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1. În legătură cu Inequality in triangle-2213 Romanian Mathematical Magazine 11/2020

Marin Chirciu[1]

Art 1808

Articolul prezinta o modalitatea unitară de demonstrare a unor relații în triunghi folosind o inegalitatea algebrică.

Lema.

Dacă $x, y > 0$, atunci

$$\frac{(x+y)^3}{x^3+y^3} \le 4.$$

Demonstrație.

Inegalitatea se scrie echivalent:

$\frac{(x+y)^3}{x^3+y^3} \le 4 \Leftrightarrow (x+y)(x-y)^2 \ge 0$, evident cu egalitate dacă și numai dacă $x = y$.

În continuare folosind **Lema** de mai sus rezolvăm o clasă de probleme.

1) In ΔABC

$$\sum \frac{1}{a^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{1}{r^2}.$$

George Apostolopoulos, Greece, RMM 11/2020

Soluție.

Avem

$$Ms = \sum \frac{1}{a^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{a^2}\cdot 4 = 4\sum \frac{1}{a^2} \overset{(2)}{\le} 4\cdot\frac{1}{4r^2} = \frac{1}{r^2} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{a^2} \le \frac{1}{4r^2}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarcă.

În aceeași clasă de probleme.

2) In ΔABC

$$\sum \frac{1}{a}\frac{(b+c)^3}{b^3+c^3} \le \frac{4p}{3Rr}.$$

Marin Chirciu

Soluție.

Avem

$$Ms = \sum \frac{1}{a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{a}\cdot 4 = 4\sum \frac{1}{bc} \overset{(2)}{\le} 4\cdot\frac{p}{3Rr} = \frac{4p}{3Rr} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{a} \le \frac{p}{3Rr}$.

3) In ΔABC

[1] *Profesor, Colegiul Național „Zinca Golescu”, Pitești*

$$\sum a\frac{(b+c)^3}{b^3+c^3}\le 8p.$$

Marin Chirciu

Soluție.

$$Ms=\sum a\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum a\cdot 4=4\sum a=4\cdot 2p=8p=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$.

4) In ΔABC

$$\sum a^3\frac{(b+c)^3}{b^3+c^3}\le 32p\left(R^2-r^2\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum a^3\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum a^3\cdot 4=4\sum a^3\overset{Tsitsifas}{\le}4\cdot 8p\left(R^2-r^2\right)=32p\left(R^2-r^2\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$.

5) In ΔABC

$$\sum a^2\frac{(b+c)^3}{b^3+c^3}\le 4\left(2R^2+r^2\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum a^2\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum a^2\cdot 4=4\sum a^2\overset{Gerretsen}{\le}4\cdot 4\left(2R^2+r^2\right)=4\left(2R^2+r^2\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$.

6) In ΔABC

$$\sum yza^2\frac{(b+c)^3}{b^3+c^3}\le 12R^2p.$$

Marin Chirciu

Soluție.

$$Ms=\sum yza^2\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum yza^2\cdot 4=4\sum yza^2\overset{(2)}{\le}4\cdot(x+y+z)^2R^2=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și (2) $\Leftrightarrow \sum yza^2\le(x+y+z)^2R^2$,

(Math.Magazin,48/1975, M.S.Klamkin, Canada).

7) In ΔABC

$$\sum a(p-a)\frac{(b+c)^3}{b^3+c^3}\le 36Rr.$$

Marin Chirciu

$$Ms=\sum a(p-a)\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum a(p-a)\cdot 4=4\sum a(p-a)\overset{(2)}{\le}4\cdot 9Rr=36Rr=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și (2) $\Leftrightarrow \sum a(p-a)=2r(4R+r)\le 9Rr$.

8) In ΔABC

$$\sum a^2(p-a)\frac{(b+c)^3}{b^3+c^3}\le 12R^2p.$$

Marin Chirciu

Soluție.

$$Ms=\sum a^2(p-a)\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum a^2(p-a)\cdot 4=4\sum a^2(p-a)\overset{(2)}{\le}4\cdot 3R^2p=12R^2p=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și (2) $\Leftrightarrow \sum a^2(p-a)=4pr(R+r)\le 3R^2p$.

9) In ΔABC

$$\sum a^3(p-a)\frac{(b+c)^3}{b^3+c^3}\le 4abcp.$$

Marin Chirciu

Soluție.

$$Ms=\sum a^3(p-a)\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum a^3(p-a)\cdot 4=4\sum a^3(p-a)\overset{(2)}{\le}4\cdot abcp=4abcp=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și (2) $\Leftrightarrow$

$\sum a^3(p-a)=p^2(4Rr+6r^2)-2r^2(4R+r)^2\le abcp$, (AMM, 1952, J.Andersson).

10) In ΔABC

$$\sum bc\frac{(b+c)^3}{b^3+c^3}\le 36(R+r)^2.$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum bc\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum bc\cdot 4=4\sum bc\overset{Carlitz}{\le}4\cdot 4R^2=36(R+r)^2=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$.

11) In ΔABC

$$\sum (b-c)^2\frac{(b+c)^3}{b^3+c^3}\le 32R(R-2r).$$

Marin Chirciu

Soluție.

$$Ms=\sum (b-c)^2\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum (b-c)^2\cdot 4=4\sum (b-c)^2\overset{Gerretsen}{\le}4\cdot 8R(R-2r)=$$

$$=32R(R-2r)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$.

12) In ΔABC

$$\sum \frac{m_a}{h_a}\frac{(b+c)^3}{b^3+c^3} \le \frac{6R}{r}.$$

Marin Chirciu

Soluție.

Avem

$$Ms = \sum \frac{m_a}{h_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{m_a}{h_a}\cdot 4 = 4\sum \frac{m_a}{h_a} \overset{(2)}{\le} 4\cdot\frac{3R}{2r} = \frac{6R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \frac{m_a}{h_a} \le \frac{R}{2r}$, (L.Panaitopol).

13) In ΔABC

$$\sum \frac{a}{b}\frac{(b+c)^3}{b^3+c^3} \le \frac{6R}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a}{b}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a}{b}\cdot 4 = 4\sum \frac{a}{b} \overset{(2)}{\le} 4\cdot\frac{3R}{2r} = \frac{6R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{a}{b} \le \frac{3R}{2r}$,(V.Băndilă, M.Lascu).

14) *In ΔABC

$$\sum \frac{a}{b}\frac{(b+c)^3}{b^3+c^3} \le \frac{4p^2}{r(4R+r)}.$$

Marin Chirciu

Soluție.

Avem

$$Ms = \sum \frac{a}{b}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a}{b}\cdot 4 = 4\sum \frac{a}{b} \overset{(2)}{\le} 4\cdot\frac{p^2}{r(4R+r)} = \frac{4p^2}{r(4R+r)} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{a}{b} \le \frac{p^2}{r(4R+r)}$,(C.Mateescu).

15) In ΔABC

$$\sum \left(\frac{b}{c}+\frac{c}{b}\right)\frac{(b+c)^3}{b^3+c^3} \le \frac{12R}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \left(\frac{b}{c}+\frac{c}{b}\right)\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \left(\frac{b}{c}+\frac{c}{b}\right)\cdot 4 = 4\sum \left(\frac{b}{c}+\frac{c}{b}\right) \overset{(2)}{\le} 4\cdot 3\frac{R}{r} = \frac{12R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \frac{b}{c}+\frac{c}{b} \le \frac{R}{r}$,(V.Băndilă).

16) In ΔABC

$$\sum \frac{1}{a}\frac{(b+c)^2}{b^3+c^3} \le \frac{1}{Rr}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{a}\frac{(b+c)^2}{b^3+c^3} = \sum \frac{1}{a(b+c)}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{a(b+c)}\cdot 4 = 4\sum \frac{1}{a(b+c)} \overset{(2)}{\le} 4\cdot\frac{1}{4Rr} =$$

$$= \frac{1}{Rr} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{a(b+c)} \le \frac{1}{4Rr}$.

17) In ΔABC

$$\sum \frac{a^2}{bc}\frac{(b+c)^3}{b^3+c^3} \le 4\cdot\left(\frac{2R}{r}-1\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a^2}{bc}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^2}{bc}\cdot 4 = 4\sum \frac{a^2}{bc} \overset{(2)}{\le} 4\cdot\left(\frac{2R}{r}-1\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{a^2}{bc} \le \frac{2R}{r}-1$,(V.Nicula).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

18) In ΔABC

$$\sum \frac{bc}{p-a}\frac{(b+c)^3}{b^3+c^3} \le \frac{108R^2}{p}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{bc}{p-a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{bc}{p-a}\cdot 4 = 4\sum \frac{bc}{p-a} \overset{(2)}{\le} 4\cdot\frac{27R^2}{p} = \frac{108R^2}{p} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{bc}{p-a} \le \frac{27R^2}{p}$,(Vezi **Lema**).

