{r}{R}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In acute ΔABC

$$\sum h_a \cos A \geq 3r\left(1 + \frac{r}{R}\right).$$

Marin Chirciu

Soluție

Tripletele (h_a, h_b, h_c) și $(\cos A, \cos B, \cos C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS=\sum h_a\cos A\overset{Chebyshev}{\geq}\frac{1}{3}\sum h_a\sum\cos A\geq\frac{1}{3}\cdot 9r\cdot\left(1+\frac{r}{R}\right)=3r\left(1+\frac{r}{R}\right)=RHS.$$

Am folosit mai sus $\sum h_a\geq 9r$ și $\sum\cos A=1+\frac{r}{R}$.

Remarca.

In acute ΔABC

$$\sum w_a\cos A\geq 3r\left(1+\frac{r}{R}\right).$$

Marin Chirciu

Soluție

Tripletele (w_a,w_b,w_c) și $(\cos A,\cos B,\cos C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS=\sum w_a\cos A\overset{Chebyshev}{\geq}\frac{1}{3}\sum w_a\sum\cos A\geq\frac{1}{3}\cdot 9r\cdot\left(1+\frac{r}{R}\right)=3r\left(1+\frac{r}{R}\right)=RHS.$$

Am folosit mai sus $\sum w_a\geq 9r$ și $\sum\cos A=1+\frac{r}{R}$.

Remarca.

In acute ΔABC

$$\sum s_a\cos A\geq 3r\left(1+\frac{r}{R}\right).$$

Marin Chirciu

Soluție

Tripletele (s_a,s_b,s_c) și $(\cos A,\cos B,\cos C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS=\sum s_a\cos A\overset{Chebyshev}{\geq}\frac{1}{3}\sum s_a\sum\cos A\geq\frac{1}{3}\cdot 9r\cdot\left(1+\frac{r}{R}\right)=3r\left(1+\frac{r}{R}\right)=RHS.$$

Am folosit mai sus $\sum s_a\geq 9r$ și $\sum\cos A=1+\frac{r}{R}$.

Remarca.

In acute ΔABC

$$\sum r_a \cos A \geq 3r\left(1+\frac{r}{R}\right).$$

Marin Chirciu

Soluție

Tripletele (r_a, r_b, r_c) și $(\cos A, \cos B, \cos C)$ in acute sunt la fel ordonate.

Cu inegalitatea lui Chebyshev obținem:

$$LHS = \sum r_a \cos A \overset{Chebyshev}{\geq} \frac{1}{3}\sum r_a \sum \cos A \geq \frac{1}{3}\cdot 9r\cdot\left(1+\frac{r}{R}\right) = 3r\left(1+\frac{r}{R}\right) = RHS .$$

Am folosit mai sus $\sum r_a \geq 9r$ și $\sum \cos A = 1+\frac{r}{R}$.

Aplicația97.

In ΔABC

$$\sum \frac{w_a\left(w_b^2+w_c^2\right)}{w_a^2+w_b w_c} \geq 9r .$$

Zaza Mzhavanadze, Georgia, RMM 2/2023

Soluție

Lema

If $x, y, z > 0$

$$\sum \frac{x\left(y^2+z^2\right)}{x^2+yz} \geq 3\sqrt[3]{xyz} .$$

$$\sum \frac{x\left(y^2+z^2\right)}{x^2+yz} \overset{CBS}{\geq} \sum \frac{x\left(y^2+z^2\right)}{\sqrt{\left(x^2+y^2\right)\left(x^2+z^2\right)}} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{x\left(y^2+z^2\right)}{\sqrt{\left(x^2+y^2\right)\left(x^2+z^2\right)}}} = 3\sqrt[3]{xyz} .$$

Folosind **Lema** pentru $(x, y, z) = (w_a, w_b, w_c)$ obținem:

$$LHS = \sum \frac{w_a\left(w_b^2+w_c^2\right)}{w_a^2+w_b w_c} \overset{Lema}{\geq} 3\sqrt[3]{\prod w_a} \geq 3\sqrt[3]{27r^3} = 9r = RHS \text{, vezi } \prod w_a \geq 27r^3 .$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \frac{h_a\left(h_b^2+h_c^2\right)}{h_a^2+h_bh_c} \geq 9r.$$

Marin Chirciu

Soluție

Lema

If $x,y,z>0$

$$\sum \frac{x\left(y^2+z^2\right)}{x^2+yz} \geq 3\sqrt[3]{xyz}.$$

Folosind **Lema** pentru $(x,y,z)=(h_a,h_b,h_c)$ obținem:

$$LHS=\sum \frac{h_a\left(h_b^2+h_c^2\right)}{h_a^2+h_bh_c} \overset{Lema}{\geq} 3\sqrt[3]{\prod h_a} \geq 3\sqrt[3]{27r^3}=9r=RHS \text{ ,vezi } \prod h_a \geq 27r^3.$$

Remarca.

In ΔABC

$$\sum \frac{m_a\left(m_b^2+m_c^2\right)}{m_a^2+m_bm_c} \geq 9r.$$

Marin Chirciu

Soluție

Lema

If $x,y,z>0$

$$\sum \frac{x\left(y^2+z^2\right)}{x^2+yz} \geq 3\sqrt[3]{xyz}.$$

Folosind **Lema** pentru $(x,y,z)=(m_a,m_b,m_c)$ obținem:

$$LHS=\sum \frac{m_a\left(m_b^2+m_c^2\right)}{m_a^2+m_bm_c} \overset{Lema}{\geq} 3\sqrt[3]{\prod m_a} \geq 3\sqrt[3]{27r^3}=9r=RHS \text{ , vezi } \prod m_a \geq 27r^3.$$

Remarca.

In ΔABC

$$\sum \frac{s_a\left(s_b^2+s_c^2\right)}{s_a^2+s_bs_c} \geq 9r.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$

$$\sum \frac{x\left(y^2+z^2\right)}{x^2+yz} \geq 3\sqrt[3]{xyz}.$$

Folosind **Lema** pentru $(x, y, z) = (s_a, s_b, s_c)$ obținem:

$$LHS = \sum \frac{s_a\left(s_b^2+s_c^2\right)}{s_a^2+s_bs_c} \overset{Lema}{\geq} 3\sqrt[3]{\prod s_a} \geq 3\sqrt[3]{27r^3} = 9r = RHS \text{ ,vezi } \prod s_a \geq 27r^3.$$

Remarca.

In ΔABC

$$\sum \frac{r_a\left(r_b^2+r_c^2\right)}{r_a^2+r_br_c} \geq 9r.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$

$$\sum \frac{x\left(y^2+z^2\right)}{x^2+yz} \geq 3\sqrt[3]{xyz}.$$

Folosind **Lema** pentru $(x, y, z) = (r_a, r_b, r_c)$ obținem:

$$LHS = \sum \frac{r_a\left(r_b^2+r_c^2\right)}{r_a^2+r_br_c} \overset{Lema}{\geq} 3\sqrt[3]{\prod r_a} \geq 3\sqrt[3]{27r^3} = 9r = RHS \text{ ,vezi } \prod r_a \geq 27r^3.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația98.

In ΔABC

$$\sum \frac{m_a^2+m_bm_c}{m_a^5\left(m_b+m_c\right)} \geq \frac{16}{27R^4}.$$

Zaza Mzhavanadze, Georgia, RMM 2/2023

Soluție

Lema

If $x,y,z>0$

$$\sum\frac{x^2+yz}{x^5(y+z)}\geq\frac{3^5}{(x+y+z)^4}.$$

Demonstrație

$$\sum\frac{x^2+yz}{x^5(y+z)}\overset{AM-GM}{\geq}\sum\frac{2x\sqrt{yz}}{x^5(y+z)}=\sum\frac{2\sqrt{yz}}{x^4(y+z)}\overset{AM-GM}{\geq}2\cdot3\sqrt[3]{\prod\frac{\sqrt{yz}}{x^4(y+z)}}=\frac{6}{xyz\sqrt[3]{\prod(y+z)}}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}\frac{6}{\left(\frac{x+y+z}{3}\right)^3\frac{\sum(y+z)}{3}}=\frac{6}{\frac{1}{3^3}\left(\sum x\right)^3\frac{2\sum x}{3}}=\frac{3^5}{\left(\sum x\right)^4}.$$

Folosind **Lema** pentru $(x,y,z)=(m_a,m_b,m_c)$ obținem:

$$LHS=\sum\frac{m_a^2+m_bm_c}{m_a^5(m_b+m_c)}\overset{Lema}{\geq}\frac{3^5}{\left(\sum m_a\right)^4}\geq\frac{3^5}{\left(\frac{9R}{2}\right)^4}=\frac{16}{27R^4}=RHS.$$

Am folosit mai sus $\sum m_a\overset{Leuenberger}{\leq}4R+r\overset{Euler}{\leq}\frac{9R}{2}$.

Remarca.

In ΔABC

$$\sum\frac{h_a^2+h_bh_c}{h_a^5(h_b+h_c)}\geq\frac{16}{27R^4}.$$

Marin Chirciu

Soluție

Lema

If $x,y,z>0$

$$\sum\frac{x^2+yz}{x^5(y+z)}\geq\frac{3^5}{(x+y+z)^4}.$$

Folosind **Lema** pentru $(x,y,z)=(h_a,h_b,h_c)$ obținem:

$$LHS=\sum\frac{h_a^2+h_bh_c}{h_a^5(h_b+h_c)}\overset{Lema}{\geq}\frac{3^5}{\left(\sum h_a\right)^4}\geq\frac{3^5}{\left(\frac{9R}{2}\right)^4}=\frac{16}{27R^4}=RHS.$$

Am folosit mai sus $\sum h_a=\frac{p^2+r^2+4Rr}{2R}\overset{Gerretsen}{\leq}4R+r\overset{Euler}{\leq}\frac{9R}{2}$.

Remarca.

In ΔABC

$$\sum \frac{w_a^2 + w_b w_c}{w_a^5 (w_b + w_c)} \geq \frac{16}{27R^4}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$

$$\sum \frac{x^2 + yz}{x^5 (y+z)} \geq \frac{3^5}{(x+y+z)^4}.$$

Folosind **Lema** pentru $(x, y, z) = (w_a, w_b, w_c)$ obținem:

$$LHS = \sum \frac{w_a^2 + w_b w_c}{w_a^5 (w_b + w_c)} \overset{Lema}{\geq} \frac{3^5}{\left(\sum w_a\right)^4} \geq \frac{3^5}{\left(\frac{9R}{2}\right)^4} = \frac{16}{27R^4} = RHS.$$

Am folosit mai sus $\sum w_a \leq \sum m_a \overset{Leuenberger}{\leq} 4R + r \overset{Euler}{\leq} \frac{9R}{2}$.

Remarca.

In ΔABC

$$\sum \frac{s_a^2 + s_b s_c}{s_a^5 (s_b + s_c)} \geq \frac{16}{27R^4}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$

$$\sum \frac{x^2 + yz}{x^5 (y+z)} \geq \frac{3^5}{(x+y+z)^4}.$$

Folosind **Lema** pentru $(x, y, z) = (s_a, s_b, s_c)$ obținem:

$$LHS = \sum \frac{s_a^2 + s_b s_c}{s_a^5 (s_b + s_c)} \overset{Lema}{\geq} \frac{3^5}{\left(\sum s_a\right)^4} \geq \frac{3^5}{\left(\frac{9R}{2}\right)^4} = \frac{16}{27R^4} = RHS.$$

Am folosit mai sus $\sum s_a \le \sum m_a \overset{Leuenberger}{\le} 4R + r \overset{Euler}{\le} \frac{9R}{2}$.

Remarca.

In ΔABC

$$\sum \frac{r_a^2 + r_b r_c}{r_a^5 (r_b + r_c)} \ge \frac{16}{27R^4}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z > 0$

$$\sum \frac{x^2 + yz}{x^5 (y+z)} \ge \frac{3^5}{(x+y+z)^4}.$$

Folosind **Lema** pentru $(x, y, z) = (r_a, r_b, r_c)$ obținem:

$$LHS = \sum \frac{r_a^2 + r_b r_c}{r_a^5 (r_b + r_c)} \overset{Lema}{\ge} \frac{3^5}{\left(\sum r_a\right)^4} \ge \frac{3^5}{\left(\frac{9R}{2}\right)^4} = \frac{16}{27R^4} = RHS.$$

Am folosit mai sus $\sum r_a = 4R + r \overset{Euler}{\le} \frac{9R}{2}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația99.

If $a, b, c > 0$

$$\sum \frac{a^8 (a^2 + bc)}{(b+c)^{10}} \ge \frac{3}{512}, n \in \mathbf{N}.$$

Zaza Mzhavanadze, Georgia, RMM 2/2023

Soluție

Lema

If $x, y, z > 0$ and $n \in \mathbf{N}$ then

$$\sum \frac{x^8 (x^2 + yz)}{(y+z)^{10}} \ge \left(\frac{x}{y+z}\right)^9.$$

Soluție

$\frac{x^8\left(x^2+yz\right)}{\left(y+z\right)^{10}}=\frac{x^8}{\left(y+z\right)^{10}}\left(x-y\right)\left(x-z\right)+\left(\frac{x}{y+z}\right)^9$,(1).

Tripletele $\left(x,y,z\right)$ și $\left(\frac{x^8}{\left(y+z\right)^{10}},\frac{y^8}{\left(z+x\right)^{10}},\frac{z^8}{\left(x+y\right)^{10}}\right)$ sunt la fel ordonate.

Are loc inegalitatea lui Schur generalizată: $\sum\frac{x^8}{\left(y+z\right)^{10}}\left(x-y\right)\left(x-z\right)\geq 0,\ n\in\mathbf{N}$, (2).

Din (1) și (2) obținem: $\sum\frac{x^8\left(x^2+yz\right)}{\left(y+z\right)^{10}}\geq\sum\left(\frac{x}{y+z}\right)^9$.

Folosind **Lema** pentru $\left(x,y,z\right)=\left(a,b,c\right)$ obținem:

$$\sum\frac{a^8\left(a^2+bc\right)}{\left(b+c\right)^{10}}\geq\sum\left(\frac{a}{b+c}\right)^9\overset{Holder}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^9}{3^8}\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^9}{3^8}=\frac{3}{2^9}.$$

Remarca.

In ΔABC

$$\sum\frac{a^n\left(a^2+bc\right)}{\left(b+c\right)^{n+2}}\geq\frac{3}{2^n},n\in\mathbf{N}.$$

Marin Chirciu

<u>Soluție</u>

<u>Lema</u>

If $x,y,z>0$ and $n\in\mathbf{N}$ then

$$\sum\frac{x^n\left(x^2+yz\right)}{\left(y+z\right)^{n+2}}\geq\left(\frac{x}{y+z}\right)^{n+1}.$$

Folosind **Lema** pentru $\left(x,y,z\right)=\left(a,b,c\right)$ obținem:

$$\sum\frac{a^n\left(a^2+bc\right)}{\left(b+c\right)^{n+2}}\geq\sum\left(\frac{a}{b+c}\right)^{n+1}\overset{Holder}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^{n+1}}{3^n}\overset{Nesbitt}{\geq}\frac{\left(\frac{3}{2}\right)^{n+1}}{3^n}=\frac{3}{2^{n+1}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \frac{h_a^n\left(h_a^2+h_b h_c\right)}{\left(h_b+h_c\right)^{n+2}} \geq \frac{3}{2^n}, n \in \mathbf{N}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z>0$ and $n \in \mathbf{N}$ then

$$\sum \frac{x^n\left(x^2+yz\right)}{(y+z)^{n+2}} \geq \left(\frac{x}{y+z}\right)^{n+1}.$$

Demonstrație.

$$\frac{x^n\left(x^2+yz\right)}{(y+z)^{n+2}}=\frac{x^n}{(y+z)^{n+2}}(x-y)(x-z)+\left(\frac{x}{y+z}\right)^{n+1},(1).$$

Tripletele (x, y, z) și $\left(\frac{x^n}{(y+z)^{n+2}}, \frac{y^n}{(z+x)^{n+2}}, \frac{z^n}{(x+y)^{n+2}}\right)$ sunt la fel ordonate.

Are loc inegalitatea lui Schur generalizată: $\sum \frac{x^n}{(y+z)^{n+2}}(x-y)(x-z) \geq 0,\ n \in \mathbf{N}$, (2).

Din (1) și (2) obținem: $\sum \frac{x^n\left(x^2+yz\right)}{(y+z)^{n+2}} \geq \sum\left(\frac{x}{y+z}\right)^{n+1}$.

Folosind **Lema** pentru $(x, y, z)=\left(h_a, h_b, h_c\right)$ obținem:

$$\sum \frac{h_a^n\left(h_a^2+h_b h_c\right)}{\left(h_b+h_c\right)^{n+2}} \geq \sum\left(\frac{h_a}{h_b+h_c}\right)^{n+1} \overset{Holder}{\geq} \frac{\left(\sum \frac{h_a}{h_b+h_c}\right)^{n+1}}{3^n} \overset{Nesbitt}{\geq} \frac{\left(\frac{3}{2}\right)^{n+1}}{3^n}=\frac{3}{2^{n+1}}.$$

Remarca.

In ΔABC

$$\sum \frac{w_a^n\left(w_a^2+w_b w_c\right)}{\left(w_b+w_c\right)^{n+2}} \geq \frac{3}{2^n}, n \in \mathbf{N}.$$

Marin Chirciu

Soluție

Lema

If $x, y, z>0$ and $n \in \mathbf{N}$ then

$$\sum \frac{x^n\left(x^2+yz\right)}{\left(y+z\right)^{n+2}} \geq \left(\frac{x}{y+z}\right)^{n+1}.$$

Folosind **Lema** pentru $(x,y,z)=(w_a,w_b,w_c)$ obținem:

$$\sum \frac{w_a^n\left(w_a^2+w_bw_c\right)}{\left(w_b+w_c\right)^{n+2}} \geq \sum\left(\frac{w_a}{w_b+w_c}\right)^{n+1} \overset{Holder}{\geq} \frac{\left(\sum \frac{w_a}{w_b+w_c}\right)^{n+1}}{3^n} \overset{Nesbitt}{\geq} \frac{\left(\frac{3}{2}\right)^{n+1}}{3^n} = \frac{3}{2^{n+1}}.$$

Egali

Remarca.

In ΔABC

$$\sum \frac{m_a^n\left(m_a^2+m_bm_c\right)}{\left(m_b+m_c\right)^{n+2}} \geq \frac{3}{2^n}, n\in \mathbf{N}.$$

Marin Chirciu

<u>Soluție</u>

<u>Lema</u>

If $x,y,z>0$ and $n\in\mathbf{N}$ then

$$\sum \frac{x^n\left(x^2+yz\right)}{\left(y+z\right)^{n+2}} \geq \left(\frac{x}{y+z}\right)^{n+1}.$$

Folosind **Lema** pentru $(x,y,z)=(m_a,m_b,m_c)$ obținem:

$$\sum \frac{m_a^n\left(m_a^2+m_bm_c\right)}{\left(m_b+m_c\right)^{n+2}} \geq \sum\left(\frac{m_a}{m_b+m_c}\right)^{n+1} \overset{Holder}{\geq} \frac{\left(\sum \frac{m_a}{m_b+m_c}\right)^{n+1}}{3^n} \overset{Nesbitt}{\geq} \frac{\left(\frac{3}{2}\right)^{n+1}}{3^n} = \frac{3}{2^{n+1}}.$$

Remarca.

In ΔABC

$$\sum \frac{s_a^n\left(s_a^2+s_bs_c\right)}{\left(s_b+s_c\right)^{n+2}} \geq \frac{3}{2^n}, n\in \mathbf{N}.$$

Marin Chirciu

<u>Soluție</u>

<u>Lema</u>

If $x,y,z>0$ and $n\in\mathbf{N}$ then

$$\sum \frac{x^n\left(x^2+yz\right)}{\left(y+z\right)^{n+2}} \geq \left(\frac{x}{y+z}\right)^{n+1}.$$

Folosind **Lema** pentru $\left(x,y,z\right)=\left(s_a,s_b,s_c\right)$ obținem:

$$\sum \frac{s_a^n\left(s_a^2+s_bs_c\right)}{\left(s_b+s_c\right)^{n+2}} \geq \sum\left(\frac{s_a}{s_b+s_c}\right)^{n+1} \overset{Holder}{\geq} \frac{\left(\sum \frac{s_a}{s_b+s_c}\right)^{n+1}}{3^n} \overset{Nesbitt}{\geq} \frac{\left(\frac{3}{2}\right)^{n+1}}{3^n} = \frac{3}{2^{n+1}}.$$

Remarca.

