Varianta 006

SUBIECTUL **I**

a) 13.
b) $2\sqrt{10}$
c) $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$.
d) $4{,}8$.
e) 1.
f) $y=-3$

SUBIECTUL II
1.
a) $\hat{2}$.
b) 16.
c) 1.
d) 8.
e) $\frac{2}{5}$.

2.
a) $5x^4+1$.
b) $-\frac{1}{3}$.
c) 1.
d) $f'(x)>0,\ \forall\ x\in \mathbf{R}$. Deci funcția f este strict crescătoare pe $\mathbf{R}$.
e) $\frac{12}{5}$.

SUBIECTUL III

a) $A+I_2=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}=B$
b) $\det B=2$; $\operatorname{rang} B=2$.
c) $A^2=A\cdot A=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}=A$.
d) Din **c)**, avem $A^2=A$.
Presupunem $A^k=A$ şi demonstrăm $A^{k+1}=A, \forall k\geq 1. k\in \mathbf{N}$.
$A^{k+1}=A^k\cdot A=A\cdot A=A^2=A$. Deci $A^n=A, \forall n\geq 1., n\in \mathbf{N}$. Atunci $A^{2007}=A$.
e) Pentru $n=1$ se obține $B=A+I_2$, evident din a), deci s-a realizat etapa de verificare.
Presupunem $B^k=I_2+\left(2^k-1\right)A, \forall\ k\in \mathbf{N}^*$ şi demonstrăm
$B^{k+1}=I_2+\left(2^{k+1}-1\right)A, \forall\ k\in \mathbf{N}^*$.

$B^{k+1}=B^k\cdot B=\left[I_2+\left(2^k-1\right)A\right]\left(A+I_2\right)=A+I_2+\left(2^k-1\right)A^2+\left(2^k-1\right)A=A+I_2+\left(2^k-1\right)A+\left(2^k-1\right)A=$

$=I_2+\left(2^{k+1}-1\right)A,$ adevarată $\forall\ k\in\ \mathbf{N}^*$.
Deci $B^n=I_2+\left(2^n-1\right)A, \forall n\in\ \mathbf{N}^*$.

f) $aA+bB+cI_2=\begin{pmatrix} b+c & a+b \\ 0 & a+2b+c \end{pmatrix}\neq\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ deoarece $0\neq 3$.
g) Matricea $X=A^n+B^n$ este inversabilă, dacă $\det\left(A^n+B^n\right)\neq 0, \forall\ n\in\mathbf{N}^*$.
Dar din **d)**,avem $A^n=A, \forall\ n\in\mathbf{N}^*$ iar din **e)** avem $B^n=I_2+\left(2^n-1\right)A$.
Obținem $A^n+B^n=\begin{pmatrix} 1 & 2^n \\ 0 & 2^n+1 \end{pmatrix}$.
Obținem $\det\left(A^n+B^n\right)=2^n+1\neq 0, \forall\ n\in\mathbf{N}^*$,deci $X=A^n+B^n$ este inversabilă, $\forall\ n\in\mathbf{N}^*$.

SUBIECTUL IV

a) $f'(x)=-x\cdot 2^{2-x^2}\ln 2$.
b)Cum $x\in[1,2]$ avem $x-1\geq 0$.
Cum $x\leq 2\Rightarrow\frac{1}{x}\geq\frac{1}{2}$.Deci $\frac{1}{x}-\frac{1}{2}\geq 0$.
Atunci $(x-1)\left(\frac{1}{x}-\frac{1}{2}\right)\geq 0$, $\forall\ x\in[1,2]$.
c) Se verifică prin calcul direct.
d)Se arată că $f(x)\in[1,2], \forall x\in[0,1]$.
Dar din **c),**avem $\frac{x}{2}+\frac{1}{x}\leq\frac{3}{2}, \forall\ x\in[1,2]$.
Deci $\frac{1}{f(x)}+\frac{f(x)}{2}\leq\frac{3}{2}, \forall x\in[0,1]$.
e) $(u+v)^2\geq 4uv$ devine $u^2-2uv+v^2\geq 0$.

Obținem $(u-v)^2\geq 0, \forall\ u,v\in\mathbf{R}$,evident adevărată.
Deci $(u+v)^2\geq 4uv\ \forall\ u,v\in\mathbf{R}$.
f) Din **d),** avem $\frac{1}{f(x)}+\frac{f(x)}{2}\leq\frac{3}{2}, \quad\forall x\in[0,1]$.
Integrând ,obținem

$$\int_0^1 \frac{1}{f(x)}dx+\frac{1}{2}\int_0^1 f(x)dx \le \int_0^1 \frac{3}{2}dx$$. Deci $$\int_0^1 \frac{1}{f(x)}dx+\frac{1}{2}\int_0^1 f(x)dx \le \frac{3}{2}.$$

g)Din **e) ,** avem $(u+v)^2 \ge 4uv$.

Atunci $\left(\int_0^1 \frac{1}{f(x)}dx+\frac{1}{2}\int_0^1 f(x)dx\right)^2 \ge 4\left(\int_0^1 \frac{1}{f(x)}dx\right)\cdot\left(\frac{1}{2}\int_0^1 f(x)dx\right)$.

Dar din **f),** prin ridicare la pătrat obținem

$$\left(\int_0^1 \frac{1}{f(x)}dx+\frac{1}{2}\int_0^1 f(x)dx\right)^2 \le \left(\frac{3}{2}\right)^2.$$

Atunci $4\left(\int_0^1 \frac{1}{f(x)}dx\right)\cdot\left(\frac{1}{2}\int_0^1 f(x)dx\right) \le \left(\int_0^1 \frac{1}{f(x)}dx+\frac{1}{2}\int_0^1 f(x)dx\right)^2 \le \frac{9}{4}$

Deducem

$$\left(\int_0^1 \frac{1}{f(x)}dx\right)\cdot\left(\int_0^1 f(x)dx\right) \le \frac{9}{8}.$$