Lema

19) In ΔABC

$$\sum \frac{bc}{p-a} \le \frac{27R^2}{p}.$$

Demonstrație.

$$\sum \frac{bc}{p-a} = \frac{p^2+(4R+r)^2}{p} \overset{Gerretsen}{\le} \frac{27R^2}{p}.$$

20) In ΔABC

$$\sum \frac{a}{p-a}\frac{(b+c)^3}{b^3+c^3} \le 8\left(\frac{2R}{r}-1\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{p-a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a}{p-a}\cdot 4=4\sum\frac{a}{p-a}=4\cdot 2\left(\frac{2R}{r}-1\right)=8\left(\frac{2R}{r}-1\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$.

21) In ΔABC

$$\sum\frac{1}{p-a}\frac{(b+c)^3}{b^3+c^3}\le\frac{18R}{S}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{p-a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{1}{p-a}\cdot 4=4\sum\frac{1}{p-a}=4\cdot\frac{4R+r}{pr}\overset{Euler}{\le}\frac{18R}{pr}=\frac{18R}{S}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$.

22) In ΔABC

$$\sum\frac{1}{a(p-a)}\frac{(b+c)^3}{b^3+c^3}\le\frac{R^2}{2r^4}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{a(p-a)}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{1}{a(p-a)}\cdot 4=4\sum\frac{1}{a(p-a)}\overset{(2)}{\le}4\cdot\frac{R^2}{8r^4}=\frac{R^2}{2r^4}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și

(2) $\Leftrightarrow \sum\frac{1}{a(p-a)}=\frac{p^2+(4R+r)^2}{4Rrp^2}=\frac{1}{4Rr}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\le\frac{R^2}{8r^4}$.

23) In ΔABC

$$\sum(p-a)\frac{(b+c)^2}{b^3+c^3}\le 3.$$

Marin Chirciu

Soluție.

$$Ms=\sum(p-a)\frac{(b+c)^2}{b^3+c^3}=\sum\frac{p-a}{b+c}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{p-a}{b+c}\cdot 4=4\sum\frac{p-a}{b+c}\overset{(2)}{\le}4\cdot\frac{3}{4}=3=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și

(2) $\Leftrightarrow \sum\frac{p-a}{b+c}=\frac{p^2+5r^2+8Rr}{2(p^2+r^2+Rr)}\overset{Gerretsen}{\le}\frac{3}{4}$.

24) In ΔABC

$$\sum\frac{(p-a)^2}{bc}\frac{(b+c)^2}{b^3+c^3}\le 4\left(1-\frac{2r}{R}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{(p-a)^2}{bc}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{(p-a)^2}{bc}\cdot 4=4\sum\frac{(p-a)^2}{bc}\overset{(2)}{=}4\cdot\left(1-\frac{2r}{R}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și

(2) $\Leftrightarrow \sum\frac{(p-a)^2}{bc}=\left(1-\frac{2r}{R}\right)$, (RMT 1936, C.Ionescu-Țiu).

25) In ΔABC

$$\sum\frac{1}{p-a}\frac{(b+c)^5}{b^3+c^3}\le\frac{32Rp}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{p-a}\frac{(b+c)^5}{b^3+c^3}=\sum\frac{(b+c)^2}{p-a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{(b+c)^2}{p-a}\cdot 4=4\sum\frac{(b+c)^2}{p-a}\overset{(2)}{\le}4\cdot\frac{8Rp}{r}=$$

$=\frac{32Rp}{r}=Md$, unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și

(2) $\Leftrightarrow \sum\frac{(b+c)^2}{p-a}=\frac{4p(R+2r)}{r}\overset{Euler}{\le}\frac{8Rp}{r}$.

26) In ΔABC

$$\sum(p+a)\frac{(b+c)^2}{b^3+c^3}\le\frac{15}{4}\left(\frac{R}{r}\right)^2.$$

Marin Chirciu

Soluție.

$$Ms=\sum(p+a)\frac{(b+c)^2}{b^3+c^3}=\sum\frac{p+a}{b+c}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{p+a}{b+c}\cdot 4=4\sum\frac{p+a}{b+c}\overset{(2)}{\le}4\cdot\frac{15}{16}\left(\frac{R}{r}\right)^2=$$

$=\frac{15}{4}\left(\frac{R}{r}\right)^2=Md$, unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și

(2) $\Leftrightarrow \sum\frac{p+a}{b+c}=\frac{3(3p^2-r^2)}{2(p^2+r^2+Rr)}\overset{Gerretsen}{\le}\frac{15}{16}\left(\frac{R}{r}\right)^2$.

27) In ΔABC

$$\sum\sqrt{b+c}\frac{(b+c)^2}{b^3+c^3}\le 4\sqrt{\frac{6Rp}{r}}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\sqrt{b+c}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\sqrt{b+c}\cdot 4=4\sum\sqrt{b+c}\overset{(2)}{=}4\cdot\sqrt{\frac{6Rp}{r}}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$ și

$(2) \Leftrightarrow \sum \sqrt{b+c} \overset{CBS}{\leq} \sqrt{\frac{6Rp}{r}}$.

28) In ΔABC

$$\sum \frac{1}{a} \frac{(b+c)^4}{b^3+c^3} \leq \frac{12R}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{a} \frac{(b+c)^4}{b^3+c^3} = \sum \frac{b+c}{a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\leq} \sum \frac{b+c}{a} \cdot 4 = 4\sum \frac{b+c}{a} \overset{(2)}{\leq} 4 \cdot \frac{3R}{r} = \frac{12R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, iar (2) $\Leftrightarrow \sum \frac{b+c}{a} \leq \frac{3R}{r}$,(vezi **Lema**).

Lema

29) In ΔABC

$$\sum \frac{b+c}{a} \leq \frac{3R}{r}.$$

Soluție.

$$\sum \frac{b+c}{a} = \frac{p^2+r^2-2Rr}{2Rr} \overset{Gerretsen}{\leq} \frac{4R^2+4Rr+3r^2+r^2-2Rr}{2Rr} = \frac{4R^2+2Rr+4r^2}{2Rr} =$$

$$= \frac{2R^2+Rr+2r^2}{Rr} \overset{Euler}{\leq} \frac{3R}{r}.$$

30) In ΔABC

$$\sum a \frac{(b+c)^2}{b^3+c^3} \leq \frac{3R}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum a \frac{(b+c)^2}{b^3+c^3} = \sum \frac{a}{b+c} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\leq} \sum \frac{a}{b+c} \cdot 4 = 4\sum \frac{a}{b+c} \overset{(2)}{\leq} 4 \cdot \frac{3R}{4r} = \frac{3R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, iar (2) $\Leftrightarrow \sum \frac{a}{b+c} \leq \frac{3R}{4r}$,(vezi **Lema**).

Lema

$$\sum \frac{a}{b+c} \leq \frac{3R}{4r}.$$

Demonstrație.

Folosim $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr}$.

Obținem $\sum \frac{a}{b+c} = \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \overset{(1)}{\leq} \frac{3R}{4r}$, unde (2) $\Leftrightarrow$

$$\Leftrightarrow \frac{2(p^2-r^2-Rr)}{p^2+r^2+2Rr} \leq \frac{3R}{4r} \Leftrightarrow 8r(p^2-r^2-Rr) \leq 3R(p^2+r^2+2Rr) \Leftrightarrow$$

$$\Leftrightarrow p^2(3R-8r) + r(6R^2+11Rr+8r^2) \geq 0.$$

Distingem cazurile.

Cazul 1). Dacă $(3R-8r)\geq 0$, inegalitatea este evidentă.

Cazul 2). Dacă $(3R-8r)<0$, inegalitatea se rescrie:

$r\left(6R^2+11Rr+8r^2\right)\geq p^2(8r-3R)$, care rezultă din inegalitatea lui Gerretsen:

$p^2\leq 4R^2+4Rr+3r^2$ și inegalitatea lui Euler $R\geq 2r$.

Rămâne să arătăm că:

$r\left(6R^2+11Rr+8r^2\right)\geq\left(4R^2+4Rr+3r^2\right)(8r-3R)\Leftrightarrow 6R^3-7R^2r-6Rr^2-8r^3\geq 0\Leftrightarrow$

$\Leftrightarrow (R-2r)\left(6R^2+5Rr+4r^2\right)\geq 0$.

31) In ΔABC

$$\sum\frac{(b+c)^2}{b^3+c^3}\leq\frac{2p}{3Rr}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{(b+c)^2}{b^3+c^3}=\sum\frac{1}{b+c}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\leq}\sum\frac{1}{b+c}\cdot 4=4\sum\frac{1}{b+c}\overset{(2)}{\leq}4\cdot\frac{p}{6Rr}=\frac{2p}{3Rr}=Md,$$

unde (1) $\Leftrightarrow\frac{(b+c)^3}{b^3+c^3}\leq 4\Leftrightarrow (b+c)(b-c)^2\geq 0$, iar (2) $\Leftrightarrow\sum\frac{1}{b+c}\leq\frac{p}{6Rr}$,(vezi **Lema**).