In ΔABC

$$\sum \frac{r_a^n\left(r_a^2+r_br_c\right)}{\left(r_b+r_c\right)^{n+2}} \geq \frac{3}{2^n}, n \in \mathbf{N}.$$

Marin Chirciu

Soluție

Lema

If $x,y,z>0$ and $n\in\mathbf{N}$ then

$$\sum \frac{x^n\left(x^2+yz\right)}{\left(y+z\right)^{n+2}} \geq \left(\frac{x}{y+z}\right)^{n+1}.$$

Folosind **Lema** pentru $\left(x,y,z\right)=\left(r_a,r_b,r_c\right)$ obținem:

$$\sum \frac{r_a^n\left(r_a^2+r_br_c\right)}{\left(r_b+r_c\right)^{n+2}} \geq \sum\left(\frac{r_a}{r_b+r_c}\right)^{n+1} \overset{Holder}{\geq} \frac{\left(\sum \frac{r_a}{r_b+r_c}\right)^{n+1}}{3^n} \overset{Nesbitt}{\geq} \frac{\left(\frac{3}{2}\right)^{n+1}}{3^n} = \frac{3}{2^{n+1}}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Aplicația100.

In ΔABC, G -centroid, R_1,R_2,R_3 -circumradii of $\Delta GBC,\Delta GCA,\Delta GAB$. Show that

$$\sum \frac{m_bm_c}{bc} = \frac{3}{4R}\left(R_1+R_2+R_3\right).$$

Ertan Yildirim, Turkey,RMM 2/2023

Soluție

Lema

In ΔABC, G -centroid, R_1 -circumradii of ΔGBC

$$R_1 = \frac{am_bm_c}{3S}.$$

Demonstrație

Folosind $R = \frac{abc}{4S}$ în ΔGBC obținem: $R_1 = \frac{GB \cdot GC \cdot BC}{4[GBC]} = \frac{\frac{2}{3}m_b \cdot \frac{2}{3}m_b \cdot a}{4 \cdot \frac{S}{3}} = \frac{am_bm_c}{3S}$.

$$R_1 + R_2 + R_3 = \sum \frac{am_bm_c}{3S} = \sum \frac{am_bm_c}{3 \cdot \frac{abc}{4R}} = \frac{4R}{3}\sum \frac{m_bm_c}{bc} \Rightarrow \sum \frac{m_bm_c}{bc} = \frac{3}{4R}(R_1 + R_2 + R_3).$$

Remarca.

In ΔABC, G -centroid, R_1, R_2, R_3 -circumradii of $\Delta GBC, \Delta GCA, \Delta GAB$. Show that

$$\frac{24r^3}{R} \le R_1R_2 + R_2R_3 + R_3R_1 \le \frac{3R^5}{8r^3}.$$

Marin Chirciu

Soluție

Lema

In ΔABC, G -centroid, R_1 -circumradii of ΔGBC

$$R_1 = \frac{am_bm_c}{3S}.$$

Inegalitatea din dreapta.

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{m_am_bm_c}{9S^2}\sum bcm_a \le \frac{\frac{Rp^2}{2}}{9p^2r^2} \cdot \frac{R^2p^2}{r} = \frac{R^3p^2}{18r^3} \overset{Mitrinovic}{\le} \frac{R^3 \cdot \frac{27R^2}{4}}{18r^3} = \frac{3R^5}{8r^3}.$$

Am folosit mai sus:

$$\sum bcm_a \overset{Panaitopol}{\le} \sum bc\frac{Rp}{a} = Rp\sum \frac{bc}{a} = Rp\frac{\sum b^2c^2}{abc} \overset{Goldstone}{\le} Rp\frac{4R^2p^2}{4Rrp} = \frac{R^2p^2}{r}$$

Inegalitatea din stânga.

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{m_am_bm_c}{9S^2}\sum bcm_a \ge \frac{rp^2}{9p^2r^2} \cdot \frac{216r^4}{R} = \frac{24r^3}{R}$$

Am folosit mai sus:

$rp^2 \le m_a m_b m_c \le \frac{Rp^2}{2}$ și $9r \le m_a + m_b + m_c \le \frac{9R}{2}$.

$$\sum bcm_a \ge \sum bch_a = \sum bc \cdot \frac{2S}{a} = 2S\sum \frac{bc}{a} = 2S\frac{\sum b^2c^2}{abc} \overset{Goldstone}{\le} 2S \cdot \frac{16r^2p^2}{4RS} = \frac{8r^2p^2}{R} \overset{Mitrinovic}{\le}$$

$$\overset{Mitrinovic}{\le} \frac{8r^2 \cdot 27r^2}{R} = \frac{216r^4}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC, G -centroid, R_1, R_2, R_3 -circumradii of $\Delta GBC, \Delta GCA, \Delta GAB$. Show that

$$12r^2 \le R_1^2 + R_2^2 + R_3^2 \le \frac{3R^7}{32r^5}.$$

Marin Chirciu

Soluție

Lema

In ΔABC, G -centroid, R_1 -circumradii of ΔGBC

$$R_1 = \frac{am_bm_c}{3S}.$$

$$R_1^2 + R_2^2 + R_3^2 = \sum \left(\frac{am_bm_c}{3S}\right)^2 = \frac{1}{9S^2}\sum a^2m_b^2m_c^2.$$

Inegalitatea din dreapta.

$$R_1^2 + R_2^2 + R_3^2 = \frac{1}{9S^2}\sum a^2m_b^2m_c^2 \le \frac{1}{9S^2}\sum a^2\left(\frac{Rp}{b}\right)^2\left(\frac{Rp}{c}\right)^2 = \frac{R^4p^4}{9p^2r^2}\sum \frac{a^2}{b^2c^2} = \frac{R^4p^2}{9r^2} \cdot \frac{\sum a^4}{a^2b^2c^2} =$$

$$= \frac{R^4p^2}{9r^2} \cdot \frac{\sum a^4}{16R^2r^2p^2} = \frac{R^2}{144r^4} \cdot \sum a^4 \le \frac{R^2}{144r^4} \cdot \frac{27R^5}{2r} = \frac{3R^7}{32r^5}.$$

Am folosit mai sus:

$m_a \le \frac{Rp}{a}$,(Panaitopol), $\sum a^4 \le (4R+r)^2 \cdot \frac{2R^3}{3r} \le \frac{27R^5}{2r}$.

Inegalitatea din stânga.

$$R_1^2 + R_2^2 + R_3^2 = \frac{1}{9S^2}\sum a^2m_b^2m_c^2 \ge \frac{1}{9S^2}\sum a^2p(p-b)p(p-c) = \frac{p^2}{9p^2r^2}\sum a^2(p-b)(p-c) =$$

$$=\frac{1}{9r^2}\cdot 4p^2r(R-r)=\frac{4p^2r(R-r)}{9r^2}\geq\frac{4\cdot 27r^2\cdot r(R-r)}{9r^2}=12r(R-r)\geq 12r\cdot r=12r^2.$$

Am folosit mai sus:

$m_a^2\geq p(p-a)$ și $\sum a^2(p-b)(p-c)=4p^2r(R-r)$.

Remarca.

In ΔABC, G -centroid, R_1,R_2,R_3 -circumradii of $\Delta GBC,\Delta GCA,\Delta GAB$. Show that

$$\frac{12r^2}{R^3}\leq\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_2}\leq\frac{3R}{4r^2}.$$

Marin Chirciu

Soluție

Lema

In ΔABC, G -centroid, R_1 -circumradii of ΔGBC

$$R_1=\frac{am_bm_c}{3S}.$$

Folosind **Lema** obținem:

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_2}=\sum\frac{3S}{am_bm_c}=3S\frac{\sum bcm_a}{abcm_am_bm_c}=\frac{3S}{4RSm_am_bm_c}\sum bcm_a=\frac{3}{4Rm_am_bm_c}\sum bcm_a.$$

Inegalitatea din dreapta.

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_2}=\frac{3}{4Rm_am_bm_c}\sum bcm_a\leq\frac{3}{4Rm_am_bm_c}\sum bc\frac{Rp}{a}=\frac{3p}{4m_am_bm_c}\sum\frac{bc}{a}\leq$$

$$\leq\frac{3p}{4\cdot rp^2}\cdot\frac{Rp}{r}=\frac{3R}{4r^2}.$$

Am folosit mai sus:

$m_a\leq\frac{Rp}{a}$,(Panaitopol), $m_am_bm_c\geq rp^2$ și $\sum\frac{bc}{a}=\frac{\sum b^2c^2}{abc}\overset{Goldstone}{\leq}\frac{4R^2p^2}{4Rrp}=\frac{Rp}{r}$.

Inegalitatea din stânga.

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_2}=\frac{3}{4Rm_am_bm_c}\sum bcm_a\geq\frac{3}{4Rm_am_bm_c}\sum bch_a=\frac{3}{4Rm_am_bm_c}\sum bc\cdot\frac{2S}{a}=$$

$$=\frac{3S}{2Rm_am_bm_c}\sum\frac{bc}{a}\geq\frac{3rp}{2R\cdot\frac{Rp^2}{2}}\cdot\frac{4rp}{R}=\frac{12r^2}{R^3}$$

Am folosit mai sus:

$$m_a m_b m_c \leq \frac{Rp^2}{2} \text{ și } \sum \frac{bc}{a} = \frac{\sum b^2c^2}{abc} \overset{Goldstone}{\geq} \frac{16r^2p^2}{4Rrp} = \frac{4rp}{R}.$$

Remarca.

In ΔABC, G -centroid, R_1, R_2, R_3 -circumradii of $\Delta GBC, \Delta GCA, \Delta GAB$. Show that

$$\frac{48r^4}{R^6} \leq \frac{1}{R_1R_2} + \frac{1}{R_2R_3} + \frac{1}{R_3R_1} \leq \frac{3R^2}{16r^4}.$$

Marin Chirciu

Soluție

Lema

In ΔABC, G -centroid, R_1 -circumradii of ΔGBC

$$R_1 = \frac{am_bm_c}{3S}.$$

Folosind **Lema** obținem:

$$\frac{1}{R_1R_2} + \frac{1}{R_2R_3} + \frac{1}{R_3R_1} = \sum \frac{3S}{bm_cm_a} \frac{3S}{cm_am_b} = \frac{9S}{4R} \cdot \frac{\sum am_bm_c}{(m_am_bm_c)^2}.$$

Inegalitatea din dreapta.

$$\frac{1}{R_1R_2} + \frac{1}{R_2R_3} + \frac{1}{R_3R_1} = \frac{9S}{4R} \cdot \frac{\sum am_bm_c}{(m_am_bm_c)^2} \leq \frac{9rp}{4R} \cdot \frac{1}{(rp^2)^2} \cdot \frac{9R^3p}{4r} = \frac{81R^2}{16r^2p^2} \leq \frac{81R^2}{16r^2 \cdot 27r^2} = \frac{3R^2}{16r^4}.$$

Am folosit mai sus:

$$\sum am_bm_c \overset{Panaitopol}{\leq} \sum a \frac{Rp}{b} \frac{Rp}{c} = R^2p^2 \sum \frac{a}{bc} = R^2p^2 \frac{\sum a^2}{abc} \overset{Leibniz}{\leq} R^2p^2 \cdot \frac{9R}{4rp} = \frac{9R^3p}{4r}.$$

$$m_a \leq \frac{Rp}{a} \text{ ,(Panaitopol), } m_am_bm_c \geq rp^2 \text{ și } \sum \frac{a}{bc} = \frac{\sum a^2}{abc} \overset{Leibniz}{\leq} \frac{9R^2}{4Rrp} = \frac{9R}{4rp}.$$

Inegalitatea din stânga.

$$\frac{1}{R_1R_2} + \frac{1}{R_2R_3} + \frac{1}{R_3R_1} = \frac{9S}{4R} \cdot \frac{\sum am_bm_c}{(m_am_bm_c)^2} \geq \frac{9rp}{4R} \cdot \frac{1}{\left(\frac{Rp^2}{2}\right)^2} \cdot \frac{36r^3p}{R} = \frac{81r^4}{R^4p^2} \geq \frac{4 \cdot 81r^4}{R^4 \cdot \frac{27R^2}{4}} = \frac{48r^4}{R^6}.$$

Am folosit mai sus:

$$\sum am_bm_c \overset{m_a\ge h_a}{\ge} \sum ah_bh_c = \sum a\frac{2S}{b}\frac{2S}{c} = 4S^2\sum\frac{a}{bc} \ge 4p^2r^2\cdot\frac{9r}{Rp} = \frac{36r^3p}{R};$$

$$\sum\frac{a}{bc} = \frac{\sum a^2}{abc} \overset{Neuberg}{\ge} \frac{36r^2}{4Rrp} = \frac{9r}{Rp} \text{ și } m_am_bm_c \le \frac{Rp^2}{2}.$$

Remarca.

In ΔABC, G -centroid, R_1, R_2, R_3 -circumradii of $\Delta GBC, \Delta GCA, \Delta GAB$. Show that

$$\frac{48r^4}{R^6} \le \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} \le \frac{9R^2}{16r^4} - \frac{6}{R^2}.$$

Marin Chirciu

Soluție

Lema

In ΔABC, G -centroid, R_1 -circumradii of ΔGBC

$$R_1 = \frac{am_bm_c}{3S}.$$

$$\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} = \sum\left(\frac{3S}{am_bm_c}\right)^2 = 9S^2\sum\frac{1}{(am_bm_c)^2} = 9S^2\cdot\frac{\sum b^2c^2m_a^2}{(abc)^2(m_am_bm_c)^2}.$$

Inegalitatea din dreapta.

$$\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} = 9S^2\cdot\frac{\sum b^2c^2m_a^2}{(abc)^2(m_am_bm_c)^2} \le \frac{9p^2r^2}{16R^2r^2p^2}\cdot\frac{1}{(rp^2)^2}\cdot\frac{R^2p^2(R^2p^2-72r^4)}{r^2} =$$

$$= \frac{9(R^2p^2-72r^4)}{16r^4p^2} = \frac{9}{16r^2}\left(\frac{R^2}{r^2} - \frac{72r^2}{p^2}\right) \le \frac{9}{16r^2}\left(\frac{R^2}{r^2} - \frac{72r^2}{\frac{27R^2}{4}}\right) = \frac{9}{16r^2}\left(\frac{R^2}{r^2} - \frac{32r^2}{3R^2}\right) =$$

$$= \frac{3(3R^4-32r^4)}{16R^2r^4} = \frac{9R^2}{16r^4} - \frac{6}{R^2}.$$

Am folosit mai sus:

$m_am_bm_c \ge rp^2$, $\sum b^2c^2m_a^2 \le \dfrac{R^2p^2(R^2p^2-72r^4)}{r^2}$, vezi:

$$\sum b^2c^2m_a^2 \overset{Panaitopol}{\le} \sum b^2c^2\frac{R^2p^2}{a^2} = R^2p^2\sum\frac{b^2c^2}{a^2} \le R^2p^2\cdot\frac{R^2p^2-72r^4}{r^2} = \frac{R^2p^2(R^2p^2-72r^4)}{r^2}.$$

$$\sum\frac{b^2c^2}{a^2}=\frac{\sum b^4c^4}{a^2b^2c^2}\overset{Goldstone}{\leq}\frac{16R^2p^2\left(R^2p^2-72r^4\right)}{16R^2r^2p^2}=\frac{R^2p^2-72r^4}{r^2}.$$

$$\sum b^4c^4=\left(\sum b^2c^2\right)^2-2a^2b^2c^2\sum a^2\overset{Goldstone}{\leq}\left(4R^2p^2\right)^2-2\cdot 16R^2r^2p^2\cdot 36r^2=$$

$$=16R^2p^2\left(R^2p^2-72r^4\right).$$

Inegalitatea din stânga.

$$\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}=9S^2\cdot\frac{\sum b^2c^2m_a^2}{(abc)^2\left(m_am_bm_c\right)^2}\geq\frac{9p^2r^2}{16R^2r^2p^2}\cdot\frac{1}{\left(\frac{Rp^2}{2}\right)^2}\cdot 144r^4p^2=\frac{36r^4}{R^4p^2}\overset{Mitrinovic}{\geq}$$

$$\overset{Mitrinovic}{\geq}\frac{324r^4}{R^4\cdot\frac{27R^2}{4}}=\frac{48r^4}{R^6}.$$

Am folosit mai sus:

$m_am_bm_c\leq\frac{Rp^2}{2}$, $\sum b^2c^2m_a^2\geq 144r^4p^2$,vezi:

$$\sum b^2c^2m_a^2\geq\sum b^2c^2h_a^2=\sum b^2c^2\frac{4S^2}{a^2}=4S^2\sum\frac{b^2c^2}{a^2}\geq 4r^2p^2\cdot 36r^2=144r^4p^2;$$

$$\sum\frac{b^2c^2}{a^2}=\frac{\sum b^4c^4}{a^2b^2c^2}\overset{Goldstone}{\geq}\frac{576R^2r^4p^2}{16R^2r^2p^2}=36r^2;$$

$$\sum b^4c^4\geq a^2b^2c^2\sum a^2\overset{Neuberg}{\geq}16R^2r^2p^2\cdot 36r^2=576R^2r^4p^2;$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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Art 5200

1 Decembrie 2023

4. O inegalitate şi aplicaţii în triunghi

De Gheorghe Ghiţă, Buzău

Articolul prezintă o inegalitate şi aplicaţii ale acesteia în special pentru triunghi.

Propoziţie

$a, b, c, x, y, z > 0 \Rightarrow$

$$\frac{ax+bz+cy}{a^2}+\frac{by+cx+az}{b^2}+\frac{cz+ay+bx}{c^2} \geq (x+y+z)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \quad (1)$$

Gheorghe Ghiţă, Buzău

Soluţie.$\sum \frac{ax+bz+cy}{a^2} \geq \sum x \sum \frac{1}{a} \Leftrightarrow \sum\left(\frac{x}{a}+\frac{bz+cy}{a^2}\right) \geq \sum x \sum \frac{1}{a} \Leftrightarrow \sum \frac{bz+cy}{a^2} \geq \sum x \sum \frac{1}{a} - \sum \frac{x}{a} \Leftrightarrow \sum \frac{bz+cy}{a^2} \geq \sum \frac{\sum x}{a} - \sum \frac{x}{a} \Leftrightarrow \sum \frac{bz+cy}{a^2} \geq \sum \frac{\sum x - x}{a} = \sum \frac{y+z}{a} \Leftrightarrow \sum \frac{bz+cy}{a^2} - \sum \frac{y+z}{a} \geq 0 \Leftrightarrow \sum\left(\frac{bz+cy}{a^2}-\frac{y+z}{a}\right) \geq 0 \Leftrightarrow$

$\sum x\left(\frac{b}{c^2}+\frac{c}{b^2}-\frac{1}{b}-\frac{1}{c}\right) \geq 0 \Leftrightarrow \sum x \frac{(b+c)(b-c)^2}{b^2c^2} \geq 0.$

Egalitatea are loc dacă şi numai dacă $a = b = c$.

Corolar. $a, b, c, x, y, z > 0 \Rightarrow$

$$\frac{ax+bz+cy}{a^2}+\frac{by+cx+az}{b^2}+\frac{cz+ay+bx}{c^2} \geq \frac{9(x+y+z)}{a+b+c} \quad (2)$$

$$\frac{ax+bz+cy}{a^2}+\frac{by+cx+az}{b^2}+\frac{cz+ay+bx}{c^2} \overset{(1)}{\geq} (x+y+z)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \overset{MA-MH}{\geq} \frac{9(x+y+z)}{a+b+c}.$$

Lemă.$\Delta ABC \Rightarrow$

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \geq \frac{2pr}{R^2}.$$

Gheorghe Ghiţă, Buzău

Demonstraţie.$\sum \cos\frac{A}{2} = \sum \sqrt{\frac{p(p-a)}{bc}} = \sum \sqrt{\frac{2p}{abc}\cdot\frac{a(p-a)}{2}} = \sqrt{\frac{2p}{abc}} \sum \sqrt{\frac{a}{2}(p-a)} \overset{MA-MG}{\geq} \sqrt{\frac{2p}{4pRr}} \cdot \sum \frac{\frac{2a}{2}(p-a)}{\frac{a}{2}+p-a} = \sqrt{\frac{1}{2Rr}} \cdot$

$\sum \frac{2a(p-a)}{2p-a} \overset{Euler}{\geq} \frac{2}{R} \sum \frac{a(p-a)}{b+c} = \frac{2}{R}\left(p\sum\frac{a}{b+c} - \sum\frac{a^2}{b+c}\right) \overset{(1)}{=} \frac{2}{R}\left(p \cdot \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} - \frac{2p(p^2-4Rr-3r^2)}{p^2+r^2+2Rr}\right) = \frac{4pr(3R+2r)}{R(p^2+r^2+2Rr)} \geq \frac{2pr}{R^2} \Leftrightarrow$

$6R^2+2Rr-r^2 \geq p^2$, adevărată, deoarece $p^2 \overset{Gerretsen}{\leq} 4R^2+4Rr+3r^2$ şi $4R^2+4Rr+3r^2 \leq 6R^2+2Rr-r^2 \Leftrightarrow (R-2r)(R+r) \geq 0$;

$$(1): \sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}, \sum \frac{a^2}{b+c} = \frac{4p^2(p^2-4Rr-3r^2)}{2p(p^2+r^2+2Rr)}.$$

Aplicaţii

A1. $a, b, c > 0 \Rightarrow$

$$(a+b+c)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Soluţia1. Se aplică (1) pentru $x = y = z$.