Lema

$$\sum\frac{1}{b+c}\leq\frac{p}{6Rr}.$$

Demonstrație.

Folosim $\sum\frac{1}{b+c}=\frac{5p^2+r^2+4Rr}{2p\left(p^2+r^2+2Rr\right)}$.

Obținem $\sum\frac{1}{b+c}=\frac{5p^2+r^2+4Rr}{2p\left(p^2+r^2+2Rr\right)}\overset{(3)}{\leq}\frac{p}{6Rr}$, unde (3) $\Leftrightarrow$

$\Leftrightarrow\frac{5p^2+r^2+4Rr}{2p\left(p^2+r^2+2Rr\right)}\leq\frac{p}{6Rr}\Leftrightarrow 3Rr\left(5p^2+r^2+4Rr\right)\leq p^2\left(p^2+r^2+2Rr\right)\Leftrightarrow$

$\Leftrightarrow p^2\left(p^2+r^2-12Rr\right)\geq 3Rr^2(4R+r)$, care rezultă din inegalitatea lui Gerretsen:

$p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

Rămâne să arătăm că:

$\left(16Rr-5r^2\right)\left(16Rr-5r^2+r^2-13Rr\right)\geq 3Rr^2(4R+r)\Leftrightarrow(16R-5r)(3R-4r)\geq 3R(4R+r)$

$\Leftrightarrow (R-2r)(18R-5r)\geq 0$.

32) In ΔABC

$$\sum r_a\frac{(b+c)^3}{b^3+c^3}\leq 18R.$$

Marin Chirciu

Soluție.

$$Ms=\sum r_a\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\leq}\sum r_a\cdot 4=4\sum r_a\overset{(2)}{\leq}4\cdot\frac{9R}{2}=18R=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum r_a = 4R + r \le \frac{9R}{2}$.

33) In ΔABC

$$\sum (r_b + r_c)\frac{(b+c)^3}{b^3+c^3} \le 9R.$$

Marin Chirciu

Soluție.

$$Ms = \sum (r_b + r_c)\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum (r_b + r_c)\cdot 4 = 4\sum (r_b + r_c) \overset{(2)}{\le} 4\cdot\frac{1}{r} = \frac{4}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$,

iar (2) $\Leftrightarrow \sum (r_b + r_c) = 2\sum r_a = 2(4R+r) \le 2\cdot\frac{9R}{2} = 9R$.

34) In ΔABC

$$\sum \frac{1}{r_b + r_c}\frac{(b+c)^3}{b^3+c^3} \le \frac{2}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum (r_b + r_c)\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum (r_b + r_c)\cdot 4 = 4\sum (r_b + r_c) \overset{(2)}{\le} 4\cdot\frac{1}{2r} = \frac{2}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow$

$$\sum \frac{1}{r_b + r_c} = \frac{1}{4R}\left[1 + \frac{(4R+r)^2}{p^2}\right] \overset{Gerretsen}{\le} \frac{1}{4R}\left[1 + \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{1}{4R}\left(1 + \frac{R+r}{r}\right) \overset{Euler}{\le} \frac{1}{4R}\cdot\frac{2R}{r} = \frac{1}{2r}.$$

35) In ΔABC

$$\sum r_b r_c \frac{(b+c)^3}{b^3+c^3} \le 4p^2.$$

Marin Chirciu

Soluție.

$$Ms = \sum r_b r_c \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum r_b r_c \cdot 4 = 4\sum r_b r_c \overset{(2)}{=} 4\cdot p^2 = 4p^2 = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum r_b r_c = p^2$.

36) In ΔABC

$$\sum \frac{r_a}{a}\frac{(b+c)^3}{b^3+c^3} \le \frac{27R}{p}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{r_a}{a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{r_a}{a}\cdot 4=4\sum\frac{r_a}{a}\overset{(2)}{\le}4\cdot\frac{27R}{4p}=\frac{27R}{p}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$,

iar (2) $\Leftrightarrow \sum\frac{r_a}{a}=\frac{p^2+(4R+r)^2}{4Rp}\overset{Gerretsen}{\le}\frac{27R}{4p}$.

37) In ΔABC

$$\sum\frac{a}{r_a}\frac{(b+c)^3}{b^3+c^3}\le\frac{36R}{p}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a}{r_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a}{r_a}\cdot 4=4\sum\frac{a}{r_a}\overset{(2)}{\le}4\cdot\frac{9R}{p}=\frac{36R}{p}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$,

iar (2) $\Leftrightarrow \sum\frac{a}{r_a}=\frac{2(4R+r)}{p}\overset{Euler}{\le}\frac{9R}{p}$.

38) In ΔABC

$$\sum\frac{a^2}{r_a}\frac{(b+c)^3}{b^3+c^3}\le 12R.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a^2}{r_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a^2}{r_a}\cdot 4=4\sum\frac{a^2}{r_a}\overset{(2)}{\le}4\cdot 3R=12R=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$,

iar (2) $\Leftrightarrow \sum\frac{a^2}{r_a}=2(R+r)\overset{Euler}{\le}3R$.

39) In ΔABC

$$\sum\frac{1}{r_a}\frac{(b+c)^3}{b^3+c^3}\le\frac{4}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{r_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{1}{r_a}\cdot 4=4\sum\frac{1}{r_a}\overset{(2)}{=}4\cdot\frac{1}{r}=\frac{4}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum\frac{1}{r_a}=\frac{1}{r}$.

40) In ΔABC

$$\sum \frac{1}{h_a}\frac{(b+c)^3}{b^3+c^3} \le \frac{4}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{h_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{h_a}\cdot 4 = 4\sum \frac{1}{h_a} \overset{(2)}{=} 4\cdot\frac{1}{r} = \frac{4}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{h_a} = \frac{1}{r}$.

41) In ΔABC

$$\sum \frac{1}{m_a}\frac{(b+c)^3}{b^3+c^3} \le \frac{4}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{m_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{m_a}\cdot 4 = 4\sum \frac{1}{m_a} \overset{(2)}{\le} 4\cdot\frac{1}{r} = \frac{4}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{m_a} \le \sum \frac{1}{h_a} = \frac{1}{r}$.

42) In ΔABC

$$\sum \frac{1}{w_a}\frac{(b+c)^3}{b^3+c^3} \le \frac{4}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{w_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{w_a}\cdot 4 = 4\sum \frac{1}{w_a} \overset{(2)}{\le} 4\cdot\frac{1}{r} = \frac{4}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{w_a} \le \sum \frac{1}{h_a} = \frac{1}{r}$.

43) In ΔABC

$$\sum \frac{w_a}{h_a}\frac{(b+c)^3}{b^3+c^3} \le 2\sqrt{\frac{2R}{r}}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{w_a}{h_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{w_a}{h_a}\cdot 4 = 4\sum \frac{w_a}{h_a} \overset{(2)}{\le} 4\cdot\sqrt{\frac{R}{2r}} = 2\sqrt{\frac{2R}{r}} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \frac{w_a}{h_a} \le \sqrt{\frac{R}{2r}}$,(vezi **Lema**).

Lema.

In ΔABC

$$\frac{w_a}{h_a} \le \sqrt{\frac{R}{2r}}.$$

Demonstrație.

Folosind $w_a = \frac{2bc}{b+c}\cos\frac{A}{2}$ și $h_a = \frac{2S}{a}$ obținem:

$$\frac{w_a}{h_a} = \frac{\frac{2bc}{b+c}\cos\frac{A}{2}}{\frac{2S}{a}} = \frac{\frac{2bc}{b+c}\sqrt{\frac{p(p-a)}{bc}}}{\frac{2S}{a}} = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{p(p-a)}\cdot\frac{a}{2pr} = \frac{\sqrt{abc}}{r\sqrt{p}}\cdot\frac{\sqrt{a(p-a)}}{b+c} =$$

$$= \frac{\sqrt{4Rrp}}{r\sqrt{p}}\cdot\frac{\sqrt{a(p-a)}}{(b+c-a)+a} = 2\sqrt{\frac{R}{r}}\cdot\frac{\sqrt{a(p-a)}}{(2p-2a)+a} \overset{AGM}{\le} 2\sqrt{\frac{R}{r}}\cdot\frac{\sqrt{a(p-a)}}{2\sqrt{(2p-2a)\cdot a}} = \sqrt{\frac{R}{2r}}.$$

44) In ΔABC

$$\sum\frac{1}{\sin\frac{A}{2}}\frac{(b+c)^3}{b^3+c^3} \le 8\left(\frac{R}{r}+1\right).$$

Marin Chirciu

Soluție.

Avem

$$Ms = \sum\frac{1}{\sin\frac{A}{2}}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum\frac{1}{\sin\frac{A}{2}}\cdot 4 = 4\sum\frac{1}{\sin\frac{A}{2}} \overset{(2)}{\le} 4\cdot 2\left(\frac{R}{r}+1\right) = 8\left(\frac{R}{r}+1\right) = Md \text{, unde (1)}$$

$$\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0 \text{, iar (2)} \Leftrightarrow \sum\frac{1}{\sin\frac{A}{2}} \le 2\left(\frac{R}{r}+1\right)\text{,(vezi } \textbf{Lema}\text{).}$$

Lema.

45) In ΔABC

$$\sum\frac{1}{\sin\frac{A}{2}} \le 2\left(\frac{R}{r}+1\right).$$

Demonstrație.

Cu inegalitatea CBS obținem:

$$\left(\sum\frac{1}{\sin\frac{A}{2}}\right)^2 = \left(\sum\frac{1}{\sqrt{\frac{(p-b)(p-c)}{bc}}}\right)^2 = \left(\sum\frac{\sqrt{bc}}{\sqrt{(p-b)(p-c)}}\right)^2 \le \sum bc\sum\frac{1}{(p-b)(p-c)} =$$

$$= \left(p^2+r^2+4Rr\right)\frac{1}{r^2} \overset{Gerretsen}{\le} \left(4R^2+4Rr+3r^2+r^2+4Rr\right)\frac{1}{r^2} = \left(4R^2+8Rr+4r^2\right)\frac{1}{r^2} =$$

$$= 4(R+r)^2\frac{1}{r^2}\text{, de unde } \sum\frac{1}{\sin\frac{A}{2}} \le 2\left(\frac{R}{r}+1\right).$$

46) In ΔABC

$$\sum \frac{1}{\cos\frac{A}{2}} \frac{(b+c)^3}{b^3+c^3} \le \frac{18R^2}{S}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{\cos\frac{A}{2}} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{\cos\frac{A}{2}} \cdot 4 = 4\sum \frac{1}{\cos\frac{A}{2}} \overset{(2)}{\le} 4 \cdot \frac{9R^2}{2S} = \frac{18R^2}{S} = Md \text{, unde (1)} \Leftrightarrow$$

$$\frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0 \text{, iar (2)} \Leftrightarrow \sum \frac{1}{\cos\frac{A}{2}} \le \frac{9R^2}{2S}$$,(vezi **Lema**).

Lema.

47) In ΔABC

$$\sum \frac{1}{\cos\frac{A}{2}} \le \frac{a^2+b^2+c^2}{2S} \overset{Leibniz}{\le} \frac{9R^2}{2S}.$$

Demonstrație.

$$\text{Avem } 2S\sum \frac{1}{\cos\frac{A}{2}} = \sum \frac{bc\sin A}{\cos\frac{A}{2}} = \sum \frac{bc \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}} = 2\sum bc\sin\frac{A}{2} \overset{(1)}{\le} a^2+b^2+c^2,$$

$$\text{unde (1)} \Leftrightarrow 2\sum bc\sin\frac{A}{2} \le a^2+b^2+c^2 \Leftrightarrow a^2 - 2\sum bc\sin\frac{A}{2} + b^2+c^2 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow a^2 - 2a\left(b\sin\frac{C}{2} + c\sin\frac{B}{2}\right) + b^2+c^2 - 2bc\sin\frac{A}{2} \ge 0 \text{, care rezultă din :}$$

$$\Delta = 4\left(b\sin\frac{C}{2} + c\sin\frac{B}{2}\right)^2 - 4\left(b^2+c^2-2bc\sin\frac{A}{2}\right) =$$

$$= 4\left[-b^2\cos^2\frac{C}{2} - c^2\cos^2\frac{B}{2} + 2bc\left(\sin\frac{A}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)\right] \overset{(2)}{=} -4\left(b\cos\frac{C}{2} - c\cos\frac{B}{2}\right)^2 \le 0,$$

$$\text{unde (2)} \Leftrightarrow \cos\frac{B}{2}\cos\frac{C}{2} = \sin\frac{A}{2} + \sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\sum \frac{1}{\cos\frac{A}{2}} \le \frac{a^2+b^2+c^2}{2S} \overset{Leibniz}{\le} \frac{9R^2}{2S}.$$

48) In ΔABC

$$\sum \left(\frac{w_a}{h_a}\right)^2 \frac{(b+c)^3}{b^3+c^3} \le \frac{6R}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \left(\frac{w_a}{h_a}\right)^2 \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \left(\frac{w_a}{h_a}\right)^2 \cdot 4 = 4\sum \left(\frac{w_a}{h_a}\right)^2 \overset{(2)}{\le} 4 \cdot \frac{3R}{2r} = \frac{6R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum\left(\frac{w_a}{h_a}\right)^2 \le \frac{3R}{2r}$,(vezi **Lema**).

Lema.

In ΔABC

$$\sum\left(\frac{w_a}{h_a}\right)^2 \le \frac{3R}{2r}.$$

Demonstrație.

Folosind $\cos\frac{B-C}{2} \ge \sqrt{\frac{2r}{R}}$ și $\cos\frac{B-C}{2} = \frac{h_a}{w_a}$ obținem:

$\frac{h_a}{w_a} = \cos\frac{B-C}{2} \ge \sqrt{\frac{2r}{R}}$, de unde $\left(\frac{h_a}{w_a}\right)^2 \ge \frac{2r}{R} \Leftrightarrow \left(\frac{w_a}{h_a}\right)^2 \le \frac{R}{2r}$, de unde $\sum\left(\frac{w_a}{h_a}\right)^2 \le \frac{3R}{2r}$.

49) In ΔABC

$$\sum\left(\frac{h_a}{l_a}\right)^2 \frac{(b+c)^3}{b^3+c^3} \le 4\cdot\left(2+\frac{3r}{2R}+\frac{r^2}{R^2}\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum\left(\frac{h_a}{l_a}\right)^2 \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum\left(\frac{h_a}{l_a}\right)^2 \cdot 4 = 4\sum\left(\frac{h_a}{l_a}\right)^2 \overset{(2)}{\le} 4\cdot\left(2+\frac{3r}{2R}+\frac{r^2}{R^2}\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum\left(\frac{w_a}{h_a}\right)^2 \le \frac{3R}{2r}$,(vezi **Lema**).

Lemă.

$$\sum\left(\frac{h_a}{l_a}\right)^2 = \frac{p^2+r^2+2Rr+4R^2}{4R^2} \overset{Gerretsen}{\le} 2+\frac{3r}{2R}+\frac{r^2}{R^2}.$$

Demonstrație.

$$\sum\left(\frac{h_a}{l_a}\right)^2 = \sum\frac{\left(\frac{2S}{a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2} = \frac{S^2}{a^2b^2c^2}\sum\frac{(b+c)^2}{\cos^2\frac{A}{2}} = \frac{1}{16R^2}\sum\frac{bc(b+c)^2}{p(p-a)} =$$

$$= \frac{1}{16R^2}\cdot 4\left(p^2+r^2+2Rr+4R^2\right) = \frac{p^2+r^2+2Rr+4R^2}{4R^2}.$$

$$\sum\left(\frac{h_a}{l_a}\right)^2 = \frac{p^2+r^2+2Rr+4R^2}{4R^2} \overset{Gerretsen}{\le} \frac{4R^2+4Rr+3r^2+r^2+2Rr+4R^2}{4R^2} = \frac{8R^2+6Rr+4r^2}{4R^2} =$$

$$= \frac{4R^2+3Rr+2r^2}{2R^2} = 2+\frac{3r}{2R}+\frac{r^2}{R^2}.$$

50) In ΔABC

$$\sum\frac{r_a}{w_a}\frac{(b+c)^3}{b^3+c^3} \le 4\cdot\left(\left(\frac{R}{r}\right)^2-1\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{r_a}{w_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{r_a}{w_a}\cdot 4=4\sum\frac{r_a}{w_a}\overset{(2)}{\le}4\cdot\left(\left(\frac{R}{r}\right)^2-1\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4\Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum\frac{r_a}{w_a}\le\left(\frac{R}{r}\right)^2-1$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum\frac{r_a}{w_a}\le\left(\frac{R}{r}\right)^2-1.$$

Demonstrație.

Folosind $h_a\le w_a$ avem $\sum\frac{r_a}{w_a}\le\sum\frac{r_a}{h_a}\overset{(2)}{\le}\left(\frac{R}{r}\right)^2-1$, unde (2) rezultă din $\sum\frac{r_a}{h_a}=\frac{2R}{r}-1$ și

$\sum\frac{r_a}{w_a}\le\left(\frac{R}{r}\right)^2-1\Leftrightarrow\frac{2R}{r}-1\le\left(\frac{R}{r}\right)^2-1\Leftrightarrow R\ge 2r$, (Euler).

51) In ΔABC

$$\sum w_ar_a\frac{(b+c)^3}{b^3+c^3}\le 4p^2.$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum w_ar_a\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum w_ar_a\cdot 4=4\sum w_ar_a\overset{(2)}{\le}4\cdot p^2=4p^2=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4\Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum w_ar_a\le p^2$,(vezi **Lema**).