Egalitatea are loc dacă şi numai dacă $a = b = c$.

Soluţia2. $a, b, c > 0 \Rightarrow x = b + c, y = c + a, z = a + b$ sunt laturile unui triunghi,$\sum a = \frac{\sum x}{2} = s(semiperimetrul)$

$$a = \frac{-x+y+z}{2}, b = \frac{x-y+z}{2}, c = \frac{x+y-z}{2}, s = a+b+c, a = s-x, b = s-y, c = s-z;$$

$$abc = (s-x)(s-y)(s-z) = r^2 s, \sum a = \sum \frac{x+y+z}{2} = p, \sum ab = r(4R+r), \sum a^2 = s^2 - 8Rr - 2r^2$$

$$\sum \frac{1}{a} = \frac{4R+r}{pr}, \sum \frac{1}{a^2} = \frac{(\sum ab)^2 - abc\sum a}{(abc)^2} = \frac{(4R+r)^2 - 2p^2}{p^2 r^2}$$

$$(a+b+c)\left(\tfrac{1}{a^2}+\tfrac{1}{b^2}+\tfrac{1}{c^2}\right) \geq 3\left(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}\right) \Leftrightarrow (R-2r)(2R+r) \geq 0.$$

A2. $a, b, c > 0 \Rightarrow$

$$\frac{2bc}{a^2}+\frac{2ca}{b^2}+\frac{2ab}{c^2}+3 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Soluţie. Se aplică (1) pentru $(x, y, z) = (a, b, c)$.

Egalitatea are loc dacă şi numai dacă $a = b = c$.

A3. $a, b, c > 0 \Rightarrow$

$$\frac{b^2+bc+c^2}{a}+\frac{c^2+ca+a^2}{b}+\frac{a^2+ab+b^2}{c} \geq abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

Soluţie. Se aplică (1) pentru $(x, y, z) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

Egalitatea are loc dacă şi numai dacă $a = b = c$.

A4. $a, b, c > 0 \Rightarrow$

$$\frac{b^2+bc+c^2}{a}+\frac{c^2+ca+a^2}{b}+\frac{a^2+ab+b^2}{c} \geq (ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Soluţie. Se aplică (1) pentru $(x, y, z) = (bc, ca, ab)$.

Egalitatea are loc dacă şi numai dacă $a = b = c$.

A5. $a,b,c>0\Rightarrow$

$$\frac{a^{n+1}+b^nc+bc^n}{a^2}+\frac{b^{n+1}+a^nc+ca^n}{b^2}+\frac{c^{n+1}+a^nb+ab^n}{c^2}\geq(a^n+b^n+c^n)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Soluţie. Se aplică (1) pentru $(x,y,z)=(a^n,b^n,c^n), n\geq 0$.

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A6. $a,b,c>0\Rightarrow$

$$\frac{ka^2+2kbc+2nab+nc^2}{a^2}+\frac{kb^2+2kca+2nbc+na^2}{b^2}+\frac{kc^2+2kab+2nca+nc^2}{c^2}$$
$$\geq(n+k)(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Soluţie. Se aplică (1) pentru $(x,y,z)=(ka+nb,kb+nc,kc+na)$.

Egalitatea are loc dacă şi numai dacă $a=b=c$.

A7. $\Delta ABC\Rightarrow$

$$\frac{ah_a+bh_c+ch_b}{a^2}+\frac{bh_b+ch_a+ah_c}{b^2}+\frac{ch_c+ah_b+bh_a}{c^2}\geq\frac{9\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ atunci:

$$\frac{ah_a+bh_c+ch_b}{a^2}+\frac{bh_b+ch_a+ah_c}{b^2}+\frac{ch_c+ah_b+bh_a}{c^2}\overset{(2)}{\geq}\frac{9(h_a+h_b+h_c)}{a+b+c}\overset{\sum h_a\geq 9r}{\geq}\frac{81r}{2p}\overset{Mitrinovic}{\geq}\frac{9\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A8. $\Delta ABC\Rightarrow$

$$\frac{am_a+bm_c+cm_b}{a^2}+\frac{bm_b+cm_a+am_c}{b^2}+\frac{cm_c+am_b+bm_a}{c^2}\geq\frac{9\sqrt{3}r}{R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ atunci:

$$\frac{am_a+bm_c+cm_b}{a^2}+\frac{bm_b+cm_a+am_c}{b^2}$$
$$+\frac{cm_c+am_b+bm_a}{c^2}\overset{(2)}{\geq}\frac{9(m_a+m_b+m_c)}{a+b+c}\overset{\sum m_a\geq 9r}{\geq}\frac{81r}{2p}\overset{Mitrinovic}{\geq}\frac{9\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A9. $\Delta ABC\Rightarrow$

$$\frac{ar_a+br_c+cr_b}{a^2}+\frac{br_b+cr_a+ar_c}{b^2}+\frac{cr_c+ar_b+br_a}{c^2}\geq\frac{9\sqrt{3}r}{R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(r_a,r_b,r_c)$ atunci:

$$\frac{ar_a+br_c+cr_b}{a^2}+\frac{br_b+cr_a+ar_c}{b^2}+\frac{cr_c+ar_b+br_a}{c^2}\overset{(2)}{\geq}\frac{9(r_a+r_b+r_c)}{a+b+c}\overset{\sum r_a=4R+r}{\cong}\frac{9(4R+r)}{2p}\overset{Euler}{\geq}\frac{81r}{2p}\overset{Mitrinovic}{\geq}\frac{9\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A10. $\Delta ABC\Rightarrow$

$$\frac{aw_a+bw_c+cw_b}{a^2}+\frac{bw_b+cw_a+aw_c}{b^2}+\frac{cw_c+aw_b+bw_a}{c^2}\geq\frac{9\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$atunci:

$$\frac{aw_a+bw_c+cw_b}{a^2}+\frac{bw_b+cw_a+aw_c}{b^2}+\frac{cw_c+aw_b+bw_a}{c^2}\overset{(2)}{\geq}\frac{9(w_a+w_b+w_c)}{a+b+c}\overset{\sum w_a\geq 9r}{\geq}\frac{81r}{2p}\overset{Mitrinovic}{\geq}\frac{9\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A11. $\Delta ABC\Rightarrow$

$$\frac{as_a+bs_c+cs_b}{a^2}+\frac{bw_b+cw_a+aw_c}{b^2}+\frac{cw_c+aw_b+bw_a}{c^2}\geq\frac{9\sqrt{3}r}{R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ atunci:

$$\frac{as_a+bs_c+cs_b}{a^2}+\frac{bs_b+cs_a+as_c}{b^2}+\frac{cs_c+as_b+bs_a}{c^2}\overset{(2)}{\geq}\frac{9(s_a+s_b+s_c)}{a+b+c}\overset{\sum s_a\geq 9r}{\geq}\frac{81r}{2p}\overset{Mitrinovic}{\geq}\frac{9\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A12. $\Delta ABC\Rightarrow$

$$\frac{asinA+bsinC+csinB}{a^2}+\frac{bsinB+csinA+asinC}{b^2}+\frac{csinC+asinB+bsinA}{c^2}\geq\frac{9}{2R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(sinA,sinB,sinC)$ atunci:

$$\frac{asinA+bsinC+csinB}{a^2}+\frac{bsinB+csinA+asinC}{b^2}+\frac{csinC+asinB+bsinA}{c^2}\overset{(2)}{\geq}\frac{9(sinA+sinB+sinC)}{a+b+c}\overset{\sum sinA=\frac{p}{R}}{=}\frac{9}{2R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A13. $\Delta ABC \Rightarrow$

$$\frac{asin\frac{A}{2}+bsin\frac{C}{2}+csin\frac{B}{2}}{a^2}+\frac{bsin\frac{B}{2}+csin\frac{A}{2}+asin\frac{C}{2}}{b^2}+\frac{csin\frac{C}{2}+asin\frac{B}{2}+bsin\frac{A}{2}}{c^2}\geq\frac{3\sqrt{3}}{2R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$:

$$\frac{asin\frac{A}{2}+bsin\frac{C}{2}+csin\frac{B}{2}}{a^2}+\frac{bsin\frac{B}{2}+csin\frac{A}{2}+asin\frac{C}{2}}{b^2}+\frac{csin\frac{C}{2}+asin\frac{B}{2}+bsin\frac{A}{2}}{c^2}\overset{(2)}{\geq}\frac{9\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)}{a+b+c}$$

$$\overset{Jensen}{\geq}\frac{27}{4p}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}}{2R}$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A14. $\Delta ABC \Rightarrow$

$$\frac{acos\frac{A}{2}+bcos\frac{C}{2}+ccos\frac{B}{2}}{a^2}+\frac{bcos\frac{B}{2}+ccos\frac{A}{2}+acos\frac{C}{2}}{b^2}+\frac{ccos\frac{C}{2}+acos\frac{B}{2}+bcos\frac{A}{2}}{c^2}\geq\frac{9r}{R^2}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{acos\frac{A}{2}+bcos\frac{C}{2}+ccos\frac{B}{2}}{a^2}+\frac{bcos\frac{B}{2}+ccos\frac{A}{2}+acos\frac{C}{2}}{b^2}+\frac{ccos\frac{C}{2}+acos\frac{B}{2}+bcos\frac{A}{2}}{c^2}\overset{(2)}{\geq}\frac{9\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)}{a+b+c}$$

$$\overset{Lemă}{\geq}\frac{9\frac{2pr}{R^2}}{2p}=\frac{9r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A15. $\Delta ABC \Rightarrow$

$$\frac{atg\frac{A}{2}+btg\frac{C}{2}+ctg\frac{B}{2}}{a^2}+\frac{btg\frac{B}{2}+ctg\frac{A}{2}+atg\frac{C}{2}}{b^2}+\frac{ctg\frac{C}{2}+atg\frac{B}{2}+btg\frac{A}{2}}{c^2}\geq\frac{6r}{R^2}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$\frac{atg\frac{A}{2}+btg\frac{C}{2}+ctg\frac{B}{2}}{a^2}+\frac{btg\frac{B}{2}+ctg\frac{A}{2}+atg\frac{C}{2}}{b^2}+\frac{ctg\frac{C}{2}+atg\frac{B}{2}+btg\frac{A}{2}}{c^2}\overset{(2)}{\geq}\frac{9\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)}{a+b+c}\overset{\sum tg\frac{A}{2}=\frac{4R+r}{p}}{\geq}$$

$$\frac{9(4R+r)}{2p^2}\overset{Euler}{\geq}\frac{81r}{2p^2}\overset{Mitrinovic}{\geq}\frac{6r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A16. $\Delta ABC\Rightarrow$

$$\frac{actg\frac{A}{2}+bctg\frac{C}{2}+cctg\frac{B}{2}}{a^2}+\frac{bctg\frac{B}{2}+cctg\frac{A}{2}+actg\frac{C}{2}}{b^2}+\frac{cctg\frac{C}{2}+actg\frac{B}{2}+bctg\frac{A}{2}}{c^2}\geq\frac{9}{2r}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$\frac{actg\frac{A}{2}+bctg\frac{C}{2}+cctg\frac{B}{2}}{a^2}+\frac{bctg\frac{B}{2}+cctg\frac{A}{2}+actg\frac{C}{2}}{b^2}$$
$$+\frac{cctg\frac{C}{2}+actg\frac{B}{2}+bctg\frac{A}{2}}{c^2}\overset{(2)}{\geq}\frac{9\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)}{a+b+c}$$

$$\overset{\sum ctg\frac{A}{2}=\frac{p}{r}}{\geq}\frac{9p}{2pr}=\frac{9}{2r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A17. $\Delta ABC\Rightarrow$

$$\frac{h_am_a+h_bm_c+h_cm_b}{h_a^2}+\frac{h_bm_b+h_cm_a+h_am_c}{h_b^2}+\frac{h_cm_c+h_bm_a+h_am_b}{h_c^2}\geq 9$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_am_a+h_bm_c+h_cm_b}{h_a^2}+\frac{h_bm_b+h_cm_a+h_am_c}{h_b^2}+\frac{h_cm_c+h_bm_a+h_am_b}{h_c^2}\overset{(2)}{\geq}\frac{9(m_a+m_b+m_c)}{h_a+h_b+h_c}\overset{\sum m_a\geq 9r;\ \sum h_a\leq\frac{9R}{2}}{\geq}9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A18. $\Delta ABC\Rightarrow$

$$\frac{h_ar_a+h_br_c+h_cr_b}{h_a^2}+\frac{h_br_b+h_cr_a+h_ar_c}{h_b^2}+\frac{h_cr_c+h_br_a+h_ar_b}{h_c^2}\geq 9$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(r_a,r_b,r_c)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_ar_a+h_br_c+h_cr_b}{h_a^2}+\frac{h_br_b+h_cr_a+h_ar_c}{h_b^2}+\frac{h_cr_c+h_br_a+h_ar_b}{h_c^2}\overset{(1)}{\geq}(r_a+r_b+r_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{\sum r_a\geq 9r;\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq} 9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A19. $\Delta ABC\Rightarrow$

$$\frac{h_aw_a+h_bw_c+h_cw_b}{h_a^2}+\frac{h_bw_b+h_cw_a+h_aw_c}{h_b^2}+\frac{h_cw_c+h_bw_a+h_aw_b}{h_c^2}\geq 9$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_aw_a+h_bw_c+h_cw_b}{h_a^2}+\frac{h_bw_b+h_cw_a+h_aw_c}{h_b^2}+\frac{h_cw_c+h_bw_a+h_aw_b}{h_c^2}\overset{(1)}{\geq}(w_a+w_b+w_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)$$

$$\overset{\sum w_a\geq 9r;\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq} 9r\cdot\frac{1}{r}=9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A20. $\Delta ABC\Rightarrow$

$$\frac{h_as_a+h_bs_c+h_cs_b}{h_a^2}+\frac{h_bs_b+h_cs_a+h_as_c}{h_b^2}+\frac{h_cs_c+h_bs_a+h_as_b}{h_c^2}\geq 9$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_as_a+h_bs_c+h_cs_b}{h_a^2}+\frac{h_bs_b+h_cs_a+h_as_c}{h_b^2}+\frac{h_cs_c+h_bs_a+h_as_b}{h_c^2}\overset{(1)}{\geq}(s_a+s_b+s_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{\sum s_a\geq 9r;\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq} 9r\cdot\frac{1}{r}$$

$$=9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A21. $\Delta ABC\Rightarrow$

$$\frac{h_a sinA+h_b sinC+h_c sinB}{h_a^2}+\frac{h_b sinB+h_c sinA+h_a sinC}{h_b^2}+\frac{h_c sinC+h_b sinA+h_a sinB}{h_c^2}\geq 9$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(sinA,sinB,sinC)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_a sinA+h_b sinC+h_c sinB}{h_a^2}+\frac{h_b sinB+h_c sinA+h_a sinC}{h_b^2}+\frac{h_c sinC+h_b sinA+h_a sinB}{h_c^2}\overset{(1)}{\geq}$$

$$(sinA+sinB+sinC)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{\sum\frac{1}{h_a}=\frac{1}{r};\ \sum sinA=\frac{p}{R}}{\geq}\frac{p}{Rr}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A22. $\Delta ABC\Rightarrow$

$$\frac{h_a sin\frac{A}{2}+h_b sin\frac{C}{2}+h_c sin\frac{B}{2}}{h_a^2}+\frac{h_b sin\frac{B}{2}+h_c sin\frac{A}{2}+h_a sin\frac{C}{2}}{h_b^2}+\frac{h_c sin\frac{C}{2}+h_a sin\frac{B}{2}+h_b sin\frac{A}{2}}{h_c^2}\geq\frac{3}{2r}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_a sin\frac{A}{2}+h_b sin\frac{C}{2}+h_c sin\frac{B}{2}}{h_a^2}+\frac{h_b sin\frac{B}{2}+h_c sin\frac{A}{2}+h_a sin\frac{C}{2}}{h_b^2}+\frac{h_c sin\frac{C}{2}+h_a sin\frac{B}{2}+h_b sin\frac{A}{2}}{h_c^2}\overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{Jensen;\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq}\frac{3}{2}\cdot\frac{1}{r}=\frac{3}{2r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A23. $\Delta ABC\Rightarrow$

$$\frac{h_a cos\frac{A}{2}+h_b cos\frac{C}{2}+h_c cos\frac{B}{2}}{h_a^2}+\frac{h_b cos\frac{B}{2}+h_c cos\frac{A}{2}+h_a cos\frac{C}{2}}{h_b^2}+\frac{h_c cos\frac{C}{2}+h_a cos\frac{B}{2}+h_b cos\frac{A}{2}}{h_c^2}\geq\frac{6\sqrt{3}r}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Se aplică $(*)$ pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ şi $(a,b,c)=(h_a,h_b,h_c)$:

$$\frac{h_a cos\frac{A}{2}+h_b cos\frac{C}{2}+h_c cos\frac{B}{2}}{h_a^2}+\frac{h_b cos\frac{B}{2}+h_c cos\frac{A}{2}+h_a cos\frac{C}{2}}{h_b^2}+\frac{h_c cos\frac{C}{2}+h_a cos\frac{B}{2}+h_b cos\frac{A}{2}}{h_c^2}\overset{(1)}{\geq}$$