Lemă.

In ΔABC

$$w_ar_a\le\frac{pa}{2}.$$

Demonstrație.

Folosind $w_a\le\sqrt{p(p-a)}$ și $r_a=\frac{S}{p-a}$ obținem :

$$w_ar_a\le\sqrt{p(p-a)}\cdot\frac{S}{p-a}=\frac{\sqrt{p(p-a)}\cdot\sqrt{p(p-a)(p-b)(p-c)}}{p-a}=p\sqrt{(p-b)(p-c)}\overset{AM-GM}{\le}$$

$\overset{AM-GM}{\le}p\cdot\frac{(p-b)+(p-c)}{2}=\frac{pa}{2}$, cu egalitate pentru $b=c$.

$$\sum w_ar_a\le\sum\frac{pa}{2}=\frac{p}{2}\sum a=\frac{p}{2}\cdot 2p=p^2.$$

52) In ΔABC

$$\sum\frac{m_a}{w_a}\frac{(b+c)^3}{b^3+c^3}\le 4\left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right).$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum\frac{m_a}{w_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{m_a}{w_a}\cdot 4=4\sum\frac{m_a}{w_a}\overset{(2)}{\le}4\cdot\left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right)=4\left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4\Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum\frac{m_a}{w_a}\le\frac{R}{r}+\frac{r}{R}+\frac{1}{2}$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum\frac{m_a}{w_a}\le\frac{p^2+r^2-2Rr}{4Rr}.$$

Demonstrație.

Din ordinea liniilor importante în triunghi $h_a\le s_a\le w_a\le m_a$ și $s_a=\frac{2bc}{b^2+c^2}m_a$ folosim:

$s_a\le w_a\Leftrightarrow\frac{2bc}{b^2+c^2}m_a\le w_a\Leftrightarrow\frac{m_a}{w_a}\le\frac{b^2+c^2}{2bc}$, (inegalitatea lui G. Tsintsifas, 1975).

Obținem:

$$\sum\frac{m_a}{w_a}\le\sum\frac{b^2+c^2}{2bc}=\frac{p^2+r^2-2Rr}{4Rr}.$$

$$\sum\frac{m_a}{w_a}\le\frac{p^2+r^2-2Rr}{4Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2-2Rr}{4Rr}=\frac{4R^2+2Rr+4r^2}{4Rr}=$$

$$=\frac{2R^2+Rr+2r^2}{2Rr}=\frac{R}{r}+\frac{r}{R}+\frac{1}{2}.$$

53) In ΔABC

$$\sum\frac{a}{b}\frac{(b+c)^3}{b^3+c^3}\le 2\left(\frac{2R}{r}+\frac{r}{R}\right).$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum\frac{a}{b}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a}{b}\cdot 4=4\sum\frac{a}{b}\overset{(2)}{\le}4\cdot\left(\frac{R}{r}+\frac{r}{2R}\right)=2\left(\frac{2R}{r}+\frac{r}{R}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4\Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum\frac{a}{b}\le\frac{R}{r}+\frac{r}{2R}$,(Vezi **Lema**).

Lema

In ΔABC

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\frac{s^2-r^2-4Rr}{3Rr}.$$

Solution.

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\frac{(a+b+c)(a^2+b^2+c^2)}{3abc}\Leftrightarrow (a+b+c)(a^2+b^2+c^2)\ge 3(ab^2+bc^2+ca^2)\Leftrightarrow$$

$$\Leftrightarrow a^3+b^3+c^3+a^2b+b^2c+c^2a\ge 2(ab^2+bc^2+ca^2)\Leftrightarrow\Leftrightarrow$$

$$(a^3+ac^2-2a^2c)+(b^3+a^2b-2ab^2)+(c^3+b^2c-2bc^2)\ge 0\Leftrightarrow$$

$\Leftrightarrow a(a-c)^2+b(b-a)^2+c(c-b)^2\geq 0$.

Equality occurs if and only if $a=b=c$.

By identities $a+b+c=2s$, $abc=4Rrs$, $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ the inequality

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{(a+b+c)(a^2+b^2+c^2)}{3abc}$ is rewrite $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{2s\cdot 2(s^2-r^2-4Rr)}{3\cdot 4Rrs}\Leftrightarrow$

$\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{s^2-r^2-4Rr}{3Rr}$.

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{s^2-r^2-4Rr}{3Rr}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-r^2-4Rr}{4Rr}=\frac{4R^2+2r^2}{4Rr}=\frac{2R^2+r^2}{2Rr}=\frac{R}{r}+\frac{r}{2R}$.

54) In ΔABC

$$\sum\frac{a}{b}\frac{(b+c)^3}{b^3+c^3}\leq 4\left(1+\frac{R}{r}\right).$$

Marin Chirciu

Soluție.

$Ms=\sum\frac{a}{b}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\leq}\sum\frac{a}{b}\cdot 4=4\sum\frac{a}{b}\overset{(2)}{\leq}4\cdot\left(1+\frac{R}{r}\right)=Md$,

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\leq 4\Leftrightarrow (b+c)(b-c)^2\geq 0$, iar (2) $\Leftrightarrow \sum\frac{a}{b}\leq 1+\frac{R}{r}$,(Shan-He Wu).

In ΔABC

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 1+\frac{R}{r}.$$

Shan-He Wu Inequality

Demonstrație.

Se folosește inegalitatea Mateescu $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{p^2}{r(4R+r)}$.

Obținem $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{p^2}{r(4R+r)}\overset{(1)}{\leq}1+\frac{R}{r}$,

unde (1) $\Leftrightarrow \frac{p^2}{r(4R+r)}\leq 1+\frac{R}{r}\Leftrightarrow p^2\leq 4R^2+5Rr+r^2$, care rezultă din inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că $4R^2+4Rr+3r^2\leq 4R^2+5Rr+r^2\Leftrightarrow R\geq 2r$, (inegalitatea lui Euler).

55) In ΔABC

$$\sum\frac{m_a}{r_a}\frac{(b+c)^3}{b^3+c^3}\leq 4\left(\frac{R}{r}+\frac{r}{R}-1\right).$$

Marin Chirciu

Soluție.

Avem

$Ms=\sum\frac{m_a}{r_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\leq}\sum\frac{m_a}{r_a}\cdot 4=4\sum\frac{m_a}{r_a}\overset{(2)}{\leq}4\cdot\left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right)=4\left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right)=Md$,

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{m_a}{w_a} \le \frac{R}{r}+\frac{r}{R}+\frac{1}{2}$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum \frac{m_a}{r_a} \le \frac{p^2+r^2-8Rr}{4r^2}.$$

Soluție.

Folosind inegalitatea lui Panaitopol $\frac{m_a}{h_a} \le \frac{R}{2r}$, (vezi GM 11/1982), obținem:

$\frac{m_a}{r_a} \le \frac{h_a \cdot \frac{R}{2r}}{r_a} = \frac{R}{2r} \cdot \frac{h_a}{r_a}$, de unde

$$\sum \frac{m_a}{r_a} \le \sum \frac{R}{2r} \cdot \frac{h_a}{r_a} = \frac{R}{2r} \sum \frac{h_a}{r_a} = \frac{R}{2r} \cdot \frac{p^2+r^2-8Rr}{2Rr} = \frac{p^2+r^2-8Rr}{4r^2},$$

care rezultă din identitatea cunoscută în triunghi $\sum \frac{h_a}{r_a} = \frac{p^2+r^2-8Rr}{2Rr}$.

$$\sum \frac{m_a}{r_a} \le \frac{p^2+r^2-8Rr}{4r^2} \overset{Gerretsen}{\le} \frac{4R^2+4Rr+3r^2+r^2-8Rr}{4Rr} = \frac{4R^2-4Rr+4r^2}{4Rr} =$$

$$= \frac{R^2-Rr+r^2}{Rr} = \frac{R}{r}+\frac{r}{R}-1.$$

56) In ΔABC

$$\sum \frac{m_a}{r_a} \frac{(b+c)^3}{b^3+c^3} \le 4\left(\frac{R}{r}+\frac{r}{R}-1\right).$$

Marin Chirciu

Soluție.

Avem

$$Ms = \sum \frac{m_a}{r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{m_a}{r_a} \cdot 4 = 4\sum \frac{m_a}{r_a} \overset{(2)}{\le} 4 \cdot \left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right) = 4\left(\frac{R}{r}+\frac{r}{R}+\frac{1}{2}\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{m_a}{w_a} \le \frac{R}{r}+\frac{r}{R}+\frac{1}{2}$,(vezi **Lema**).

Lemă.

1) In ΔABC

$$\frac{m_a}{m_b}+\frac{m_b}{m_a} \le \frac{R}{r}.$$

Adil Abdullayev, Baku, Azerbaijan

Soluție.

Demonstrăm rezultatele ajutătoare:

Lema 1.

2) In ΔABC

$$\frac{m_a}{m_b}+\frac{m_b}{m_a} \le 2 \cdot \frac{m_c}{h_c}.$$

Soluție.

Folosind inegalitatea lui Tereshin $m_a \geq \frac{b^2+c^2}{4R}$ obținem succesiv:

$$m_c \geq \frac{a^2+b^2}{4R} \Leftrightarrow 4R \cdot m_c \geq a^2+b^2 \Leftrightarrow \frac{4RS}{S} \cdot m_c \geq a^2+b^2 \Leftrightarrow \frac{abc}{S} \cdot m_c \geq a^2+b^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{c}{S} \cdot m_c \geq \frac{a^2+b^2}{ab} \Leftrightarrow \frac{c \cdot m_c}{S} \geq \frac{a}{b}+\frac{b}{a}, (1).$$

Folosind principiul dualității, aplicăm inegalitatea (1) în triunghiul cu laturile de lungimi $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ și implicit arie $\frac{S}{3}$ și mediane $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ obținem:

$$\frac{\frac{2m_c}{3} \cdot \frac{c}{2}}{\frac{S}{3}} \geq \frac{\frac{2m_a}{3}}{\frac{2m_b}{3}} + \frac{\frac{2m_b}{3}}{\frac{2m_a}{3}} \Leftrightarrow \frac{c \cdot m_c}{S} \geq \frac{m_a}{m_b}+\frac{m_b}{m_a} \Leftrightarrow 2 \cdot \frac{m_c}{h_c} \geq \frac{m_a}{m_b}+\frac{m_b}{m_a}.$$

Lema 2.

3) In ΔABC

$$\frac{m_a}{h_a} \leq \frac{R}{2r}.$$

GM 11/1982, Laurențiu Panaitopol

Demonstrație.

Folosim rezultatul cunoscut în triunghi:

În orice triunghi are loc inegalitatea normală:

$$Max(m_a \sin A, m_b \sin B, m_c \sin C) \leq \frac{p}{2}.$$

Într-adevăr:

Fie A' mijlocul laturii (BC), iar M, N proiecțiile lui A' pe AB, AC respectiv.

Fie apoi P, Q simetricele lui A' față de AB, AC respectiv.

Deoarece $m(AMA') = m(ANA') = 90^\circ$ rezultă că penctele A, M, A', N se află pe cercul de diametru $[AA']$.

Obținem:

$MN = \frac{1}{2}PQ = AA' \sin A = m_a \sin A$, de unde $PQ = 2m_a \sin A$.

Dacă B', C' sunt respectiv mijloacele laturilor $[AC], [AB]$ lungimea liniei poligonale $PC'B'Q$ este p și ea majorează lungimea segmentului $[PQ]$.

De aici rezultă că $2m_a \sin A \leq p$ și deci $Max(m_a \sin A, m_b \sin B, m_c \sin C) \leq \frac{p}{2}$.

Din $2m_a \sin A \leq p$ rezultă :

$$m_a \sin A \leq \frac{p}{2} \Leftrightarrow m_a \cdot \frac{a}{2R} \leq \frac{p}{2} \Leftrightarrow m_a \cdot a \leq Rp \Leftrightarrow \frac{m_a}{h_a} \leq \frac{R}{2r}.$$

Să trecem la rezolvarea problemei din enunț.

Folosind **Lemele** obținem:

$$M_s = \frac{m_a}{m_b}+\frac{m_b}{m_a} \overset{Lema1}{\leq} 2 \cdot \frac{m_c}{h_c} \overset{Lema2}{\leq} 2 \cdot \frac{R}{2r} = M_d,$$

Folosind Lemele de mai sus rezultă:

$$\frac{m_a}{m_b}+\frac{m_b}{m_a}\overset{Lema1}{\leq}2\cdot\frac{m_c}{h_c}\overset{Lema2}{\leq}2\cdot\frac{R}{2r}=\frac{R}{r}.$$

57) In ΔABC

$$\sum\frac{r_a}{a}\frac{(b+c)^3}{b^3+c^3}\leq\frac{27R}{p}.$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum\frac{r_a}{a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\leq}\sum\frac{r_a}{a}\cdot 4=4\sum\frac{r_a}{a}\overset{(2)}{\leq}4\cdot\frac{27R}{4p}=\frac{27R}{p}=Md,$$

unde (1) $\Leftrightarrow\frac{(b+c)^3}{b^3+c^3}\leq 4\Leftrightarrow(b+c)(b-c)^2\geq 0$, iar (2) $\Leftrightarrow\sum\frac{r_a}{a}\leq\frac{27R}{4p}$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum\frac{r_a}{a}\leq\frac{27R}{4p}.$$

Demonstrație.

Avem $\sum\frac{r_a}{a}=\frac{p^2+(4R+r)^2}{4Rp}\overset{Gerretsen}{\leq}\frac{\frac{R(4R+r)^2}{2(2R-r)}+(4R+r)^2}{4Rp}=\frac{(4R+r)^2}{4Rp}\left(\frac{R}{2(2R-r)}+1\right)=$

$$=\frac{(4R+r)^2}{4Rp}\cdot\frac{5R-2r}{2(2R-r)}\overset{Euler}{\leq}\frac{(4R+r)^2}{4Rp}\cdot\frac{4}{3}=\frac{(4R+r)^2}{3Rp}\overset{Euler}{\leq}\frac{27R}{4p}$$

58) In ΔABC

$$\sum\frac{a}{r_a}\frac{(b+c)^3}{b^3+c^3}\leq\frac{36R}{p}.$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum\frac{a}{r_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\leq}\sum\frac{a}{r_a}\cdot 4=4\sum\frac{a}{r_a}\overset{(2)}{\leq}4\cdot\frac{9R}{p}=\frac{36R}{p}=Md,$$

unde (1) $\Leftrightarrow\frac{(b+c)^3}{b^3+c^3}\leq 4\Leftrightarrow(b+c)(b-c)^2\geq 0$, iar (2) $\Leftrightarrow\sum\frac{a}{r_a}\leq\frac{9R}{p}$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum\frac{a}{r_a}\leq\frac{9R}{p}.$$

Demonstrație.

Avem $\sum\frac{a}{r_a}=\frac{2(4R+r)}{p}\overset{Euler}{\leq}\frac{9R}{p}$.

59) In ΔABC

$$\sum \frac{a^2}{r_a}\frac{(b+c)^3}{b^3+c^3} \le 24R.$$

Marin Chirciu

Soluție.
Avem

$$Ms = \sum \frac{a^2}{r_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^2}{r_a}\cdot 4 = 4\sum \frac{a^2}{r_a} \overset{(2)}{\le} 4\cdot 6R = 24R = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{a^2}{r_a} \le 6R$,(vezi **Lema**).

Lemă.
In ΔABC

$$\sum \frac{a^2}{r_a} \le 6R.$$

Demonstrație.
Avem $\sum \frac{a^2}{r_a} = 4(R+r) \overset{Euler}{\le} 6R$.

60) In ΔABC

$$\sum \frac{1}{r_b r_c}\frac{(b+c)^3}{b^3+c^3} \le \frac{4}{3r^2}.$$

Marin Chirciu

Soluție.
Avem

$$Ms = \sum \frac{1}{r_b r_c}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{r_b r_c}\cdot 4 = 4\sum \frac{1}{r_b r_c} \overset{(2)}{\le} 4\cdot \frac{1}{3r^2} = \frac{4}{3r^2} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{1}{r_b r_c} \le \frac{1}{3r^2}$,(vezi **Lema**).

Lemă.
In ΔABC

$$\sum \frac{1}{r_b r_c} \le \frac{1}{3r^2}.$$

Demonstrație.
Avem $\sum \frac{1}{r_b r_c} = \frac{4R+r}{rp^2} \overset{Gerresten}{\le} \frac{1}{3r^2}$.

61) In ΔABC

$$\sum \frac{a}{r_a}\frac{(b+c)^4}{b^3+c^3} \le 48R.$$

Marin Chirciu

Soluție.
Avem

$$Ms=\sum\frac{a}{r_a}\frac{(b+c)^4}{b^3+c^3}=\sum\frac{a(b+c)}{r_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a(b+c)}{r_a}\cdot 4=4\sum\frac{a(b+c)}{r_a}\overset{(2)}{\le}4\cdot 12R=48R=Md$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum\frac{a(b+c)}{r_a}=12R$.

62) In ΔABC

$$\sum\frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3}\le\frac{4p}{r}.$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum\frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{r_b+r_c}{a}\cdot 4=4\sum\frac{r_b+r_c}{a}\overset{(2)}{\le}4\cdot\frac{p}{r}=\frac{4p}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum\frac{r_b+r_c}{a}=\frac{p}{r}$.

63) In ΔABC

$$\sum r_a\cos^2\frac{A}{2}\frac{(b+c)^3}{b^3+c^3}\le\frac{27R}{2}.$$

Marin Chirciu

Soluție.