$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{Lema;\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq}\frac{2pr}{R^2}\cdot\frac{1}{r}=\frac{2p}{R^2}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A24. $\Delta ABC \Rightarrow$

$$\frac{h_a tg\frac{A}{2}+h_b tg\frac{C}{2}+h_c tg\frac{B}{2}}{h_a^2}+\frac{h_b tg\frac{B}{2}+h_c tg\frac{A}{2}+h_a tg\frac{C}{2}}{h_b^2}+\frac{h_c tg\frac{C}{2}+h_a tg\frac{B}{2}+h_b tg\frac{A}{2}}{h_c^2}\geq\frac{9}{p}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_a tg\frac{A}{2}+h_b tg\frac{C}{2}+h_c tg\frac{B}{2}}{h_a^2}+\frac{h_b tg\frac{B}{2}+h_c tg\frac{A}{2}+h_a tg\frac{C}{2}}{h_b^2}+\frac{h_c tg\frac{C}{2}+h_a tg\frac{B}{2}+h_b tg\frac{A}{2}}{h_c^2}\overset{(1)}{\geq}$$

$$\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{\sum tg\frac{A}{2}=\frac{4R+r}{p};\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq}\frac{4R+r}{p}\cdot\frac{1}{r}\overset{Euler}{\geq}\frac{9}{p}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A25. $\Delta ABC \Rightarrow$

$$\frac{h_a ctg\frac{A}{2}+h_b ctg\frac{C}{2}+h_c ctg\frac{B}{2}}{h_a^2}+\frac{h_b ctg\frac{B}{2}+h_c ctg\frac{A}{2}+h_a ctg\frac{C}{2}}{h_b^2}+\frac{h_c tcg\frac{C}{2}+h_a ctg\frac{B}{2}+h_b ctg\frac{A}{2}}{h_c^2}\geq\frac{3\sqrt{3}}{r}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=(h_a,h_b,h_c)$ atunci:

$$\frac{h_a ctg\frac{A}{2}+h_b ctg\frac{C}{2}+h_c ctg\frac{B}{2}}{h_a^2}+\frac{h_b ctg\frac{B}{2}+h_c ctg\frac{A}{2}+h_a ctg\frac{C}{2}}{h_b^2}+\frac{h_c tcg\frac{C}{2}+h_a ctg\frac{B}{2}+h_b ctg\frac{A}{2}}{h_c^2}\overset{(1)}{\geq}$$

$$\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\overset{\sum ctg\frac{A}{2}=\frac{p}{r};\ \sum\frac{1}{h_a}=\frac{1}{r}}{\geq}\frac{p}{r^2}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}}{r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A26. $\Delta ABC \Rightarrow$

$$\frac{am_a+cm_b+mb_c}{m_a^2}+\frac{bm_b+am_c+cm_a}{m_b^2}+\frac{cm_c+bm_a+am_b}{m_c^2}\geq\frac{12\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{am_a+cm_b+mb_c}{m_a^2}+\frac{bm_b+am_c+cm_a}{m_b^2}+r\frac{cm_c+bm_a+am_b}{m_c^2}\overset{(1)}{\geq}(a+b+c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\sum\frac{1}{m_a}\geq\frac{2}{R}}{\geq}\frac{4p}{R}$$

$$\overset{Mitrinovic}{\geq}\frac{12\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A27. $\Delta ABC \Rightarrow$

$$\frac{m_ah_a+m_bh_c+m_ch_b}{m_a^2}+\frac{m_bh_b+m_ch_a+m_ah_c}{m_b^2}+\frac{m_ch_c+m_ah_b+m_bh_a}{m_c^2}\geq\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_ah_a+m_bh_c+m_ch_b}{m_a^2}+\frac{m_bh_b+m_ch_a+m_ah_c}{m_b^2}+\frac{m_ch_c+m_ah_b+m_bh_a}{m_c^2}\overset{(1)}{\geq}(h_a+h_b+h_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)$$

$$\overset{\substack{\sum h_a\geq 9r;\\ \sum\frac{1}{m_a}\geq\frac{2}{R}}}{\geq}\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A28. $\Delta ABC \Rightarrow$

$$\frac{m_ar_a+m_br_c+m_cr_b}{m_a^2}+\frac{m_br_b+m_cr_a+m_ar_c}{m_b^2}+\frac{m_cr_c+m_ar_b+m_br_a}{m_c^2}\geq.\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(r_a,r_b,r_c)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_ar_a+m_br_c+m_cr_b}{m_a^2}+\frac{m_br_b+m_cr_a+m_ar_c}{m_b^2}$$

$$+\frac{m_cr_c+m_ar_b+m_br_a}{m_c^2}\overset{(1)}{\geq}(r_a+r_b+r_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\substack{\sum r_a\geq 9r;\\ \sum\frac{1}{m_a}\geq\frac{2}{R}}}{\geq}\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A29. $\Delta ABC \Rightarrow$

$$\frac{m_aw_a+m_bw_c+m_cw_b}{m_a^2}+\frac{m_bw_b+m_cw_a+m_aw_c}{m_b^2}+\frac{m_cw_c+m_aw_b+m_bw_a}{m_c^2}\geq\frac{18r}{R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=(m_a,m_b,m_c)$:

$$\frac{m_aw_a+m_bw_c+m_cw_b}{m_a^2}+\frac{m_bw_b+m_cw_a+m_aw_c}{m_b^2}+\frac{m_cw_c+m_aw_b+m_bw_a}{m_c^2}\overset{(1)}{\geq}(w_a+w_b+w_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\sum w_a\geq 9r;\ \sum\frac{1}{m_a}\geq\frac{2}{R}}{\geq}\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A30. $\Delta ABC\Rightarrow$

$$\frac{m_as_a+m_bs_c+m_cs_b}{m_a^2}+\frac{m_bs_b+m_cs_a+m_as_c}{m_b^2}+\frac{m_cs_c+m_as_b+m_bs_a}{m_c^2}\geq\frac{18r}{R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Se aplică $(*)$ pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=(m_a,m_b,m_c)$:

$$\frac{m_as_a+m_bs_c+m_cs_b}{m_a^2}+\frac{m_bs_b+m_cs_a+m_as_c}{m_b^2}+\frac{m_cs_c+m_as_b+m_bs_a}{m_c^2}\overset{(1)}{\geq}(s_a+s_b+s_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\sum w_a\geq 9r;\ \sum\frac{1}{m_a}\geq\frac{2}{R}}{\geq}\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A31. $\Delta ABC\Rightarrow$

$$\frac{m_a sinA+m_b sinC+m_c\sin B}{m_a^2}+\frac{m_b sinB+m_c sinA+m_a sinC}{m_b^2}+\frac{m_c sinC+m_a sinB+m_b sinA}{m_c^2}\geq\frac{6\sqrt{3}r}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(sinA,sinB,sinC)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_a sinA+m_b sinC+m_c\sin B}{m_a^2}+\frac{m_b sinB+m_c sinA+m_a sinC}{m_b^2}+\frac{m_c sinC+m_a sinB+m_b sinA}{m_c^2}\overset{(1)}{\geq}$$

$$(sinA+sinB+sinC)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\sum sinA=\frac{p}{R};\ \sum\frac{1}{m_a}\geq\frac{2}{R}}{\geq}\frac{2p}{R^2}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A32. $\Delta ABC \Rightarrow$

$$\frac{m_a sin\frac{A}{2}+m_b sin\frac{C}{2}+m_c \sin\frac{B}{2}}{m_a^2}+\frac{m_b sin\frac{B}{2}+m_c sin\frac{A}{2}+m_a sin\frac{C}{2}}{m_b^2}+\frac{m_c sin\frac{C}{2}+m_a sin\frac{B}{2}+m_b sin\frac{A}{2}}{m_c^2}\geq\frac{3}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_a sin\frac{A}{2}+m_b sin\frac{C}{2}+m_c \sin\frac{B}{2}}{m_a^2}+\frac{m_b sin\frac{B}{2}+m_c sin\frac{A}{2}+m_a sin\frac{C}{2}}{m_b^2}+\frac{m_c sin\frac{C}{2}+m_a sin\frac{B}{2}+m_b sin\frac{A}{2}}{m_c^2}\overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\substack{Jensen\\ \sum\frac{1}{m_a}\geq\frac{2}{R}}}{\geq}\frac{3}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A33. $\Delta ABC \Rightarrow$

$$\frac{m_a cos\frac{A}{2}+m_b cos\frac{C}{2}+m_c \cos\frac{B}{2}}{m_a^2}+\frac{m_b cos\frac{B}{2}+m_c cos\frac{A}{2}+m_a cos\frac{C}{2}}{m_b^2}+\frac{m_c cos\frac{C}{2}+m_a cos\frac{B}{2}+m_b cos\frac{A}{2}}{m_c^2}\geq\frac{12\sqrt{3}r^2}{R^3}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_a cos\frac{A}{2}+m_b cos\frac{C}{2}+m_c \cos\frac{B}{2}}{m_a^2}+\frac{m_b cos\frac{B}{2}+m_c cos\frac{A}{2}+m_a cos\frac{C}{2}}{m_b^2}+\frac{m_c cos\frac{C}{2}+m_a cos\frac{B}{2}+m_b cos\frac{A}{2}}{m_c^2}\overset{(1)}{\geq}$$

$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\substack{Lema;\\ \sum\frac{1}{m_a}\geq\frac{2}{R}}}{\geq}\frac{4pr}{R^3}\overset{Mitrinovic}{\geq}\frac{12\sqrt{3}r^2}{R^3}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A34. $\Delta ABC \Rightarrow$

$$\frac{m_a tg\frac{A}{2}+m_b tg\frac{C}{2}+m_c \mathrm{tg}\frac{B}{2}}{m_a^2}+\frac{m_b tg\frac{B}{2}+m_c tg\frac{A}{2}+m_a tg\frac{C}{2}}{m_b^2}+\frac{m_c tg\frac{C}{2}+m_a tg\frac{B}{2}+m_b tg\frac{A}{2}}{m_c^2}\geq\frac{4\sqrt{3}r}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_a tg\frac{A}{2}+m_b tg\frac{C}{2}+m_c\,\mathrm{tg}\frac{B}{2}}{m_a^2}+\frac{m_b tg\frac{B}{2}+m_c tg\frac{A}{2}+m_a tg\frac{C}{2}}{m_b^2}+\frac{m_c tg\frac{C}{2}+m_a tg\frac{B}{2}+m_b tg\frac{A}{2}}{m_c^2}\overset{(1)}{\geq}$$

$$\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\substack{\sum tg\frac{A}{2}=\frac{4R+r}{p};\\ \sum\frac{1}{m_a}\geq\frac{2}{R}}}{\geq}\frac{2(4R+r)}{pR}\overset{Euler}{\geq}\frac{18r}{pR}\overset{Mitrinovic}{\geq}\frac{4\sqrt{3}r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A35. $\Delta ABC\Rightarrow$

$$\frac{m_a ctg\frac{A}{2}+m_b ctg\frac{C}{2}+m_c\,\mathrm{ctg}\frac{B}{2}}{m_a^2}+\frac{m_b ctg\frac{B}{2}+m_c ctg\frac{A}{2}+m_a ctg\frac{C}{2}}{m_b^2}+\frac{m_c ctg\frac{C}{2}+m_a ctg\frac{B}{2}+m_b ctg\frac{A}{2}}{m_c^2}\geq\frac{6\sqrt{3}}{R}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=(m_a,m_b,m_c)$ atunci:

$$\frac{m_a ctg\frac{A}{2}+m_b ctg\frac{C}{2}+m_c\,\mathrm{ctg}\frac{B}{2}}{m_a^2}+\frac{m_b ctg\frac{B}{2}+m_c ctg\frac{A}{2}+m_a ctg\frac{C}{2}}{m_b^2}+\frac{m_c ctg\frac{C}{2}+m_a ctg\frac{B}{2}+m_b ctg\frac{A}{2}}{m_c^2}\overset{(1)}{\geq}$$

$$\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{\substack{\sum ctg\frac{A}{2}=\frac{p}{r};\\ \sum\frac{1}{m_a}\geq\frac{2}{R}}}{\geq}\frac{2p}{Rr}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A36. $\Delta ABC\Rightarrow$

$$\frac{ar_a+cr_b+br_c}{r_a^2}+\frac{br_b+ar_c+cr_a}{r_b^2}+\frac{cr_c+br_a+ar_b}{r_c^2}\geq 6\sqrt{3}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=(r_a,r_b,r_c)$ atunci:

$$\frac{ar_a+cr_b+br_c}{r_a^2}+\frac{br_b+ar_c+cr_a}{r_b^2}+\frac{cr_c+br_a+ar_b}{r_c^2}\overset{(1)}{\geq}(a+b+c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\sum\frac{1}{r_a}=\frac{1}{r}}{\geq}\frac{2p}{r}\overset{Mitrinovic}{\geq}6\sqrt{3}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A37. $\Delta ABC\Rightarrow$

$$\frac{r_ah_a+r_bh_c+r_ch_b}{r_a^2}+\frac{r_bh_b+r_ch_a+r_ah_c}{r_b^2}+\frac{r_ch_c+r_ah_b+r_bh_a}{r_c^2}\geq 9.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=(r_a,r_b,r_c)$ atunci:

$$\frac{r_ah_a+r_bh_c+r_ch_b}{r_a^2}+\frac{r_bh_b+r_ch_a+r_ah_c}{r_b^2}+\frac{r_ch_c+r_ah_b+r_bh_a}{r_c^2}\overset{(1)}{\geq}(h_a+h_b+h_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\sum h_a\geq 9r;\ \sum\frac{1}{r_a}=\frac{1}{r}}{\geq}9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A38. $\Delta ABC\Rightarrow$

$$\frac{r_am_a+r_bm_c+r_cm_b}{r_a^2}+\frac{r_bm_b+r_cm_a+r_am_c}{r_b^2}+\frac{r_cm_c+r_am_b+r_bm_a}{r_c^2}\geq 9.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=(r_a,r_b,r_c)$ atunci:

$$\frac{r_am_a+r_bm_c+r_cm_b}{r_a^2}+\frac{r_bm_b+r_cm_a+r_am_c}{r_b^2}+\frac{r_cm_c+r_am_b+r_bm_a}{r_c^2}\overset{(1)}{\geq}(m_a+m_b+m_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\sum m_a\geq 9r;\ \sum\frac{1}{r_a}=\frac{1}{r}}{\geq}9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A39. $\Delta ABC\Rightarrow$

$$\frac{r_aw_a+r_bw_c+r_cw_b}{r_a^2}+\frac{r_bw_b+r_cw_a+r_aw_c}{r_b^2}+\frac{r_cw_c+r_aw_b+r_bw_a}{r_c^2}\geq 9.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=(r_a,r_b,r_c)$ atunci:

$$\frac{r_aw_a+r_bw_c+r_cw_b}{r_a^2}+\frac{r_bw_b+r_cw_a+r_aw_c}{r_b^2}+\frac{r_cw_c+r_aw_b+r_bw_a}{r_c^2}\overset{(1)}{\geq}(w_a+w_b+w_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\sum w_a\geq 9r;\ \sum\frac{1}{r_a}=\frac{1}{r}}{\geq}9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A40. $\Delta ABC\Rightarrow$

$$\frac{r_as_a+r_bs_c+r_cs_b}{r_a^2}+\frac{r_bs_b+r_cs_a+r_as_c}{r_b^2}+\frac{r_cs_c+r_as_b+r_bs_a}{r_c^2}\geq 9.$$

Gheorghe Ghiţ, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=(r_a,r_b,r_c)$ atunci:

$$\frac{r_as_a+r_bs_c+r_cs_b}{r_a^2}+\frac{r_bs_b+r_cs_a+r_as_c}{r_b^2}+\frac{r_cs_c+r_as_b+r_bs_a}{r_c^2}\overset{(1)}{\geq}(s_a+s_b+s_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\sum s_a\geq 9r;\ \sum\frac{1}{r_a}=\frac{1}{r}}{\geq}9.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A41. $\Delta ABC \Rightarrow$

$$\frac{r_a sinA + r_b sinC + r_c sinB}{r_a^2} + \frac{r_b sinB + r_c sinA + r_a sinC}{r_b^2} + \frac{r_c sinC + r_a sinB + r_b sinA}{r_c^2} \geq \frac{3\sqrt{3}}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (sinA, sinB, sinC)$ şi $(a, b, c) = (r_a, r_b, r_c)$ atunci:

$$\frac{r_a sinA + r_b sinC + r_c sinB}{r_a^2} + \frac{r_b sinB + r_c sinA + r_a sinC}{r_b^2} + \frac{r_c sinC + r_a sinB + r_b sinA}{r_c^2} \overset{(1)}{\geq}$$

$$(sinA + sinB + sinC)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \overset{\sum sinA = \frac{p}{R};\ \sum \frac{1}{r_a} = \frac{1}{r}}{\geq} \frac{p}{Rr} \overset{Mitrinovic}{\geq} \frac{3\sqrt{3}}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A42. $\Delta ABC \Rightarrow$

$$\frac{r_a sin\frac{A}{2} + r_b sin\frac{C}{2} + r_c sin\frac{B}{2}}{r_a^2} + \frac{r_b sin\frac{B}{2} + r_c sin\frac{A}{2} + r_a sin\frac{C}{2}}{r_b^2} + \frac{r_c sin\frac{C}{2} + r_a sin\frac{B}{2} + r_b sin\frac{A}{2}}{r_c^2} \geq . \frac{3}{2r}$$

Gheorghe Ghiţă

Soluţie. Pentru $(x, y, z) = \left(sin\frac{A}{2}, sin\frac{B}{2}, sin\frac{C}{2}\right)$ şi $(a, b, c) = (r_a, r_b, r_c)$

$$\frac{r_a sin\frac{A}{2} + r_b sin\frac{C}{2} + r_c sin\frac{B}{2}}{r_a^2} + \frac{r_b sin\frac{B}{2} + r_c sin\frac{A}{2} + r_a sin\frac{C}{2}}{r_b^2} + \frac{r_c sin\frac{C}{2} + r_a sin\frac{B}{2} + r_b sin\frac{A}{2}}{r_c^2} \overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2} + sin\frac{B}{2} + sin\frac{C}{2}\right)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \overset{Jensen;\ \sum \frac{1}{r_a} = \frac{1}{r}}{\leq} \frac{3}{2r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A43. $\Delta ABC \Rightarrow$

$$\frac{r_a cos\frac{A}{2} + r_b cos\frac{C}{2} + r_c cos\frac{B}{2}}{r_a^2} + \frac{r_b cos\frac{B}{2} + r_c cos\frac{A}{2} + r_a cos\frac{C}{2}}{r_b^2} + \frac{r_c cos\frac{C}{2} + r_a cos\frac{B}{2} + r_b cos\frac{A}{2}}{r_c^2} \geq \frac{6\sqrt{3}r}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = \left(cos\frac{A}{2}, cos\frac{B}{2}, cos\frac{C}{2}\right)$ şi $(a, b, c) = (r_a, r_b, r_c)$ atunci:

$$\frac{r_a cos\frac{A}{2}+r_b cos\frac{C}{2}+r_c cos\frac{B}{2}}{r_a^2}+\frac{r_b cos\frac{B}{2}+r_c cos\frac{A}{2}+r_a cos\frac{C}{2}}{r_b^2}+\frac{r_c cos\frac{C}{2}+r_a cos\frac{B}{2}+r_b cos\frac{A}{2}}{r_c^2}\overset{(1)}{\geq}$$

$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\substack{Lema;\\ \sum\frac{1}{r_a}=\frac{1}{r}}}{\geq}\frac{2p}{R^2}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A44. $\Delta ABC \Rightarrow$

$$\frac{r_a tg\frac{A}{2}+r_b tg\frac{C}{2}+r_c tg\frac{B}{2}}{r_a^2}+\frac{r_b tg\frac{B}{2}+r_c tg\frac{A}{2}+r_a tg\frac{C}{2}}{r_b^2}+\frac{r_c tg\frac{C}{2}+r_a tg\frac{B}{2}+r_b tg\frac{A}{2}}{r_c^2}\geq\frac{2\sqrt{3}}{r}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=(r_a,r_b,r_c)$ atunci:

$$\frac{r_a tg\frac{A}{2}+r_b tg\frac{C}{2}+r_c tg\frac{B}{2}}{r_a^2}+\frac{r_b tg\frac{B}{2}+r_c tg\frac{A}{2}+r_a tg\frac{C}{2}}{r_b^2}+\frac{r_c tg\frac{C}{2}+r_a tg\frac{B}{2}+r_b tg\frac{A}{2}}{r_c^2}\overset{(1)}{\geq}$$

$$\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\substack{\sum tg\frac{A}{2}=\frac{4R+r}{p};\\ \sum\frac{1}{r_a}=\frac{1}{r}}}{\geq}\frac{4R+r}{pr}\overset{Euler}{\geq}\frac{9}{p}\overset{Mitrinovic}{\geq}\frac{2\sqrt{3}}{r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A45. $\Delta ABC \Rightarrow$

$$\frac{r_a ctg\frac{A}{2}+r_b ctg\frac{C}{2}+r_c ctg\frac{B}{2}}{r_a^2}+\frac{r_b ctg\frac{B}{2}+r_c ctg\frac{A}{2}+r_a ctg\frac{C}{2}}{r_b^2}+\frac{r_c ctg\frac{C}{2}+r_a ctg\frac{B}{2}+r_b\, ctg\frac{A}{2}}{r_c^2}\geq\frac{3\sqrt{3}}{r}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=(r_a,r_b,r_c)$

$$\frac{r_a ctg\frac{A}{2}+r_b ctg\frac{C}{2}+r_c ctg\frac{B}{2}}{r_a^2}+\frac{r_b ctg\frac{B}{2}+r_c ctg\frac{A}{2}+r_a ctg\frac{C}{2}}{r_b^2}+\frac{r_c ctg\frac{C}{2}+r_a ctg\frac{B}{2}+r_b\, ctg\frac{A}{2}}{r_c^2}\overset{(1)}{\geq}$$

$$\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\overset{\substack{\sum ctg\frac{A}{2}=\frac{p}{r};\\ \sum\frac{1}{r_a}=\frac{1}{r}}}{\geq}\frac{p}{r^2}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}}{r}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A46. $\Delta ABC \Rightarrow$

$$\frac{aw_a+cw_b+bw_c}{w_a^2}+\frac{bw_b+aw_c+cw_a}{w_b^2}+\frac{cw_c+bw_a+aw_b}{w_c^2}\geq\frac{12\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{aw_a+cw_b+bw_c}{w_a^2}+\frac{bw_b+aw_c+cw_a}{w_b^2}+\frac{cw_c+bw_a+aw_b}{w_c^2}\overset{(1)}{\geq}(a+b+c)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\overset{\sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq}\frac{4p}{R}\overset{Mitrinovic}{\geq}\frac{12\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A47. $\Delta ABC \Rightarrow$

$$\frac{w_ah_a+w_bh_c+w_ch_b}{w_a^2}+\frac{w_bh_b+w_ch_a+w_ah_c}{w_b^2}+\frac{w_ch_c+w_ah_b+w_bh_a}{w_c^2}\geq\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{w_ah_a+w_bh_c+w_ch_b}{w_a^2}+\frac{w_bh_b+w_ch_a+w_ah_c}{w_b^2}+\frac{w_ch_c+w_ah_b+w_bh_a}{w_c^2}\overset{(1)}{\geq}(h_a+h_b+h_c)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)$$