Avem

$$Ms=\sum r_a\cos^2\frac{A}{2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum r_a\cos^2\frac{A}{2}\cdot 4=4\sum r_a\cos^2\frac{A}{2}\overset{(2)}{\le}4\cdot\frac{27R}{8}=\frac{27R}{2}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2) $\Leftrightarrow \sum r_a\cos^2\frac{A}{2}\le\frac{27R}{8}$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum r_a\cos^2\frac{A}{2}\le\frac{27R}{8}.$$

Demonstrație.

Avem $\sum r_a\cos^2\frac{A}{2}=\frac{p^2}{2R}\overset{Mitrinovic}{\le}\frac{\frac{27R^2}{4}}{2R}=\frac{27R}{8}$.

64) In ΔABC

$$\sum r_a\sin^2\frac{A}{2}\frac{(b+c)^3}{b^3+c^3}\le\frac{2(8R^2-14Rr+5r^2)}{R}.$$

Marin Chirciu

Soluție.

$$Ms=\sum r_a\sin^2\frac{A}{2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum r_a\sin^2\frac{A}{2}\cdot 4=4\sum r_a\sin^2\frac{A}{2}\overset{(2)}{\le}4\cdot\frac{8R^2-14Rr+5r^2}{2R}=$$

$=\frac{2(8R^2-14Rr+5r^2)}{R}=Md$, unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, iar (2)

$\Leftrightarrow \sum r_a \sin^2 \frac{A}{2} \le \frac{8R^2 - 14Rr + 5r^2}{2R}$,(vezi **Lema**).

Lemă.

In ΔABC

$$\sum r_a \sin^2 \frac{A}{2} \le \frac{8R^2 - 14Rr + 5r^2}{2R}.$$

Demonstrație.

Avem $\sum r_a \sin^2 \frac{A}{2} = \frac{8R^2 + 2Rr - p^2}{2R} \overset{Gerretsen}{\le} \frac{8R^2 + 2Rr - 16Rr + 5r^2}{2R} = \frac{8R^2 - 14Rr + 5r^2}{2R}$.

65) In ΔABC

$$\sum r_a \cot^2 \frac{A}{2} \frac{(b+c)^3}{b^3 + c^3} \le \frac{27R^2}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum r_a \cot^2 \frac{A}{2} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum r_a \cot^2 \frac{A}{2} \cdot 4 = 4\sum r_a \cot^2 \frac{A}{2} \overset{(2)}{\le} 4 \cdot \frac{p^2}{r} \overset{Mitrinovic}{\le} \frac{4}{r} \cdot \frac{27R^2}{4} =$$

$$= \frac{27R^2}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum r_a \cot^2 \frac{A}{2} = \frac{p^2}{r}$.

66) In ΔABC

$$\sum \frac{\sin^2 \frac{A}{2}}{r_a} \frac{(b+c)^3}{b^3+c^3} \le \frac{2}{R}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{\sin^2 \frac{A}{2}}{r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{\sin^2 \frac{A}{2}}{r_a} \cdot 4 = 4\sum \frac{\sin^2 \frac{A}{2}}{r_a} \overset{(2)}{\le} 4 \cdot \frac{1}{2R} = \frac{2}{R} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{\sin^2 \frac{A}{2}}{r_a} = \frac{1}{2R}$.

67) In ΔABC

$$\sum \frac{\cos^2 \frac{A}{2}}{r_a} \frac{(b+c)^3}{b^3+c^3} \le \frac{2}{r}\left(2 - \frac{r}{R}\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{\cos^2 \frac{A}{2}}{r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{\cos^2 \frac{A}{2}}{r_a} \cdot 4 = 4\sum \frac{\cos^2 \frac{A}{2}}{r_a} \overset{(2)}{\le} 4 \cdot \frac{2R-r}{2Rr} = \frac{2}{r}\left(2 - \frac{r}{R}\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, iar (2) $\Leftrightarrow \sum \frac{\cos^2 \frac{A}{2}}{r_a} = \frac{2R-r}{2Rr}$.

68) In ΔABC

$$\sum \frac{\tan^2 \frac{A}{2}}{r_a} \frac{(b+c)^3}{b^3+c^3} \le \frac{2R}{3r^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{\tan^2 \frac{A}{2}}{r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{\tan^2 \frac{A}{2}}{r_a} \cdot 4 = 4\sum \frac{\tan^2 \frac{A}{2}}{r_a} \overset{(2)}{\le} 4 \cdot \frac{4R+r}{p^2} \overset{(3)}{\le} \frac{2R}{3r^2} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{\tan^2 \frac{A}{2}}{r_a} = \frac{4R+r}{p^2}$, iar (3) rezultă din inegalitatea lui Euler $r \le \frac{R}{2}$ și inegalitatea lui Mitrinovic $p^2 \ge 27r^2$.

69) In ΔABC

$$\sum \frac{a^3}{r_a} \frac{(b+c)^3}{b^3+c^3} \le 16Rp.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a^3}{r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^3}{r_a} \cdot 4 = 4\sum \frac{a^3}{r_a} \overset{(2)}{\le} 4 \cdot \frac{abc}{r} \overset{(3)}{\le} 4 \cdot \frac{4Rrp}{r} = 16Rp = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{a^3}{r_a} \overset{Andersson}{\le} \frac{abc}{r}$, iar (3) rezultă din $abc = 4Rrp$.

70) In ΔABC

$$\sum \frac{r_b r_c}{bc} \frac{(b+c)^3}{b^3+c^3} \le 9.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{r_b r_c}{bc} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_b r_c}{bc} \cdot 4 = 4\sum \frac{r_b r_c}{bc} \overset{(2)}{\le} 4 \cdot \frac{9}{4} = 9 = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{r_b r_c}{bc} \le \frac{9}{4}$.

71) In ΔABC

$$\sum \frac{a^2}{r_b r_c} \frac{(b+c)^3}{b^3+c^3} \le 16\left(\frac{R}{r} - 1\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a^2}{r_b r_c} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^2}{r_b r_c} \cdot 4 = 4\sum \frac{a^2}{r_b r_c} \overset{(2)}{=} 4 \cdot 4\left(\frac{R}{r} - 1\right) = 16\left(\frac{R}{r} - 1\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{a^2}{r_b r_c} = 4\left(\frac{R}{r}-1\right)$.

72) In ΔABC

$$\sum \frac{r_a}{2r_a^2+r_b r_c}\frac{(b+c)^3}{b^3+c^3} \le \frac{4}{3r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{r_a}{2r_a^2+r_b r_c}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a}{2r_a^2+r_b r_c}\cdot 4 = 4\sum \frac{r_a}{2r_a^2+r_b r_c} \overset{(2)}{\le} 4\cdot\frac{1}{3r} = \frac{4}{3r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{r_a}{2r_a^2+r_b r_c} \le \frac{1}{3r}$.

73) In ΔABC

$$\sum \frac{1}{2r_a^2+r_b r_c}\frac{(b+c)^3}{b^3+c^3} \le \frac{4}{9r^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{2r_a^2+r_b r_c}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{2r_a^2+r_b r_c}\cdot 4 = 4\sum \frac{1}{2r_a^2+r_b r_c} \overset{(2)}{\le} 4\cdot\frac{1}{9r^2} = \frac{4}{9r^2} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{1}{2r_a^2+r_b r_c} \le \frac{1}{9r^2}$.

74) In ΔABC

$$\sum \frac{1}{r+r_a}\frac{(b+c)^3}{b^3+c^3} \le \frac{3}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{r+r_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{r+r_a}\cdot 4 = 4\sum \frac{1}{r+r_a} \overset{(2)}{\le} 4\cdot\frac{3}{4r} = \frac{3}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{1}{r+r_a} \le \frac{3}{4r}$.

75) In ΔABC

$$\sum \sqrt{rr_a}\frac{(b+c)^3}{b^3+c^3} \le 4p.$$

Marin Chirciu

Soluție.

$$Ms = \sum \sqrt{rr_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \sqrt{rr_a}\cdot 4 = 4\sum \sqrt{rr_a} \overset{(2)}{\le} 4\cdot p = 4p = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \sqrt{rr_a} \le p$.

76) In ΔABC

$$\sum \frac{p^2-r_a^2}{p^2+r_a^2}\frac{(b+c)^3}{b^3+c^3} \le 4\left(1+\frac{r}{R}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{p^2-r_a^2}{p^2+r_a^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{p^2-r_a^2}{p^2+r_a^2}\cdot 4 = 4\sum \frac{p^2-r_a^2}{p^2+r_a^2} \overset{(2)}{=} 4\cdot\left(1+\frac{r}{R}\right)=4\left(1+\frac{r}{R}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{p^2-r_a^2}{p^2+r_a^2}=1+\frac{r}{R}$,(GM 1940, Emilian Mereuță).