$$\overset{\substack{\sum h_a\geq 9r;\\ \sum\frac{1}{w_a}\geq\frac{2}{R}}}{\geq}\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A48. $\Delta ABC \Rightarrow$

$$\frac{w_am_a+w_bm_c+w_cm_b}{w_a^2}+\frac{w_bm_b+w_cm_a+w_am_c}{w_b^2}+\frac{w_cm_c+w_am_b+w_bm_a}{w_c^2}\geq\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{w_am_a+w_bm_c+w_cm_b}{w_a^2}+\frac{w_bm_b+w_cm_a+w_am_c}{w_b^2}+\frac{w_cm_c+w_am_b+w_bm_a}{w_c^2}\overset{(1)}{\geq}(m_a+m_b+m_c)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)$$

$$\overset{\sum m_a \geq 9r;\ \sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq} \frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A49. $\Delta ABC \Rightarrow$

$$\frac{w_a r_a + w_b r_c + w_c r_b}{w_a^2} + \frac{w_b r_b + w_c r_a + w_a r_c}{w_b^2} + \frac{w_c r_c + w_a r_b + w_b r_a}{w_c^2} \geq \frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = (r_a, r_b, r_c)$ şi $(a,b,c) = (w_a, w_b, w_c)$ atunci:

$$\frac{w_a r_a + w_b r_c + w_c r_b}{w_a^2} + \frac{w_b r_b + w_c r_a + w_a r_c}{w_b^2} + \frac{w_c r_c + w_a r_b + w_b r_a}{w_c^2} \overset{(1)}{\geq} (r_a + r_b + r_c)\left(\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c}\right) \overset{\sum r_a \geq 9r;\ \sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq} \frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A50. $\Delta ABC \Rightarrow$

$$\frac{w_a s_a + w_b s_c + w_c s_b}{w_a^2} + \frac{w_b s_b + w_c s_a + w_a s_c}{w_b^2} + \frac{w_c s_c + w_a s_b + w_b s_a}{w_c^2} \geq \frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = (s_a, s_b, s_c)$ şi $(a,b,c) = (w_a, w_b, w_c)$ atunci:

$$\frac{w_a s_a + w_b s_c + w_c s_b}{w_a^2} + \frac{w_b s_b + w_c s_a + w_a s_c}{w_b^2} + \frac{w_c s_c + w_a s_b + w_b s_a}{w_c^2} \overset{(1)}{\geq} (s_a + s_b + s_c)\left(\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c}\right) \overset{\sum s_a \geq 9r;\ \sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq}$$

$$\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A51. $\Delta ABC \Rightarrow$

$$\frac{w_a sinA + w_b sinC + w_c sinB}{w_a^2} + \frac{w_b sinB + w_c sinA + w_a sinC}{w_b^2} + \frac{w_c sinC + w_a sinB + w_b sinA}{w_c^2} \geq \frac{6\sqrt{3}r}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = (sinA, sinB, sinC)$ şi $(a,b,c) = (w_a, w_b, w_c)$ atunci:

$$\frac{w_a sinA + w_b sinC + w_c sinB}{w_a^2} + \frac{w_b sinB + w_c sinA + w_a sinC}{w_b^2} + \frac{w_c sinC + w_a sinB + w_b sinA}{w_c^2} \overset{(1)}{\geq}$$

$$(sinA+sinB+sinC)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\overset{\sum sinA=\frac{p}{R};\ \sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq}\frac{2p}{R^2}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A52. $\Delta ABC\Rightarrow$

$$\frac{w_a sin\frac{A}{2}+w_b sin\frac{C}{2}+w_c sin\frac{B}{2}}{w_a^2}+\frac{w_b sin\frac{B}{2}+w_c sin\frac{A}{2}+w_a sin\frac{C}{2}}{w_b^2}+\frac{w_c sin\frac{C}{2}+w_a sin\frac{B}{2}+w_b sin\frac{A}{2}}{w_c^2}\geq\frac{3}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{w_a sin\frac{A}{2}+w_b sin\frac{C}{2}+w_c sin\frac{B}{2}}{w_a^2}+\frac{w_b sin\frac{B}{2}+w_c sin\frac{A}{2}+w_a sin\frac{C}{2}}{w_b^2}+\frac{w_c sin\frac{C}{2}+w_a sin\frac{B}{2}+w_b sin\frac{A}{2}}{w_c^2}\overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\overset{Jensen;\ \sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq}\frac{3}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A53. $\Delta ABC\Rightarrow$

$$\frac{w_a cos\frac{A}{2}+w_b cos\frac{C}{2}+w_c cos\frac{B}{2}}{w_a^2}+\frac{w_b cos\frac{B}{2}+w_c cos\frac{A}{2}+w_a cos\frac{C}{2}}{w_b^2}+\frac{w_c cos\frac{C}{2}+w_a cos\frac{B}{2}+w_b cos\frac{A}{2}}{w_c^2}\geq\frac{12\sqrt{3}r^2}{R^3}$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{w_a cos\frac{A}{2}+w_b cos\frac{C}{2}+w_c cos\frac{B}{2}}{w_a^2}+\frac{w_b cos\frac{B}{2}+w_c cos\frac{A}{2}+w_a cos\frac{C}{2}}{w_b^2}+\frac{w_c cos\frac{C}{2}+w_a cos\frac{B}{2}+w_b cos\frac{A}{2}}{w_c^2}\overset{(1)}{\geq}$$

$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\overset{Lema;\ \sum\frac{1}{w_a}\geq\frac{2}{R}}{\geq}\frac{4pr}{R^3}\overset{Mitrinovic}{\geq}\frac{12\sqrt{3}r^2}{R^3}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A54. $\Delta ABC\Rightarrow$

$$\frac{w_a tg\frac{A}{2}+w_b tg\frac{C}{2}+w_c tg\frac{B}{2}}{w_a^2}+\frac{w_b tg\frac{B}{2}+w_c tg\frac{A}{2}+w_a tg\frac{C}{2}}{w_b^2}+\frac{w_c tg\frac{C}{2}+w_a tg\frac{B}{2}+w_b tg\frac{A}{2}}{w_c^2}\geq\frac{4\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{w_a tg\frac{A}{2}+w_b tg\frac{C}{2}+w_c tg\frac{B}{2}}{w_a^2}+\frac{w_b tg\frac{B}{2}+w_c tg\frac{A}{2}+w_a tg\frac{C}{2}}{w_b^2}+\frac{w_c tg\frac{C}{2}+w_a tg\frac{B}{2}+w_b tg\frac{A}{2}}{w_c^2}\overset{(1)}{\geq}$$

$$\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\overset{\substack{\sum tg\frac{A}{2}=\frac{4R+r}{p};\\ \sum\frac{1}{w_a}\geq\frac{2}{R}}}{\geq}\frac{8R+2r}{pR}\overset{Euler}{\geq}\frac{18r}{pR}\overset{Mitrinovic}{\geq}\frac{4\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A55. $\Delta ABC\Rightarrow$

$$\frac{w_a ctg\frac{A}{2}+w_b ctg\frac{C}{2}+w_c ctg\frac{B}{2}}{w_a^2}+\frac{w_b ctg\frac{B}{2}+w_c ctg\frac{A}{2}+w_a ctg\frac{C}{2}}{w_b^2}+\frac{w_c ctg\frac{C}{2}+w_a ctg\frac{B}{2}+w_b ctg\frac{A}{2}}{w_c^2}\geq\frac{6\sqrt{3}}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=(w_a,w_b,w_c)$ atunci:

$$\frac{w_a ctg\frac{A}{2}+w_b ctg\frac{C}{2}+w_c ctg\frac{B}{2}}{w_a^2}+\frac{w_b ctg\frac{B}{2}+w_c ctg\frac{A}{2}+w_a ctg\frac{C}{2}}{w_b^2}+\frac{w_c ctg\frac{C}{2}+w_a ctg\frac{B}{2}+w_b ctg\frac{A}{2}}{w_c^2}\overset{(1)}{\geq}$$

$$\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\overset{\substack{\sum ctg\frac{A}{2}=\frac{p}{r};\\ \sum\frac{1}{w_a}\geq\frac{2}{R}}}{\geq}\frac{2p}{Rr}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A56. $\Delta ABC\Rightarrow$

$$\frac{as_a+cs_b+bs_c}{s_a^2}+\frac{bs_b+as_c+cs_a}{s_b^2}+\frac{cs_c+bs_a+as_b}{s_c^2}\geq\frac{12\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{as_a+cs_b+bs_c}{s_a^2}+\frac{bs_b+as_c+cs_a}{s_b^2}+\frac{cs_c+bs_a+as_b}{s_c^2}\overset{(1)}{\geq}(a+b+c)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\sum\frac{1}{s_a}\geq\frac{2}{R}}{\geq}\frac{4p}{R}\overset{Mitrinovic}{\geq}\frac{12\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A57. $\Delta ABC \Rightarrow$

$$\frac{s_ah_a+s_bh_c+s_ch_b}{s_a^2}+\frac{s_bh_b+s_ch_a+s_ah_c}{s_b^2}+\frac{s_ch_c+s_ah_b+s_bh_a}{s_c^2}\geq\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{s_ah_a+s_bh_c+s_ch_b}{s_a^2}+\frac{s_bh_b+s_ch_a+s_ah_c}{s_b^2}+\frac{s_ch_c+s_ah_b+s_bh_a}{s_c^2}\overset{(1)}{\geq}(h_a+h_b+h_c)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{\sum h_a\geq 9r\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A58. $\Delta ABC \Rightarrow$

$$\frac{s_am_a+s_bm_c+s_cm_b}{s_a^2}+\frac{s_bm_b+s_cm_a+s_am_c}{s_b^2}+\frac{s_cm_c+s_am_b+s_bm_a}{s_c^2}\geq\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{s_am_a+s_bm_c+s_cm_b}{s_a^2}+\frac{s_bm_b+s_cm_a+s_am_c}{s_b^2}$$

$$+\frac{s_cm_c+s_am_b+s_bm_a}{s_c^2}\overset{(1)}{\geq}(m_a+m_b+m_c)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{\sum m_a\geq 9r;\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}$$

$$=\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A59. $\Delta ABC \Rightarrow$

$$\frac{s_ar_a+s_br_c+s_cr_b}{s_a^2}+\frac{s_br_b+s_cr_a+s_ar_c}{s_b^2}+\frac{s_cr_c+s_ar_b+s_br_a}{s_c^2}\geq\frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(r_a,r_b,r_c)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{s_ar_a+s_br_c+s_cr_b}{s_a^2}+\frac{s_br_b+s_cr_a+s_ar_c}{s_b^2}+\frac{s_cr_c+s_ar_b+s_br_a}{s_c^2}\overset{(1)}{\geq}(r_a+r_b+r_c)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{\sum r_a\geq 9r;\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}=\frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A60. $\Delta ABC \Rightarrow$

$$\frac{s_a w_a + s_b w_c + s_c w_b}{s_a^2} + \frac{s_b w_b + s_c w_a + s_a w_c}{s_b^2} + \frac{s_c w_c + s_a w_b + s_b w_a}{s_c^2} \geq \frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (w_a, w_b, w_c)$ şi $(a, b, c) = (s_a, s_b, s_c)$ atunci:

$$\frac{s_a w_a + s_b w_c + s_c w_b}{s_a^2} + \frac{s_b w_b + s_c w_a + s_a w_c}{s_b^2}$$

$$+ \frac{s_c w_c + s_a w_b + s_b w_a}{s_c^2} \overset{(1)}{\geq} (w_a + w_b + w_c)\left(\frac{1}{s_a} + \frac{1}{s_b} + \frac{1}{s_c}\right) \overset{\substack{\sum w_a \geq 9r; \\ \sum \frac{1}{s_a} \geq \frac{2}{R}}}{\geq} \frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A61. $\Delta ABC \Rightarrow$

$$\frac{s_a sinA + s_b sinC + s_c sinB}{s_a^2} + \frac{s_b sinB + s_c sinA + s_a simC}{s_b^2} + \frac{s_c sinC + s_a sinB + s_b sinA}{s_c^2} \geq \frac{6\sqrt{3}r}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (sinA, sinB, sinC)$ şi $(a, b, c) = (s_a, s_b, s_c)$ atunci:

$$\frac{s_a sinA + s_b sinC + s_c sinB}{s_a^2} + \frac{s_b sinB + s_c sinA + s_a simC}{s_b^2} + \frac{s_c sinC + s_a sinB + s_b sinA}{s_c^2} \overset{(1)}{\geq}$$

$$(sinA + sinB + sinC)\left(\frac{1}{s_a} + \frac{1}{s_b} + \frac{1}{s_c}\right) \overset{\substack{\sum sinA = \frac{p}{R}; \\ \sum \frac{1}{s_a} \geq \frac{2}{R}}}{\geq} = \frac{2p}{R^2} \overset{Mitrinovic}{\geq} \frac{6\sqrt{3}r}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A62. $\Delta ABC \Rightarrow$

$$\frac{s_a sin\frac{A}{2} + s_b sin\frac{C}{2} + s_c sin\frac{B}{2}}{s_a^2} + \frac{s_b sin\frac{B}{2} + s_c sin\frac{A}{2} + s_a sin\frac{C}{2}}{s_b^2} + \frac{s_c sin\frac{C}{2} + s_a sin\frac{B}{2} + s_b sin\frac{A}{2}}{s_c^2} \geq \frac{3}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = \left(sin\frac{A}{2}, sin\frac{B}{2}, sin\frac{C}{2}\right)$ şi $(a, b, c) = (s_a, s_b, s_c)$ atunci:

$$\frac{s_a sin\frac{A}{2} + s_b sin\frac{C}{2} + s_c sin\frac{B}{2}}{s_a^2} + \frac{s_b sin\frac{B}{2} + s_c sin\frac{A}{2} + s_a sin\frac{C}{2}}{s_b^2} + \frac{s_c sin\frac{C}{2} + s_a sin\frac{B}{2} + s_b sin\frac{A}{2}}{s_c^2} \overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{Jensen;\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}=\frac{3}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A63. $\Delta ABC \Rightarrow$

$$\frac{s_a cos\frac{A}{2}+s_b cos\frac{C}{2}+s_c cos\frac{B}{2}}{s_a^2}+\frac{s_b cos\frac{B}{2}+s_c cos\frac{A}{2}+s_a cos\frac{C}{2}}{s_b^2}+\frac{s_c cos\frac{C}{2}+s_a cos\frac{B}{2}+s_b cos\frac{A}{2}}{s_c^2}\geq\frac{12\sqrt{3}r^2}{R^3}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{s_a cos\frac{A}{2}+s_b cos\frac{C}{2}+s_c cos\frac{B}{2}}{s_a^2}+\frac{s_b cos\frac{B}{2}+s_c cos\frac{A}{2}+s_a cos\frac{C}{2}}{s_b^2}+\frac{s_c cos\frac{C}{2}+s_a cos\frac{B}{2}+s_b cos\frac{A}{2}}{s_c^2}\overset{(1)}{\geq}$$

$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{Lema;\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}\frac{12\sqrt{3}r^2}{R^3}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A64. $\Delta ABC \Rightarrow$

$$\frac{s_a tg\frac{A}{2}+s_b tg\frac{C}{2}+s_c tg\frac{B}{2}}{s_a^2}+\frac{s_b tg\frac{B}{2}+s_c tg\frac{A}{2}+s_a tg\frac{C}{2}}{s_b^2}+\frac{s_c tg\frac{C}{2}+s_a tg\frac{B}{2}+s_b tg\frac{A}{2}}{s_c^2}\geq\frac{4\sqrt{3}r}{3R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{s_a tg\frac{A}{2}+s_b tg\frac{C}{2}+s_c tg\frac{B}{2}}{s_a^2}+\frac{s_b tg\frac{B}{2}+s_c tg\frac{A}{2}+s_a tg\frac{C}{2}}{s_b^2}+\frac{s_c tg\frac{C}{2}+s_a tg\frac{B}{2}+s_b tg\frac{A}{2}}{s_c^2}\overset{(1)}{\geq}$$

$$\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{\sum tg\frac{A}{2}=\frac{4R+r}{p};\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}\frac{8R+2r}{pR}\overset{Euler}{\geq}\frac{18r}{pR}\overset{Mitrinovic}{\geq}\frac{4\sqrt{3}r}{3R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A65. $\Delta ABC \Rightarrow$

$$\frac{s_a ctg\frac{A}{2}+s_b ctg\frac{C}{2}+s_c ctg\frac{B}{2}}{s_a^2}+\frac{s_b ctg\frac{B}{2}+s_c ctg\frac{A}{2}+s_a ctg\frac{C}{2}}{s_b^2}+\frac{s_c ctg\frac{C}{2}+s_a ctg\frac{B}{2}+s_b ctg\frac{A}{2}}{s_c^2}\geq\frac{6\sqrt{3}}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=(s_a,s_b,s_c)$ atunci:

$$\frac{s_a ctg\frac{A}{2}+s_b ctg\frac{C}{2}+s_c ctg\frac{B}{2}}{s_a^2}+\frac{s_b ctg\frac{B}{2}+s_c ctg\frac{A}{2}+s_a ctg\frac{C}{2}}{s_b^2}+\frac{s_c ctg\frac{C}{2}+s_a ctg\frac{B}{2}+s_b ctg\frac{A}{2}}{s_c^2}\overset{(1)}{\geq}$$

$$\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c}\right)\overset{\substack{\sum ctg\frac{A}{2}=\frac{p}{r};\\ \sum\frac{1}{s_a}\geq\frac{2}{R}}}{\geq}\frac{2p}{Rr}\overset{Mitrinovic}{\geq}\frac{6\sqrt{3}}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A66. $\Delta ABC\Rightarrow$

$$\frac{h_a sinA+h_c sinB+h_b sinC}{sin^2A}+\frac{h_b sinB+h_a sinC+h_c sinA}{sin^2B}+\frac{h_c sinC+h_b sinA+h_a sinB}{sin^2C}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=(sinA,sinB,sinC)$ atunci:

$$\frac{h_a sinA+h_c sinB+h_b sinC}{sin^2A}+\frac{h_b sinB+h_a sinC+h_c sinA}{sin^2B}+\frac{h_c sinC+h_b sinA+h_a sinB}{sin^2C}\overset{(1)}{\geq}$$

$$(h_a+h_b+h_c)\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)\overset{\substack{\sum h_a\geq 9r;\\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}}{\geq}\frac{81Rr}{p}\overset{Mitrinovic}{\geq}18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A67. $\Delta ABC\Rightarrow$

$$\frac{m_a sinA+m_c sinB+m_b sinC}{sin^2A}+\frac{m_b sinB+m_a sinC+m_c sinA}{sin^2B}+\frac{m_c sinC+m_b sinA+m_a sinB}{sin^2C}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=(sinA,sinB,sinC)$ atunci:

$$\frac{m_a sinA+m_c sinB+m_b sinC}{sin^2A}+\frac{m_b sinB+m_a sinC+m_c sinA}{sin^2B}+\frac{m_c sinC+m_b sinA+m_a sinB}{sin^2C}\overset{(1)}{\geq}$$

$$(m_a+m_b+m_c)\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)\overset{\substack{\sum m_a\geq 9r;\\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}}{\geq}\frac{81Rr}{p}\overset{Mitrinovic}{\geq}18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A68. $\Delta ABC \Rightarrow$

$$\frac{r_a sinA + r_c sinB + r_b sinC}{sin^2 A} + \frac{r_b sinB + r_a sinC + r_c sinA}{sin^2 B} + \frac{r_c sinC + r_b sinA + r_a sinB}{sin^2 C} \geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = (r_a, r_b, r_c)$ şi $(a,b,c) = (sinA, sinB, sinC)$ atunci:

$$\frac{r_a sinA + r_c sinB + r_b sinC}{sin^2 A} + \frac{r_b sinB + r_a sinC + r_c sinA}{sin^2 B} + \frac{r_c sinC + r_b sinA + r_a sinB}{sin^2 C} \overset{(1)}{\geq}$$

$$(r_a + r_b + r_c)\left(\frac{1}{sinA} + \frac{1}{sinB} + \frac{1}{sinC}\right) \overset{\sum r_a = 4R+r;\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}{\geq} \frac{9R(4R+r)}{p} \overset{Euler}{\geq} \frac{81Rr}{p} \overset{Mitrinovic}{\geq} 18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A69. $\Delta ABC \Rightarrow$

$$\frac{w_a sinA + w_c sinB + w_b sinC}{sin^2 A} + \frac{w_b sinB + w_a sinC + w_c sinA}{sin^2 B} + \frac{w_c sinC + w_b sinA + w_a sinB}{sin^2 C} \geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = (w_a, w_b, w_c)$ şi $(a,b,c) = (sinA, sinB, sinC)$ atunci:

$$\frac{w_a sinA + w_c sinB + w_b sinC}{sin^2 A} + \frac{w_b sinB + w_a sinC + w_c sinA}{sin^2 B} + \frac{w_c sinC + w_b sinA + w_a sinB}{sin^2 C} \overset{(1)}{\geq}$$

$$(w_a + w_b + w_c)\left(\frac{1}{sinA} + \frac{1}{sinB} + \frac{1}{sinC}\right) \overset{\sum w_a \geq 9r;\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}{\geq} \frac{81Rr}{p} \overset{Mitrinovic}{\geq} 18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A70. $\Delta ABC \Rightarrow$

$$\frac{s_a sinA + s_c sinB + s_b sinC}{sin^2 A} + \frac{s_b sinB + s_a sinC + s_c sinA}{sin^2 B} + \frac{s_c sinC + s_b sinA + s_a sinB}{sin^2 C} \geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = (s_a, s_b, s_c)$ şi $(a,b,c) = (sinA, sinB, sinC)$ atunci:

$$\frac{s_a sinA + s_c sinB + s_b sinC}{sin^2 A} + \frac{s_b sinB + s_a sinC + s_c sinA}{sin^2 B} + \frac{s_c sinC + s_b sinA + s_a sinB}{sin^2 C} \overset{(1)}{\geq}$$

$$(s_a + s_b + s_c)\left(\frac{1}{sinA} + \frac{1}{sinB} + \frac{1}{sinC}\right) \overset{\sum s_a \geq 9r;\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}{\geq} \frac{81Rr}{p} \overset{Mitrinovic}{\geq} 18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A71. $\Delta ABC \Rightarrow$

$$\frac{sin\frac{A}{2}sinA + sin\frac{C}{2}sinB + sin\frac{B}{2}sinC}{sin^2A} + \frac{sin\frac{B}{2}sinB + sin\frac{A}{2}sinC + sin\frac{C}{2}sinA}{sin^2B} + \frac{sin\frac{C}{2}sinC + sin\frac{B}{2}sinA + sin\frac{A}{2}sinB}{sin^2C} \geq 6\sqrt{3}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = \left(sin\frac{A}{2}, sin\frac{B}{2}, sin\frac{C}{2}\right)$ şi $(a,b,c) = (sinA, sinB, sinC)$ atunci:

$$\frac{sin\frac{A}{2}sinA + sin\frac{C}{2}sinB + sin\frac{B}{2}sinC}{sin^2A} + \frac{sin\frac{B}{2}sinB + sin\frac{A}{2}sinC + sin\frac{C}{2}sinA}{sin^2B} + \frac{sin\frac{C}{2}sinC + sin\frac{B}{2}sinA + sin\frac{A}{2}sinB}{sin^2C} \overset{(1)}{\geq} \left(sin\frac{A}{2} + sin\frac{B}{2} + sin\frac{C}{2}\right)\left(\frac{1}{sinA} + \frac{1}{sinB} + \frac{1}{sinC}\right) \overset{Jensen;\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}{\geq} \frac{27R}{2p} \overset{Mitrinovic}{\geq} 6\sqrt{3}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A72. $\Delta ABC \Rightarrow$

$$\frac{cos\frac{A}{2}sinA + cos\frac{C}{2}sinB + cos\frac{B}{2}sinC}{sin^2A} + \frac{cos\frac{B}{2}sinB + cos\frac{A}{2}sinC + cos\frac{C}{2}sinA}{sin^2B} + \frac{cos\frac{C}{2}sinC + cos\frac{B}{2}sinA + cos\frac{A}{2}sinB}{sin^2C} \geq \frac{18r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z) = \left(cos\frac{A}{2}, cos\frac{B}{2}, cos\frac{C}{2}\right)$ şi $(a,b,c) = (sinA, sinB, sinC)$ atunci:

$$\frac{cos\frac{A}{2}sinA + cos\frac{C}{2}sinB + cos\frac{B}{2}sinC}{sin^2A} + \frac{cos\frac{B}{2}sinB + cos\frac{A}{2}sinC + cos\frac{C}{2}sinA}{sin^2B} + \frac{cos\frac{C}{2}sinC + cos\frac{B}{2}sinA + cos\frac{A}{2}sinB}{sin^2C} \overset{(1)}{\geq} \left(cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2}\right)\left(\frac{1}{sinA} + \frac{1}{sinB} + \frac{1}{sinC}\right) \overset{Lema;\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}{\geq} \frac{18r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A73. $\Delta ABC \Rightarrow$

$$\frac{tg\frac{A}{2}sinA+tg\frac{C}{2}sinB+tg\frac{B}{2}sinC}{sin^2A}+\frac{tg\frac{B}{2}sinB+tg\frac{A}{2}sinC+tg\frac{C}{2}sinA}{sin^2B}+\frac{tg\frac{C}{2}sinC+tg\frac{B}{2}sinA+tg\frac{A}{2}sinB}{sin^2C}\geq\frac{12r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Se aplică $(*)$ pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=(sinA,sinB,sinC)$:

$$\frac{tg\frac{A}{2}sinA+tg\frac{C}{2}sinB+tg\frac{B}{2}sinC}{sin^2A}+\frac{tg\frac{B}{2}sinB+tg\frac{A}{2}sinC+tg\frac{C}{2}sinA}{sin^2B}+\frac{tg\frac{C}{2}sinC+tg\frac{B}{2}sinA+tg\frac{A}{2}sinB}{sin^2C}$$

$$\overset{(1)}{\geq}\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)\overset{\substack{\sum tg\frac{A}{3}=\frac{4R+r}{p};\\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}}{\overbrace{\geq}}\frac{9R(4R+r)}{p^2}\overset{Euler}{\overbrace{\geq}}\frac{81Rr}{p^2}\overset{Mitinovic}{\overbrace{\geq}}\frac{12r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A74. $\Delta ABC\Rightarrow$

$$\frac{ctg\frac{A}{2}sinA+ctg\frac{C}{2}sinB+ctg\frac{B}{2}sinC}{sin^2A}+\frac{ctg\frac{B}{2}sinB+ctg\frac{A}{2}sinC+ctg\frac{C}{2}sinA}{sin^2B}+$$

$$\frac{ctg\frac{C}{2}sinC+ctg\frac{B}{2}sinA+ctg\frac{A}{2}sinB}{sin^2C}\geq 18.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=(sinA,sinB,sinC)$ atunci:

$$\frac{ctg\frac{A}{2}sinA+ctg\frac{C}{2}sinB+ctg\frac{B}{2}sinC}{sin^2A}+\frac{ctg\frac{B}{2}sinB+ctg\frac{A}{2}sinC+ctg\frac{C}{2}sinA}{sin^2B}+$$

$$\frac{ctg\frac{C}{2}sinC+ctg\frac{B}{2}sinA+ctg\frac{A}{2}sinB}{sin^2C}\overset{(1)}{\overbrace{\geq}}\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{sinA}+\frac{1}{sinB}+\frac{1}{sinC}\right)\overset{\substack{\sum ctg\frac{A}{3}=\frac{p}{r};\\ \sum\frac{1}{sinA}\geq\frac{9R}{p}}}{\overbrace{\geq}}\frac{9R}{r}\overset{Eu.ler}{\overbrace{\geq}}18$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A75. $\Delta ABC\Rightarrow$

$$\frac{asin\frac{A}{2}+csin\frac{B}{2}+bsin\frac{C}{2}}{sin^2\frac{A}{2}}+\frac{bsin\frac{B}{2}+asin\frac{C}{2}+csin\frac{A}{2}}{sin^2\frac{B}{2}}+\frac{csin\frac{C}{2}+bsin\frac{A}{2}+asin\frac{B}{2}}{sin^2\frac{C}{2}}\geq 36\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (a, b, c)$ şi $(a, b, c) = \left(\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}\right)$ atunci:

$$\frac{a\sin\frac{A}{2} + c\sin\frac{B}{2} + b\sin\frac{C}{2}}{\sin^2\frac{A}{2}} + \frac{b\sin\frac{B}{2} + a\sin\frac{C}{2} + c\sin\frac{A}{2}}{\sin^2\frac{B}{2}} + \frac{c\sin\frac{C}{2} + b\sin\frac{A}{2} + a\sin\frac{B}{2}}{\sin^2\frac{C}{2}} \overset{(1)}{\geq}$$

$$(a+b+c)\left(\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}\right) \overset{\sum\frac{1}{\sin\frac{A}{2}}\geq 6}{\geq} 12p \overset{Mitrinovic}{\geq} 36\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A76. $\Delta ABC \Rightarrow$

$$\frac{h_a\sin\frac{A}{2} + h_c\sin\frac{B}{2} + h_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}} + \frac{h_b\sin\frac{B}{2} + h_a\sin\frac{C}{2} + h_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}} + \frac{h_c\sin\frac{C}{2} + h_b\sin\frac{A}{2} + h_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}} \geq 54r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Se aplică $(*)$ pentru $(x, y, z) = (h_a, h_b, h_c)$ şi $(a, b, c) = \left(\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}\right)$:

$$\frac{h_a\sin\frac{A}{2} + h_c\sin\frac{B}{2} + h_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}} + \frac{h_b\sin\frac{B}{2} + h_a\sin\frac{C}{2} + h_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}} + \frac{h_c\sin\frac{C}{2} + h_b\sin\frac{A}{2} + h_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}} \overset{(1)}{\geq}$$

$$(h_a + h_b + h_c)\left(\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}}\right) \overset{\substack{\sum\frac{1}{\sin\frac{A}{2}}\geq 6;\\ \sum h_a \geq 9r}}{\geq} 54r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A77. $\Delta ABC \Rightarrow$

$$\frac{m_a\sin\frac{A}{2} + m_c\sin\frac{B}{2} + m_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}} + \frac{m_b\sin\frac{B}{2} + m_a\sin\frac{C}{2} + m_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}} + \frac{m_c\sin\frac{C}{2} + m_b\sin\frac{A}{2} + m_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}} \geq 54r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (m_a, m_b, m_c)$ şi $(a, b, c) = \left(\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}\right)$ atunci:

$$\frac{m_a\sin\frac{A}{2} + m_c\sin\frac{B}{2} + m_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}} + \frac{m_b\sin\frac{B}{2} + m_a\sin\frac{C}{2} + m_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}} + \frac{m_c\sin\frac{C}{2} + m_b\sin\frac{A}{2} + m_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}} \overset{(1)}{\geq}$$

$$(m_a+m_b+m)\left(\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)\overset{\sum\frac{1}{\sin\frac{A}{2}}\geq 6;\ \sum m_a\geq 9r}{\geq} 54r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A78. $\Delta ABC \Rightarrow$

$$\frac{r_a\sin\frac{A}{2}+r_c\sin\frac{B}{2}+r_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}}+\frac{r_b\sin\frac{B}{2}+r_a\sin\frac{C}{2}+r_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{r_c\sin\frac{C}{2}+r_b\sin\frac{A}{2}+r_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}}\geq 54r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Se aplică $(*)$ pentru $(x,y,z)=(r_a,r_b,r_c)$ şi $(a,b,c)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$:

$$\frac{r_a\sin\frac{A}{2}+r_c\sin\frac{B}{2}+r_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}}+\frac{r_b\sin\frac{B}{2}+r_a\sin\frac{C}{2}+r_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{r_c\sin\frac{C}{2}+r_b\sin\frac{A}{2}+r_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(r_a+r_b+r_c)\left(\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)\overset{\sum r_a=4R+r;\ \sum\frac{1}{\sin\frac{A}{2}}\geq 6}{\geq} 6(4R+r)\overset{Euler}{\geq} 54r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A79. $\Delta ABC \Rightarrow$

$$\frac{w_a\sin\frac{A}{2}+w_c\sin\frac{B}{2}+w_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}}+\frac{w_b\sin\frac{B}{2}+w_a\sin\frac{C}{2}+w_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{w_c\sin\frac{C}{2}+w_b\sin\frac{A}{2}+w_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}}\geq 54r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ atunci:

$$\frac{w_a\sin\frac{A}{2}+w_c\sin\frac{B}{2}+w_b\sin\frac{C}{2}}{\sin^2\frac{A}{2}}+\frac{w_b\sin\frac{B}{2}+w_a\sin\frac{C}{2}+w_c\sin\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{w_c\sin\frac{C}{2}+w_b\sin\frac{A}{2}+w_a\sin\frac{B}{2}}{\sin^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(w_a+w_b+w_c)\left(\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)\overset{\sum\frac{1}{\sin\frac{A}{2}}\geq 6;\ \sum w_a\geq 9r}{\geq} 54r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A80. $\Delta ABC \Rightarrow$

$$\frac{s_a sin\frac{A}{2}+s_c sin\frac{B}{2}+s_b sin\frac{C}{2}}{sin^2\frac{A}{2}}+\frac{s_b sin\frac{B}{2}+s_a sin\frac{C}{2}+s_c sin\frac{A}{2}}{sin^2\frac{B}{2}}+\frac{s_c sin\frac{C}{2}+s_b sin\frac{A}{2}+s_a sin\frac{B}{2}}{sin^2\frac{C}{2}}\geq 54r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ atunci:

$$\frac{s_a sin\frac{A}{2}+s_c sin\frac{B}{2}+s_b sin\frac{C}{2}}{sin^2\frac{A}{2}}+\frac{s_b sin\frac{B}{2}+s_a sin\frac{C}{2}+s_c sin\frac{A}{2}}{sin^2\frac{B}{2}}+\frac{s_c sin\frac{C}{2}+s_b sin\frac{A}{2}+s_a sin\frac{B}{2}}{sin^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(s_a+s_b+s_c)\left(\frac{1}{sin\frac{A}{2}}+\frac{1}{sin\frac{B}{2}}+\frac{1}{sin\frac{C}{2}}\right)\overset{\substack{\sum\frac{1}{sin\frac{A}{2}}\geq 6;\\ \sum s_a\geq 9r}}{\geq} 54r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A81. $\Delta ABC \Rightarrow$

$$\frac{sinAsin\frac{A}{2}+sinCsin\frac{B}{2}+sinBsin\frac{C}{2}}{sin^2\frac{A}{2}}+\frac{sinBsin\frac{B}{2}+sinAsin\frac{C}{2}+sinCsin\frac{A}{2}}{sin^2\frac{B}{2}}+$$

$$\frac{sinCsin\frac{C}{2}+sinB\,sin\frac{A}{2}+sinAsin\frac{B}{2}}{sin^2\frac{C}{2}}\geq\frac{18\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(sinA,sinB,sinC)$ şi $(a,b,c)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ atunci:

$$\frac{sinAsin\frac{A}{2}+sinCsin\frac{B}{2}+sinBsin\frac{C}{2}}{sin^2\frac{A}{2}}+\frac{sinBsin\frac{B}{2}+sinAsin\frac{C}{2}+sinCsin\frac{A}{2}}{sin^2\frac{B}{2}}+$$

$$\frac{sinCsin\frac{C}{2}+sinB\,sin\frac{A}{2}+sinAsin\frac{B}{2}}{sin^2\frac{C}{2}}\overset{(1)}{\geq}(sinA+sinB+sinC)\left(\frac{1}{sin\frac{A}{2}}+\frac{1}{sin\frac{B}{2}}+\frac{1}{sin\frac{C}{2}}\right)\overset{\substack{\sum sinA=\frac{p}{R};\\ \sum\frac{1}{sin\frac{A}{2}}\geq 6}}{\geq}\frac{6p}{R}\overset{Mitrinovic}{\geq}$$

$$\frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A82. $\Delta ABC \Rightarrow$

$$\frac{\cos\frac{A}{2}\sin\frac{A}{2}+\cos\frac{C}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{A}{2}}+\frac{\cos\frac{B}{2}\sin\frac{B}{2}+\cos\frac{A}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{A}{2}}{\sin^2\frac{B}{2}}+$$

$$\frac{\cos\frac{C}{2}\sin\frac{C}{2}+\cos\frac{B}{2}\sin\frac{A}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\sin^2\frac{C}{2}}\geq\frac{72\sqrt{3}r^3}{R^3}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right)$ şi $(a,b,c)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ atunci:

$$\frac{\cos\frac{A}{2}\sin\frac{A}{2}+\cos\frac{C}{2}\sin\frac{B}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{A}{2}}+\frac{\cos\frac{B}{2}\sin\frac{B}{2}+\cos\frac{A}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{A}{2}}{\sin^2\frac{B}{2}}+$$

$$\frac{\cos\frac{C}{2}\sin\frac{C}{2}+\cos\frac{B}{2}\sin\frac{A}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\sin^2\frac{C}{2}}\overset{(1)}{\geq}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\left(\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)\overset{\substack{Lemă;\\ \sum\frac{1}{\sin\frac{A}{2}}\geq 6}}{\geq}$$

$$\frac{12pr}{R^2}\overset{Mitrinovic}{\geq}\frac{36\sqrt{3}r^2}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A83. $\Delta ABC \Rightarrow$

$$\frac{\sin\frac{A}{2}tg\frac{A}{2}+\sin\frac{B}{2}tg\frac{C}{2}+\sin\frac{C}{2}tg\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\sin\frac{B}{2}tg\frac{B}{2}+\sin\frac{C}{2}tg\frac{A}{2}+\sin\frac{A}{2}tg\frac{C}{2}}{\sin^2\frac{B}{2}}+$$

$$\frac{\sin\frac{C}{2}tg\frac{C}{2}+\sin\frac{A}{2}tg\frac{B}{2}+\sin\frac{B}{2}tg\frac{A}{2}}{\sin^2\frac{C}{2}}\geq\frac{12\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Se aplică $(*)$ pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$:

$$\frac{\sin\frac{A}{2}tg\frac{A}{2}+\sin\frac{B}{2}tg\frac{C}{2}+\sin\frac{C}{2}tg\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\sin\frac{B}{2}tg\frac{B}{2}+\sin\frac{C}{2}tg\frac{A}{2}+\sin\frac{A}{2}tg\frac{C}{2}}{\sin^2\frac{B}{2}}+$$

$$\frac{\sin\frac{C}{2}tg\frac{C}{2}+\sin\frac{A}{2}tg\frac{B}{2}+\sin\frac{B}{2}tg\frac{A}{2}}{\sin^2\frac{C}{2}}\overset{(1)}{\geq}\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)\overset{\sum tg\frac{A}{2}=\frac{4R+r}{p};\ \sum\frac{1}{\sin\frac{A}{2}}\geq 6}{\geq}\frac{6(4R+r)}{p}$$

$$\overset{Euler;\ Mitrinovic}{\geq}\frac{12\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A84. $\Delta ABC\Rightarrow$

$$\frac{\sin\frac{A}{2}ctg\frac{A}{2}+\sin\frac{B}{2}ctg\frac{C}{2}+\sin\frac{C}{2}ctg\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\sin\frac{B}{2}ctg\frac{B}{2}+\sin\frac{C}{2}ctg\frac{A}{2}+\sin\frac{A}{2}ctg\frac{C}{2}}{\sin^2\frac{B}{2}}+$$

$$\frac{\sin\frac{C}{2}ctg\frac{C}{2}+\sin\frac{A}{2}ctg\frac{B}{2}+\sin\frac{B}{2}ctg\frac{A}{2}}{\sin^2\frac{C}{2}}\geq\frac{18\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ atunci:

$$\frac{\sin\frac{A}{2}ctg\frac{A}{2}+\sin\frac{B}{2}ctg\frac{C}{2}+\sin\frac{C}{2}ctg\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\sin\frac{B}{2}ctg\frac{B}{2}+\sin\frac{C}{2}ctg\frac{A}{2}+\sin\frac{A}{2}ctg\frac{C}{2}}{\sin^2\frac{B}{2}}+$$

$$\frac{\sin\frac{C}{2}ctg\frac{C}{2}+\sin\frac{A}{2}ctg\frac{B}{2}+\sin\frac{B}{2}ctg\frac{A}{2}}{\sin^2\frac{C}{2}}\overset{(1)}{\geq}\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right)\overset{\sum ctg\frac{A}{2}=\frac{p}{R};\ \sum\frac{1}{\sin\frac{A}{2}}\geq 6}{\geq}\frac{6p}{R}\overset{Mitrinovic}{\geq}$$

$$\frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A85. $\Delta ABC\Rightarrow$

$$\frac{acos\frac{A}{2}+ccos\frac{B}{2}+bcos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{bcos\frac{B}{2}+acos\frac{C}{2}+ccos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{ccos\frac{C}{2}+bcos\frac{A}{2}+acos\frac{B}{2}}{cos^2\frac{C}{2}}\geq 24r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{acos\frac{A}{2}+ccos\frac{B}{2}+bcos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{bcos\frac{B}{2}+acos\frac{C}{2}+ccos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{ccos\frac{C}{2}+bcos\frac{A}{2}+acos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(a+b+c)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right)\overset{\sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq} 4p\sqrt{3}\geq 24r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A86. $\Delta ABC\Rightarrow$

$$\frac{h_acos\frac{A}{2}+h_ccos\frac{B}{2}+h_bcos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{h_bcos\frac{B}{2}+h_ccos\frac{C}{2}+h_acos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{h_ccos\frac{C}{2}+h_bcos\frac{A}{2}+h_acos\frac{B}{2}}{cos^2\frac{C}{2}}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{h_acos\frac{A}{2}+h_ccos\frac{B}{2}+h_bcos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{h_bcos\frac{B}{2}+h_ccos\frac{C}{2}+h_acos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{h_ccos\frac{C}{2}+h_bcos\frac{A}{2}+h_acos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(h_a+h_b+h_c)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right)\overset{\substack{\sum h_a\geq 9r;\\ \sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}}{\geq} 18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A87. $\Delta ABC\Rightarrow$

$$\frac{m_acos\frac{A}{2}+m_ccos\frac{B}{2}+m_bcos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{m_bcos\frac{B}{2}+m_ccos\frac{C}{2}+mcos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{m_Ccos\frac{C}{2}+m_bcos\frac{A}{2}+m_acos\frac{B}{2}}{cos^2\frac{C}{2}}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{m_a cos\frac{A}{2}+m_c cos\frac{B}{2}+m_b cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{m_b cos\frac{B}{2}+m_a cos\frac{C}{2}+m_x cos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{m_C cos\frac{C}{2}+m_b cos\frac{A}{2}+m_a cos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(m_a+m_b+m_c)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right)\overset{\sum m_a\geq 9r;\ \sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq} 18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A88. $\Delta ABC\Rightarrow$

$$\frac{r_a cos\frac{A}{2}+r_c cos\frac{B}{2}+r_b cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{r_b cos\frac{B}{2}+r_a cos\frac{C}{2}+r_c cos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{m_C cos\frac{C}{2}+m_b cos\frac{A}{2}+m_a cos\frac{B}{2}}{cos^2\frac{C}{2}}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(r_a,r_b,r_c)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{r_a cos\frac{A}{2}+r_c cos\frac{B}{2}+r_b cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{r_b cos\frac{B}{2}+r_a cos\frac{C}{2}+r_c cos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{m_C cos\frac{C}{2}+m_b cos\frac{A}{2}+m_a cos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(r_a+r_b+r_c)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right)\overset{\sum r_a=4R+r;\ \sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq} 2(4R+r)\sqrt{3}\overset{Euler}{\geq} 18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A89. $\Delta ABC\Rightarrow$

$$\frac{w_a cos\frac{A}{2}+w_c cos\frac{B}{2}+w_b cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{w_b cos\frac{B}{2}+w_a cos\frac{C}{2}+w_c cos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{w_C cos\frac{C}{2}+w_b cos\frac{A}{2}+w_a cos\frac{B}{2}}{cos^2\frac{C}{2}}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{w_a cos\frac{A}{2}+w_c cos\frac{B}{2}+w_b cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{w_b cos\frac{B}{2}+w_a cos\frac{C}{2}+w_c cos\frac{A}{2}}{cos^2\frac{B}{2}}+\frac{w_C cos\frac{C}{2}+w_b cos\frac{A}{2}+w_a cos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(w_a+w_b+w_c)\left(\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\right)\overset{\sum w_a\geq 9r;\ \sum\frac{1}{\cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq}18\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A90. $\Delta ABC\Rightarrow$

$$\frac{s_a\cos\frac{A}{2}+s_c\cos\frac{B}{2}+s_b\cos\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{s_b\cos\frac{B}{2}+s_a\cos\frac{C}{2}+s_c\cos\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{s_c\cos\frac{C}{2}+s_b\cos\frac{A}{2}+s_a\cos\frac{B}{2}}{\cos^2\frac{C}{2}}\geq 18\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right)$ atunci:

$$\frac{s_a\cos\frac{A}{2}+s_c\cos\frac{B}{2}+s_b\cos\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{s_b\cos\frac{B}{2}+s_a\cos\frac{C}{2}+s_c\cos\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{s_c\cos\frac{C}{2}+s_b\cos\frac{A}{2}+s_a\cos\frac{B}{2}}{\cos^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(s_a+s_b+s_c)\left(\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\right)\overset{\sum s_a\geq 9r;\ \sum\frac{1}{\cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq}18\sqrt{3}r.$$

A91. $\Delta ABC\Rightarrow$

$$\frac{\sin A\cos\frac{A}{2}+\sin C\cos\frac{B}{2}+\sin B\cos\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{\sin B\cos\frac{B}{2}+\sin A\cos\frac{C}{2}+\sin C\cos\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{\sin C\cos\frac{C}{2}+\sin B\cos\frac{A}{2}+\sin A\cos\frac{B}{2}}{\cos^2\frac{C}{2}}$$

$\geq 18r.$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(\sin A,\sin B,\sin C)$ şi $(a,b,c)=\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right)$ atunci:

$$\frac{\sin A\cos\frac{A}{2}+\sin C\cos\frac{B}{2}+\sin B\cos\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{\sin B\cos\frac{B}{2}+\sin A\cos\frac{C}{2}+\sin C\cos\frac{A}{2}}{\cos^2\frac{B}{2}}+\frac{\sin C\cos\frac{C}{2}+\sin B\cos\frac{A}{2}+\sin A\cos\frac{B}{2}}{\cos^2\frac{C}{2}}$$

$$\overset{(1)}{\geq} (sinA+sinB+sinC)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right) \overset{\sum sinA=\frac{p}{R};\ \sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq} \frac{2\sqrt{3}p}{R} \overset{Mitrinovic}{\geq} 18r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Gheorghe Ghiţă, Buzău

A92. $\Delta ABC \Rightarrow$

$$\frac{tg\frac{A}{2}cos\frac{A}{2}+tg\frac{C}{2}cos\frac{B}{2}+tg\frac{B}{2}cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{tg\frac{B}{2}cos\frac{B}{2}+tg\frac{A}{2}cos\frac{C}{2}+tg\frac{C}{2}cos\frac{A}{2}}{cos^2\frac{B}{2}}+$$

$$\frac{tg\frac{C}{2}cos\frac{C}{2}+tg\frac{B}{2}cos\frac{A}{2}+tg\frac{A}{2}cos\frac{B}{2}}{cos^2\frac{C}{2}}\geq\frac{12r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{tg\frac{A}{2}cos\frac{A}{2}+tg\frac{C}{2}cos\frac{B}{2}+tg\frac{B}{2}cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{tg\frac{B}{2}cos\frac{B}{2}+tg\frac{A}{2}cos\frac{C}{2}+tg\frac{C}{2}cos\frac{A}{2}}{cos^2\frac{B}{2}}+$$

$$+\frac{tg\frac{C}{2}cos\frac{C}{2}+tg\frac{B}{2}cos\frac{A}{2}+tg\frac{A}{2}cos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}\right.$$

$$\left.+\frac{1}{cos\frac{C}{2}}\right)\overset{\sum tg\frac{A}{2}=\frac{4R+r}{p};\ \sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq}\frac{2(4R+r)\sqrt{3}}{p}$$

$$\overset{Euler}{\geq}\frac{18\sqrt{3}r}{p}\overset{Mitrinovic}{\geq}\frac{12r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Gheorghe Ghiţă, Buzău

A93. $\Delta ABC \Rightarrow$

$$\frac{ctg\frac{A}{2}cos\frac{A}{2}+ctg\frac{C}{2}cos\frac{B}{2}+ctg\frac{B}{2}cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{ctg\frac{B}{2}cos\frac{B}{2}+ctg\frac{A}{2}cos\frac{C}{2}+ctg\frac{C}{2}cos\frac{A}{2}}{cos^2\frac{B}{2}}+$$

$$\frac{ctg\frac{C}{2}cos\frac{C}{2}+ctg\frac{B}{2}cos\frac{A}{2}+ctg\frac{A}{2}cos\frac{B}{2}}{cos^2\frac{C}{2}}\geq 18.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ atunci:

$$\frac{ctg\frac{A}{2}cos\frac{A}{2}+ctg\frac{C}{2}cos\frac{B}{2}+ctg\frac{B}{2}cos\frac{C}{2}}{cos^2\frac{A}{2}}+\frac{ctg\frac{B}{2}cos\frac{B}{2}+ctg\frac{A}{2}cos\frac{C}{2}+ctg\frac{C}{2}cos\frac{A}{2}}{cos^2\frac{B}{2}}+$$

$$\frac{ctg\frac{C}{2}cos\frac{C}{2}+ctg\frac{B}{2}cos\frac{A}{2}+ctg\frac{A}{2}cos\frac{B}{2}}{cos^2\frac{C}{2}}\overset{(1)}{\geq}\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}}\right)\overset{\sum ctg\frac{A}{2}=\frac{p}{r};\ \sum\frac{1}{cos\frac{A}{2}}\geq 2\sqrt{3}}{\geq}$$

$$\frac{2\sqrt{3}p}{r}\overset{Mitrinovic}{\geq}18.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A94. $\Delta ABC\Rightarrow$

$$ctg^2\frac{A}{2}\left(atg\frac{A}{2}+ctg\frac{B}{2}+btg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(btg\frac{B}{2}+atg\frac{C}{2}+ctg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(ctg\frac{C}{2}+btg\frac{A}{2}+atg\frac{B}{2}\right)\geq 54r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(atg\frac{A}{2}+ctg\frac{B}{2}+btg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(btg\frac{B}{2}+atg\frac{C}{2}+ctg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(ctg\frac{C}{2}+btg\frac{A}{2}+atg\frac{B}{2}\right)$$

$$=\frac{atg\frac{A}{2}+ctg\frac{B}{2}+btg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{btg\frac{B}{2}+atg\frac{C}{2}+ctg\frac{A}{2}}{tg^2\frac{B}{2}}$$

$$+\frac{ctg\frac{C}{2}+btg\frac{A}{2}+tag\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}(a+b+c)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)$$

$$=(a+b+c)\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\overset{\sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}{\geq}\frac{2p^2}{r}\overset{Mitrinovic}{\geq}54r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A95. $\Delta ABC\Rightarrow$

$$ctg^2\frac{A}{2}\left(h_a tg\frac{A}{2}+h_c tg\frac{B}{2}+h_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(h_b tg\frac{B}{2}+h_a tg\frac{C}{2}+h_c tg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(h_c tg\frac{C}{2}+h_c tg\frac{A}{2}+h_a tg\frac{B}{2}\right)$$
$$\geq 27\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(h_a tg\frac{A}{2}+h_c tg\frac{B}{2}+h_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(h_b tg\frac{B}{2}+h_a tg\frac{C}{2}+h_c tg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(h_c tg\frac{C}{2}+h_c tg\frac{A}{2}+h_a tg\frac{B}{2}\right)$$
$$=\frac{h_a tg\frac{A}{2}+h_c tg\frac{B}{2}+h_b tg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{h_b tg\frac{B}{2}+h_a tg\frac{C}{2}+h_c tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{h_c tg\frac{C}{2}+h_c tg\frac{A}{2}+h_a tg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}$$
$$(h_a+h_b+h_c)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overset{\sum h_a\geq 9r;\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}{\geq} 9p \overset{;Mitrinovic}{\geq} 27\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A96. $\Delta ABC\Rightarrow$

$$ctg^2\frac{A}{2}\left(m_a tg\frac{A}{2}+m_c tg\frac{B}{2}+m_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(m_b tg\frac{B}{2}+m_a tg\frac{C}{2}+m_c tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(m_c tg\frac{C}{2}+m_c tg\frac{A}{2}+m_a tg\frac{B}{2}\right)\geq 27\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(m_a,m_b,m_c)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(m_a tg\frac{A}{2}+m_c tg\frac{B}{2}+m_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(m_b tg\frac{B}{2}+m_a tg\frac{C}{2}+m_c tg\frac{A}{2}\right)+$$
$$ctg^2\frac{C}{2}\left(m_c tg\frac{C}{2}+m_c tg\frac{A}{2}+m_a tg\frac{B}{2}\right)=$$
$$\frac{m_a tg\frac{A}{2}+m_c tg\frac{B}{2}+m_b tg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{m_b tg\frac{B}{2}+m_a tg\frac{C}{2}+m_c tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{m_c tg\frac{C}{2}+m_c tg\frac{A}{2}+m_a tg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}$$
$$(m_a+m_b+m_c)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overset{\sum m_a\geq 9r;\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}{\geq} 9p \overset{;Mitrinovic}{\geq} 27\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A97. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(r_a tg\frac{A}{2}+r_c tg\frac{B}{2}+r_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(r_b tg\frac{B}{2}+r_a tg\frac{C}{2}+r_c tg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(r_c tg\frac{C}{2}+r_c tg\frac{A}{2}+r_a tg\frac{B}{2}\right)\geq 27\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(r_a,r_b,r_c)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(r_a tg\frac{A}{2}+r_c tg\frac{B}{2}+r_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(r_b tg\frac{B}{2}+r_a tg\frac{C}{2}+r_c tg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(r_c tg\frac{C}{2}+r_c tg\frac{A}{2}+r_a tg\frac{B}{2}\right)=$$

$$\frac{r_a tg\frac{A}{2}+r_c tg\frac{B}{2}+r_b tg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{r_b tg\frac{B}{2}+r_a tg\frac{C}{2}+r_c tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{r_c tg\frac{C}{2}+r_c tg\frac{A}{2}+r_a tg\frac{B}{2}}{tg^2\frac{C}{2}}\overbrace{\geq}^{(1)}$$

$$(r_a+r_b+r_c)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overbrace{\geq}^{\sum r_a\geq 9r;\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}9p\overbrace{\geq}^{Mitrinovic}27\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A98. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(w_a tg\frac{A}{2}+w_c tg\frac{B}{2}+w_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(w_b tg\frac{B}{2}+w_a tg\frac{C}{2}+w_c tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(w_c tg\frac{C}{2}+w_c tg\frac{A}{2}+w_a tg\frac{B}{2}\right)\geq 27\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(w_a tg\frac{A}{2}+w_c tg\frac{B}{2}+w_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(w_b tg\frac{B}{2}+w_a tg\frac{C}{2}+w_c tg\frac{A}{2}\right)$$

$$+ctg^2\frac{C}{2}\left(w_c tg\frac{C}{2}+w_c tg\frac{A}{2}+w_a tg\frac{B}{2}\right)=$$

$$\frac{w_a tg\frac{A}{2}+w_c tg\frac{B}{2}+w_b tg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{w_b tg\frac{B}{2}+w_a tg\frac{C}{2}+w_c tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{w_c tg\frac{C}{2}+w_c tg\frac{A}{2}+w_a tg\frac{B}{2}}{tg^2\frac{C}{2}}\overbrace{\geq}^{(1)}$$

$$(w_a+w_b+w_c)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overbrace{\geq}^{\sum w_a\geq 9r;\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}9p\overbrace{\geq}^{;Mitrinovic}27\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

99. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(s_a tg\frac{A}{2}+s_c tg\frac{B}{2}+s_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(s_b tg\frac{B}{2}+s_a tg\frac{C}{2}+s_c tg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(s_c tg\frac{C}{2}+s_c tg\frac{A}{2}+r_a tg\frac{B}{2}\right)\geq 27\sqrt{3}r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(s_a tg\frac{A}{2}+s_c tg\frac{B}{2}+s_b tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(s_b tg\frac{B}{2}+s_a tg\frac{C}{2}+s_c tg\frac{A}{2}\right)+ctg^2\frac{C}{2}\left(s_c tg\frac{C}{2}+s_c tg\frac{A}{2}+r_a tg\frac{B}{2}\right)=$$

$$\frac{s_a tg\frac{A}{2}+s_c tg\frac{B}{2}+s_b tg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{s_b tg\frac{B}{2}+s_a tg\frac{C}{2}+s_c tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{s_c tg\frac{C}{2}+s_c tg\frac{A}{2}+s_a tg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(s_a+s_b+s_c)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overset{\sum s_a\geq 9r;\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}{\geq} 9p \overset{;Mitrinovic}{\geq} 27\sqrt{3}r.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

100. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(sinAtg\frac{A}{2}+sinCtg\frac{B}{2}+sinBtg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(sinBtg\frac{B}{2}+sinAtg\frac{C}{2}+sinCtg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(sinCtg\frac{C}{2}+sinBtg\frac{A}{2}+sinAtg\frac{B}{2}\right)\geq\frac{27r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(sinA,sinB,sinC)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(sinAtg\frac{A}{2}+sinCtg\frac{B}{2}+sinBtg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(sinBtg\frac{B}{2}+sinAtg\frac{C}{2}+sinCtg\frac{A}{2}\right)+$$

$$ctg^2\frac{C}{2}\left(sinCtg\frac{C}{2}+sinBtg\frac{A}{2}+sinAtg\frac{B}{2}\right)$$

$$=\frac{sinAtg\frac{A}{2}+sinCtg\frac{B}{2}+sinBtg\frac{C}{2}}{tg^2\frac{A}{2}}+\frac{inBtg\frac{B}{2}+sinAtg\frac{C}{2}+sinCtg\frac{A}{2}}{tg^2\frac{B}{2}}$$

$$+\frac{sinCtg\frac{C}{2}+sinBtg\frac{A}{2}+sinAtg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}(sinA+sinB+sinC)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overset{\sum sinA=\frac{p}{R};\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}{\geq}\frac{p^2}{Rr}\overset{Mitrinovic}{\geq}\frac{27r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

101. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(sin\frac{A}{2}tg\frac{A}{2}+sin\frac{C}{2}tg\frac{B}{2}+sin\frac{B}{2}tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(sin\frac{B}{2}tg\frac{B}{2}+sin\frac{A}{2}tg\frac{C}{2}+sin\frac{C}{2}tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(sin\frac{C}{2}tg\frac{C}{2}+sin\frac{B}{2}tg\frac{A}{2}+sin\frac{A}{2}tg\frac{B}{2}\right)\geq\frac{9\sqrt{3}}{2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(sin\frac{A}{2}tg\frac{A}{2}+sin\frac{C}{2}tg\frac{B}{2}+sin\frac{B}{2}tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(sin\frac{B}{2}tg\frac{B}{2}+sin\frac{A}{2}tg\frac{C}{2}+sin\frac{C}{2}tg\frac{A}{2}\right)+$$

$$+ctg^2\frac{C}{2}\left(sin\frac{C}{2}tg\frac{C}{2}+sin\frac{B}{2}tg\frac{A}{2}+sin\frac{A}{2}tg\frac{B}{2}\right)=\frac{sin\frac{A}{2}tg\frac{A}{2}+sin\frac{C}{2}tg\frac{B}{2}+sin\frac{B}{2}tg\frac{C}{2}}{tg^2\frac{A}{2}}+$$

$$\frac{sin\frac{B}{2}tg\frac{B}{2}+sin\frac{A}{2}tg\frac{C}{2}+sin\frac{C}{2}tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{sin\frac{C}{2}tg\frac{C}{2}+sin\frac{B}{2}tg\frac{A}{2}+sin\frac{A}{2}tg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overset{\sum sin\frac{A}{2}\geq\frac{3}{2};\ \sum\frac{1}{tg\frac{A}{2}}=\frac{p}{r}}{\geq}\frac{3p}{2r}\overset{Mitrinovic}{\geq}\frac{9\sqrt{3}}{2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

102. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(cos\frac{A}{2}tg\frac{A}{2}+cos\frac{C}{2}tg\frac{B}{2}+cos\frac{B}{2}tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(cos\frac{B}{2}tg\frac{B}{2}+cos\frac{A}{2}tg\frac{C}{2}+cos\frac{C}{2}tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(cos\frac{C}{2}tg\frac{C}{2}+cos\frac{B}{2}tg\frac{A}{2}+cos\frac{A}{2}tg\frac{B}{2}\right)\geq\frac{54r^2}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(cos\frac{A}{2}tg\frac{A}{2}+cos\frac{C}{2}tg\frac{B}{2}+cos\frac{B}{2}tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(cos\frac{B}{2}tg\frac{B}{2}+cos\frac{A}{2}tg\frac{C}{2}+cos\frac{C}{2}tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(cos\frac{C}{2}tg\frac{C}{2}+cos\frac{B}{2}tg\frac{A}{2}+cos\frac{A}{2}tg\frac{B}{2}\right)=\frac{cos\frac{A}{2}tg\frac{A}{2}+cos\frac{C}{2}tg\frac{B}{2}+cos\frac{B}{2}tg\frac{C}{2}}{tg^2\frac{A}{2}}+$$
$$\frac{cos\frac{B}{2}tg\frac{B}{2}+cos\frac{A}{2}tg\frac{C}{2}+cos\frac{C}{2}tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{cos\frac{C}{2}tg\frac{C}{2}+cos\frac{B}{2}tg\frac{A}{2}+cos\frac{A}{2}tg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}$$
$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)\overset{Lema;\ \sum ctg\frac{A}{2}=\frac{p}{r}}{\geq}\frac{2p^2}{R^2}\overset{Mitrinovic}{\geq}\frac{54r^2}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

103. $\Delta ABC \Rightarrow$

$$ctg^2\frac{A}{2}\left(1+ctg\frac{C}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(1+ctg\frac{A}{2}tg\frac{C}{2}+ctg\frac{C}{2}tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(1+ctg\frac{B}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{B}{2}\right)\geq 27.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ şi $(a,b,c)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ atunci:

$$ctg^2\frac{A}{2}\left(1+ctg\frac{C}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{C}{2}\right)+ctg^2\frac{B}{2}\left(1+ctg\frac{A}{2}tg\frac{C}{2}+ctg\frac{C}{2}tg\frac{A}{2}\right)$$
$$+ctg^2\frac{C}{2}\left(1+ctg\frac{B}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{B}{2}\right)=\frac{ctg\frac{A}{2}tg\frac{A}{2}+ctg\frac{C}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{C}{2}}{tg^2\frac{A}{2}}+$$
$$\frac{ctg\frac{B}{2}tg\frac{B}{2}+ctg\frac{A}{2}tg\frac{C}{2}+ctg\frac{C}{2}tg\frac{A}{2}}{tg^2\frac{B}{2}}+\frac{ctg\frac{C}{2}tg\frac{C}{2}+ctg\frac{B}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{B}{2}}{tg^2\frac{C}{2}}\overset{(1)}{\geq}$$
$$\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)\left(\frac{1}{tg\frac{A}{2}}+\frac{1}{tg\frac{B}{2}}+\frac{1}{tg\frac{C}{2}}\right)=\left(ctg\frac{A}{2}+ctg\frac{B}{2}+ctg\frac{C}{2}\right)^2\overset{\sum ctg\frac{A}{2}=\frac{p}{r}}{=}\frac{p^2}{r^2}\overset{Mitrinovic}{\geq}27.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

104. $\Delta ABC \Rightarrow$

$$tg^2\frac{A}{2}\left(actg\frac{A}{2}+cctg\frac{B}{2}+cbtg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(bctg\frac{B}{2}+actg\frac{C}{2}+cctg\frac{A}{2}\right)+tg^2\frac{C}{2}\left(cctg\frac{C}{2}+bctg\frac{A}{2}+actg\frac{B}{2}\right)\geq$$
$$18r.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(a,b,c)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(actg\frac{A}{2}+cctg\frac{B}{2}+cbtg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(bctg\frac{B}{2}+actg\frac{C}{2}+cctg\frac{A}{2}\right)+tg^2\frac{C}{2}\left(cctg\frac{C}{2}+bctg\frac{A}{2}+actg\frac{B}{2}\right)=$$

$$=\frac{actg\frac{A}{2}+cctg\frac{B}{2}+cbtg\frac{C}{2}}{ctg^2\frac{A}{2}}+\frac{bctg\frac{B}{2}+actg\frac{C}{2}+cctg\frac{A}{2}}{ctg^2\frac{B}{2}}+\frac{cctg\frac{C}{2}+bctg\frac{A}{2}+actg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(a+b+c)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\sum\frac{1}{ctg\frac{A}{2}}=\frac{4R+r}{p}}{=}8R+2r\overset{Mitrinovic}{\geq}18r.$$

A105. $\Delta ABC\Rightarrow$

$$tg^2\frac{A}{2}\left(h_actg\frac{A}{2}+h_cctg\frac{B}{2}+h_bctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(h_bctg\frac{B}{2}+h_actg\frac{C}{2}+h_ctcg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(h_cctg\frac{C}{2}+h_cctg\frac{A}{2}+h_actg\frac{B}{2}\right)\geq\frac{18\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(h_a,h_b,h_c)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(h_actg\frac{A}{2}+h_cctg\frac{B}{2}+h_bctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(h_bctg\frac{B}{2}+h_actg\frac{C}{2}+h_cctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(h_cctg\frac{C}{2}+h_cctg\frac{A}{2}+h_actg\frac{B}{2}\right)$$

$$=\frac{h_actg\frac{A}{2}+h_cctg\frac{B}{2}+h_bctg\frac{C}{2}}{ctg^2\frac{A}{2}}+\frac{h_bctg\frac{B}{2}+h_actg\frac{C}{2}+h_cctg\frac{A}{2}}{ctg^2\frac{B}{2}}+\frac{h_cctg\frac{C}{2}+h_cctg\frac{A}{2}+h_actg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(h_a+h_b+h_c)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\substack{\sum h_a\geq 9r;\\ \sum\frac{1}{ctg\frac{A}{2}}=\frac{4R+r}{p}}}{\geq}\frac{9r(4R+r)}{p}\overset{Euler}{\geq}\frac{81r^2}{p}\overset{Mitrinovic}{\geq}\frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A106. $\Delta ABC\Rightarrow$

$$tg^2\frac{A}{2}\left(m_actg\frac{A}{2}+m_cctg\frac{B}{2}+m_bctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(m_bctg\frac{B}{2}+m_actg\frac{C}{2}+m_cctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(m_cctg\frac{C}{2}+m_cctg\frac{A}{2}+m_actg\frac{B}{2}\right)\geq\frac{18\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (m_a, m_b, m_c)$ şi $(a, b, c) = \left(ctg\frac{A}{2}, ctg\frac{B}{2}, ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(m_a ctg\frac{A}{2} + m_c ctg\frac{B}{2} + m_b ctg\frac{C}{2}\right) + tg^2\frac{B}{2}\left(m_b ctg\frac{B}{2} + m_a ctg\frac{C}{2} + m_c ctg\frac{A}{2}\right) +$$

$$tg^2\frac{C}{2}\left(m_c ctg\frac{C}{2} + m_c ctg\frac{A}{2} + m_a ctg\frac{B}{2}\right) =$$

$$\frac{m_a ctg\frac{A}{2} + m_c ctg\frac{B}{2} + m_b ctg\frac{C}{2}}{ctg^2\frac{A}{2}} + \frac{m_b ctg\frac{B}{2} + m_a ctg\frac{C}{2} + m_c ctg\frac{A}{2}}{ctg^2\frac{B}{2}} + \frac{m_c ctg\frac{C}{2} + m_c ctg\frac{A}{2} + m_a tg\frac{B}{2}}{ctg^2\frac{C}{2}} \overset{(1)}{\geq}$$

$$(m_a + m_b + m_c)\left(\frac{1}{ctg\frac{A}{2}} + \frac{1}{ctg\frac{B}{2}} + \frac{1}{ctg\frac{C}{2}}\right) \overset{\substack{\sum m_a \geq 9r;\\ \sum\frac{1}{tcg\frac{A}{2}} = \frac{4R+r}{p}}}{\overbrace{\geq}} \frac{9r(4R+r)}{p} \overset{Euler}{\overbrace{\geq}} \frac{81r^2}{p} \overset{Mitrinovic}{\overbrace{\geq}} \frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A107. $\Delta ABC \Rightarrow$

$$tg^2\frac{A}{2}\left(r_a ctg\frac{A}{2} + r_c ctg\frac{B}{2} + r_b ctg\frac{C}{2}\right) + tg^2\frac{B}{2}\left(r_b ctg\frac{B}{2} + r_a ctg\frac{C}{2} + r_c ctg\frac{A}{2}\right)$$
$$+ tg^2\frac{C}{2}\left(r_c ctg\frac{C}{2} + r_c ctg\frac{A}{2} + r_a ctg\frac{B}{2}\right) \geq \frac{18\sqrt{3}r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x, y, z) = (r_a, r_b, r_c)$ şi $(a, b, c) = \left(ctg\frac{A}{2}, ctg\frac{B}{2}, ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(r_a ctg\frac{A}{2} + r_c ctg\frac{B}{2} + r_b ctg\frac{C}{2}\right) + tg^2\frac{B}{2}\left(r_b ctg\frac{B}{2} + r_a ctg\frac{C}{2} + r_c ctg\frac{A}{2}\right)$$
$$+ tg^2\frac{C}{2}\left(r_c ctg\frac{C}{2} + r_c ctg\frac{A}{2} + r_a ctg\frac{B}{2}\right) =$$

$$\frac{r_a ctg\frac{A}{2} + r_c ctg\frac{B}{2} + r_b ctg\frac{C}{2}}{ctg^2\frac{A}{2}} + \frac{r_b ctg\frac{B}{2} + r_a ctg\frac{C}{2} + r_c ctg\frac{A}{2}}{ctg^2\frac{B}{2}} + \frac{r_c ctg\frac{C}{2} + r_c ctg\frac{A}{2} + r_a ctg\frac{B}{2}}{ctg^2\frac{C}{2}} \overset{(1)}{\geq}$$

$$(r_a + r_b + r_c)\left(\frac{1}{ctg\frac{A}{2}} + \frac{1}{ctg\frac{B}{2}} + c\frac{1}{tg\frac{C}{2}}\right) \overset{\substack{\sum r_a \geq 9r;\\ \sum\frac{1}{ctg\frac{A}{2}} = \frac{4R+r}{p}}}{\overbrace{\geq}} \frac{9(4R+r)}{p} \overset{Euler}{\overbrace{\geq}} \frac{81r}{p} \overset{Mitrinovic}{\overbrace{\geq}} \frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

A108. $\Delta ABC \Rightarrow$

$$tg^2\frac{A}{2}\left(w_a ctg\frac{A}{2}+w_c ctg\frac{B}{2}+w_b ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(w_b ctg\frac{B}{2}+w_a ctg\frac{C}{2}+w_c ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(w_c ctg\frac{C}{2}+w_c ctg\frac{A}{2}+w_a ctg\frac{B}{2}\right)\geq.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(w_a,w_b,w_c)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(w_a ctg\frac{A}{2}+w_c ctg\frac{B}{2}+w_b ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(w_b ctg\frac{B}{2}+w_a ctg\frac{C}{2}+w_c ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(w_c ctg\frac{C}{2}+w_c ctg\frac{A}{2}+w_a ctg\frac{B}{2}\right)=$$
$$\frac{w_a ctg\frac{A}{2}+w_c ctg\frac{B}{2}+w_b ctg\frac{C}{2}}{ctg^2\frac{A}{2}}+\frac{w_b ctg\frac{B}{2}+w_a ctg\frac{C}{2}+w_c ctg\frac{A}{2}}{ctg^2\frac{B}{2}}+\frac{w_c ctg\frac{C}{2}+w_c ctg\frac{A}{2}+w_a ctg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}$$
$$(w_a+w_b+w_c)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\substack{\sum w_a\geq 9r;\\ \sum\frac{1}{ctg\frac{A}{2}}=\frac{4R+r}{p}}}{\geq}\frac{9(4R+r)}{p}\overset{Euler}{\geq}\frac{81r}{p}\overset{Mitrinovic}{\geq}\frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

109. $\Delta ABC\Rightarrow$

$$tg^2\frac{A}{2}\left(s_a ctg\frac{A}{2}+s_c ctg\frac{B}{2}+s_b ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(s_b ctg\frac{B}{2}+s_a ctg\frac{C}{2}+s_c ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(s_c ctg\frac{C}{2}+s_c ctg\frac{A}{2}+r_a ctg\frac{B}{2}\right)\geq$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(s_a,s_b,s_c)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(s_a ctg\frac{A}{2}+s_c ctg\frac{B}{2}+s_b ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(s_b ctg\frac{B}{2}+s_a ctg\frac{C}{2}+s_c ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(s_c ctg\frac{C}{2}+s_c ctg\frac{A}{2}+r_a ctg\frac{B}{2}\right)=$$
$$\frac{s_a ctg\frac{A}{2}+s_c ctg\frac{B}{2}+s_b ctg\frac{C}{2}}{ctg^2\frac{A}{2}}+\frac{s_b ctg\frac{B}{2}+s_a ctg\frac{C}{2}+s_c ctg\frac{A}{2}}{ctg^2\frac{B}{2}}+\frac{s_c ctg\frac{C}{2}+s_c ctg\frac{A}{2}+s_a ctg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$(s_a+s_b+s_c)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\sum s_a\geq 9r;\ \sum\frac{1}{ctg\frac{A}{2}}=\frac{p}{r}}{\geq}\frac{9(4R+r)}{p}\overset{Euler}{\geq}\frac{81r}{p}\overset{Mitrinovic}{\geq}\frac{18\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

110. $\Delta ABC \Rightarrow$

$$tg^2\frac{A}{2}\left(sinActg\frac{A}{2}+sinCctg\frac{B}{2}+sinBctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(sinBctg\frac{B}{2}+sinActg\frac{C}{2}+sinCtcg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(sinCctg\frac{C}{2}+sinBtcg\frac{A}{2}+sinActg\frac{B}{2}\right)\geq\frac{9r}{R}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=(sinA,sinB,sinC)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(sinActg\frac{A}{2}+sinCctg\frac{B}{2}+sinBctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(sinBctg\frac{B}{2}+sinActg\frac{C}{2}+sinCctg\frac{A}{2}\right)+$$
$$tg^2\frac{C}{2}\left(sinCctg\frac{C}{2}+sinBctg\frac{A}{2}+sinActg\frac{B}{2}\right)$$
$$=\frac{sinActg\frac{A}{2}+sinCctg\frac{B}{2}+sinBctg\frac{C}{2}}{ctg^2\frac{A}{2}}+\frac{inBctg\frac{B}{2}+sinActg\frac{C}{2}+sinCctg\frac{A}{2}}{ctg^2\frac{B}{2}}+$$
$$\frac{sinCctg\frac{C}{2}+sinBctg\frac{A}{2}+sinActg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}(sinA+sinB+sinC)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\sum sinA=\frac{p}{R};\ \sum\frac{1}{ctg\frac{A}{2}}=\frac{4R+r}{p}}{\geq}\frac{4R+r}{R}\overset{Euler}{\geq}$$
$$\frac{9r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

111. $\Delta ABC \Rightarrow$

$$tg^2\frac{A}{2}\left(sin\frac{A}{2}ctg\frac{A}{2}+sin\frac{C}{2}ctg\frac{B}{2}+sin\frac{B}{2}ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(sin\frac{B}{2}ctg\frac{B}{2}+sin\frac{A}{2}ctg\frac{C}{2}+sin\frac{C}{2}ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(sin\frac{C}{2}ctg\frac{C}{2}+sin\frac{B}{2}ctg\frac{A}{2}+sin\frac{A}{2}ctg\frac{B}{2}\right)\geq.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(sin\frac{A}{2}ctg\frac{A}{2}+sin\frac{C}{2}ctg\frac{B}{2}+sin\frac{B}{2}ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(sin\frac{B}{2}ctg\frac{B}{2}+sin\frac{A}{2}ctg\frac{C}{2}+sin\frac{C}{2}ctg\frac{A}{2}\right)+$$

$$+tg^2\frac{C}{2}\left(sin\frac{C}{2}ctg\frac{C}{2}+sin\frac{B}{2}ctg\frac{A}{2}+sin\frac{A}{2}ctg\frac{B}{2}\right)=\frac{sin\frac{A}{2}tg\frac{A}{2}+sin\frac{C}{2}tg\frac{B}{2}+sin\frac{B}{2}tg\frac{C}{2}}{ctg^2\frac{A}{2}}+$$

$$\frac{sin\frac{B}{2}ctg\frac{B}{2}+sin\frac{A}{2}ctg\frac{C}{2}+sin\frac{C}{2}ctg\frac{A}{2}}{ctg^2\frac{B}{2}}+\frac{sin\frac{C}{2}ctg\frac{C}{2}+sin\frac{B}{2}ctg\frac{A}{2}+sin\frac{A}{2}ctg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\substack{\sum sin\frac{A}{2}\geq\frac{3}{2};\\ \sum\frac{1}{ctg\frac{A}{2}}=\frac{4R+r}{p}}}{\geq}\frac{3(4R+r)}{2p}\overset{Euler}{\geq}\frac{27r}{2p}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}r}{R}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

112. $\Delta ABC\Rightarrow$

$$tg^2\frac{A}{2}\left(cos\frac{A}{2}ctg\frac{A}{2}+cos\frac{C}{2}ctg\frac{B}{2}+cos\frac{B}{2}ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(cos\frac{B}{2}ctg\frac{B}{2}+cos\frac{A}{2}ctg\frac{C}{2}+cos\frac{C}{2}ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(cos\frac{C}{2}ctg\frac{C}{2}+cos\frac{B}{2}ctg\frac{A}{2}+cos\frac{A}{2}ctg\frac{B}{2}\right)\geq\frac{18r^2}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(cos\frac{A}{2},cos\frac{B}{2},cos\frac{C}{2}\right)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(cos\frac{A}{2}ctg\frac{A}{2}+cos\frac{C}{2}ctg\frac{B}{2}+cos\frac{B}{2}ctg\frac{C}{2}\right)+tg^2\frac{B}{2}\left(cos\frac{B}{2}ctg\frac{B}{2}+cos\frac{A}{2}ctg\frac{C}{2}+cos\frac{C}{2}ctg\frac{A}{2}\right)$$
$$+tg^2\frac{C}{2}\left(cos\frac{C}{2}ctg\frac{C}{2}+cos\frac{B}{2}ctg\frac{A}{2}+cos\frac{A}{2}ctg\frac{B}{2}\right)=\frac{cos\frac{A}{2}ctg\frac{A}{2}+cos\frac{C}{2}ctg\frac{B}{2}+cos\frac{B}{2}ctg\frac{C}{2}}{ctg^2\frac{A}{2}}+$$

$$\frac{cos\frac{B}{2}ctg\frac{B}{2}+cos\frac{A}{2}ctg\frac{C}{2}+cos\frac{C}{2}ctg\frac{A}{2}}{ctg^2\frac{B}{2}}+\frac{cos\frac{C}{2}ctg\frac{C}{2}+cos\frac{B}{2}ctg\frac{A}{2}+cos\frac{A}{2}ctg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}$$

$$\left(cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2}\right)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)\overset{\substack{Lema;\\ \sum\frac{1}{ctg\frac{A}{2}}=\frac{4R+r}{p}}}{\geq}\frac{2r(4R+r)}{R^2}\overset{Euler}{\geq}\frac{18r^2}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

113. $\Delta ABC\Rightarrow$

$$tg^2\frac{A}{2}\left(1+ctg\frac{B}{2}tg\frac{C}{2}+ctg\frac{C}{2}tg\frac{B}{2}\right)+tg^2\frac{B}{2}\left(1+ctg\frac{C}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{C}{2}\right)+tg^2\frac{C}{2}\left(1+ctg\frac{A}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{A}{2}\right)\geq\frac{6r^2}{R^2}.$$

Gheorghe Ghiţă, Buzău

Soluţie. Pentru $(x,y,z)=\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$ şi $(a,b,c)=\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)$ atunci:

$$tg^2\frac{A}{2}\left(1+ctg\frac{B}{2}tg\frac{C}{2}+ctg\frac{C}{2}tg\frac{B}{2}\right)+tg^2\frac{B}{2}\left(1+ctg\frac{C}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{C}{2}\right)+tg^2\frac{C}{2}\left(1+ctg\frac{A}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{A}{2}\right)$$

$$=\frac{ctg\frac{A}{2}tg\frac{A}{2}+ctg\frac{C}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{C}{2}}{ctg^2\frac{A}{2}}+\frac{ctg\frac{B}{2}tg\frac{B}{2}+ctg\frac{C}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{C}{2}}{ctg^2\frac{B}{2}}+$$

$$\frac{ctg\frac{C}{2}tg\frac{C}{2}+ctg\frac{A}{2}tg\frac{B}{2}+ctg\frac{B}{2}tg\frac{A}{2}+ctg\frac{A}{2}tg\frac{B}{2}}{ctg^2\frac{C}{2}}\overset{(1)}{\geq}\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\left(\frac{1}{ctg\frac{A}{2}}+\frac{1}{ctg\frac{B}{2}}+\frac{1}{ctg\frac{C}{2}}\right)=$$

$$\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)^2\overset{\sum tg\frac{A}{2}=\frac{4R+r}{p}}{=}\frac{(4R+r)^2}{p^2}\overset{Euler}{\geq}\frac{81r^2}{p^2}\overset{Mitrinovic}{\geq}\frac{6r^2}{R^2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

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