77) In ΔABC

$$\sum \frac{r_a}{p^2+r_a^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{1}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_a}{p^2+r_a^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a}{p^2+r_a^2}\cdot 4 = 4\sum \frac{r_a}{p^2+r_a^2} \overset{(2)}{=} 4\cdot\frac{1}{4r}=\frac{1}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{r_a}{p^2+r_a^2}\le\frac{1}{4r}$.

78) In ΔABC

$$\sum \sqrt{\frac{a}{r_a}}\frac{(b+c)^3}{b^3+c^3} \le 6\sqrt{\frac{p}{r}}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \sqrt{\frac{a}{r_a}}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \sqrt{\frac{a}{r_a}}\cdot 4 = 4\sum \sqrt{\frac{a}{r_a}} \overset{(2)}{\le} 4\cdot\frac{3}{2}\sqrt{\frac{p}{r}}=6\sqrt{\frac{p}{r}}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \sqrt{\frac{a}{r_a}}\le\frac{3}{2}\sqrt{\frac{p}{r}}$,(GM 1964, I.I.Cristea).

79) In ΔABC

$$\sum \frac{(r_b+r_c)^2}{r_br_b}\frac{(b+c)^3}{b^3+c^3} \le 16\left(1+\frac{R}{r}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{(r_b+r_c)^2}{r_br_b}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(r_b+r_c)^2}{r_br_b}\cdot 4 = 4\sum \frac{(r_b+r_c)^2}{r_br_b} \overset{(2)}{\le} 4\cdot 4\left(1+\frac{R}{r}\right)=16\left(1+\frac{R}{r}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{(r_b+r_c)^2}{r_br_b}=4\left(1+\frac{R}{r}\right)$.

80) In ΔABC

$$\sum\sqrt{a\cdot\frac{p}{2}}\frac{(b+c)^3}{b^3+c^3}\le 4(4R+r).$$

Marin Chirciu

Soluție.

$$Ms=\sum\sqrt{a\cdot\frac{p}{2}}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\sqrt{a\cdot\frac{p}{2}}\cdot 4=4\sum\sqrt{a\cdot\frac{p}{2}}\overset{(2)}{\le}4\cdot(4R+r)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\sqrt{a\cdot\frac{p}{2}}=4R+r$, (Crux Math.17/1991, George Tsitsifas, Greece).

81) In ΔABC

$$\sum\sqrt{(b+c)p}\frac{(b+c)^3}{b^3+c^3}\le 8(4R+r).$$

Marin Chirciu

Soluție.

$$Ms=\sum\sqrt{(b+c)p}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\sqrt{(b+c)p}\cdot 4=4\sum\sqrt{(b+c)p}\overset{(2)}{\le}4\cdot 2(4R+r)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\sqrt{(b+c)p}=2(4R+r)$.

82) In ΔABC

$$\sum\frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3}\le\frac{4p}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{r_b+r_c}{a}\cdot 4=4\sum\frac{r_b+r_c}{a}\overset{(2)}{\le}4\cdot\frac{p}{r}=Md,$$

83) In ΔABC

$$\sum\frac{1}{r_b+r_c}\frac{(b+c)^3}{b^3+c^3}\le\frac{2}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{r_b+r_c}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{1}{r_b+r_c}\cdot 4=4\sum\frac{1}{r_b+r_c}\overset{(2)}{\le}4\cdot\frac{1}{2r}=\frac{2}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow$

$$\sum\frac{1}{r_b+r_c}=\frac{1}{4R}\left[1+\left(\frac{4R+r}{p}\right)^2\right]=\frac{1}{4R}\left[1+\frac{(4R+r)^2}{p^2}\right]\overset{Gerretsen}{\le}\frac{1}{4R}\left[1+\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{4R}\left[1+\frac{R+r}{r}\right]=\frac{1}{4R}\left(2+\frac{R}{r}\right)\overset{Euler}{\le}\frac{1}{2r}.$$

84) In ΔABC

$$\sum \frac{r_a}{r_b+r_c}\frac{(b+c)^3}{b^3+c^3} \le 4\left(\frac{R}{r}-\frac{1}{2}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_a}{r_b+r_c}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a}{r_b+r_c}\cdot 4 = 4\sum \frac{r_a}{r_b+r_c} \overset{(2)}{\le} 4\cdot\left(\frac{R}{r}-\frac{1}{2}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\sum \frac{r_a}{r_b+r_c}\le \frac{R}{r}-\frac{1}{2}$, (GM 7-8/1997, Marian Ursărescu, Roman), Vezi **Lema**.

Lema

85) In ΔABC

$$\sum \frac{r_a}{r_b+r_c}\le \frac{R}{r}-\frac{1}{2}.$$

GM 7-8/1997, Marian Ursărescu, Roman

Demonstrație.

$$\Leftrightarrow \sum \frac{r_a}{r_b+r_c}=\frac{1}{4R}\left[r-8R+(4R+r)\frac{(4R+r)^2}{p^2}\right] \overset{Gerretsen}{\le} \frac{1}{4R}\left[r-8R+(4R+r)\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{4R}\left[r-8R+(4R+r)\frac{R+r}{r}\right]=\frac{1}{4R}\frac{4R^2-3Rr+2r^2}{r} \overset{Euler}{\le} \frac{R}{r}-\frac{1}{2}.$$

86) In ΔABC

$$\sum \frac{r_b+r_c}{r_a}\frac{(b+c)^3}{b^3+c^3}\le 8\left(\frac{2R}{r}-1\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_b+r_c}{r_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_b+r_c}{r_a}\cdot 4 = 4\sum \frac{r_b+r_c}{r_a} \overset{(2)}{\le} 4\cdot\frac{2(2R-r)}{r}=8\left(\frac{2R}{r}-1\right)Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\sum \frac{r_b+r_c}{r_a}=\frac{2(2R-r)}{r}$.

87) In ΔABC

$$\sum \frac{2R-r_a}{bc}\frac{(b+c)^3}{b^3+c^3}\le \frac{2}{R}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{2R-r_a}{bc}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{2R-r_a}{bc}\cdot 4 = 4\sum \frac{2R-r_a}{bc} \overset{(2)}{\le} 4\cdot\frac{1}{2R}=\frac{2}{R}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\sum \frac{2R-r_a}{bc} = \frac{1}{2R}$.

88) In ΔABC

$$\sum \frac{1}{r+r_a} \frac{(b+c)^3}{b^3+c^3} \le \frac{3}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{r+r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{r+r_a} \cdot 4 = 4\sum \frac{1}{r+r_a} \overset{(2)}{\le} 4 \cdot \frac{3}{4r} = \frac{3}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\sum \frac{1}{r+r_a} \le \frac{3}{4r}$, vezi **Lema**.

Lema

89) In ΔABC

$$\sum \frac{1}{r+r_a} \le \frac{3}{4r}.$$

Demonstrație.

$$\Leftrightarrow \sum \frac{1}{r+r_a} = \frac{p^2+5r^2+8Rr}{2r\left(p^2+r^2+2Rr\right)} \overset{Gerretsen}{\le} \frac{3}{4r}.$$

90) In ΔABC

$$\sum \frac{r_a^2}{r+r_a} \frac{(b+c)^3}{b^3+c^3} \le \frac{p^2}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{r_a^2}{r+r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a^2}{r+r_a} \cdot 4 = 4\sum \frac{r_a^2}{r+r_a} \overset{(2)}{\le} 4 \cdot \frac{p^2}{4r} = \frac{p^2}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\sum \frac{r_a^2}{r+r_a} \le \frac{p^2}{4r}$, (Kazahstan, 2012).

91) In ΔABC

$$\sum ar_a \frac{(b+c)^3}{b^3+c^3} \le 8p(2R-r).$$

Marin Chirciu

Soluție.

$$Ms = \sum ar_a \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum ar_a \cdot 4 = 4\sum ar_a \overset{(2)}{\le} 4 \cdot 2p(2R-r) = 8p(2R-r) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\sum ar_a = 2p(2R-r)$.

92) In ΔABC

$$\sum \frac{a^2}{r_a}\frac{(b+c)^3}{b^3+c^3} \le 16(R+r).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{a^2}{r_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^2}{r_a}\cdot 4 = 4\sum \frac{a^2}{r_a} \overset{(2)}{\le} 4\cdot 4(R+r)=16(R+r)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\sum \frac{a^2}{r_a}=4(R+r)$.

93) In ΔABC

$$\sum \frac{r_a}{bc}\frac{(b+c)^3}{b^3+c^3} \le 2\left(\frac{2}{r}-\frac{1}{R}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_a}{bc}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a}{bc}\cdot 4 = 4\sum \frac{a^2}{r_a} \overset{(2)}{\le} 4\cdot \frac{2R-r}{2Rr}=\frac{2(2R-r)}{Rr}=2\left(\frac{2}{r}-\frac{1}{R}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\sum \frac{r_a}{bc}=\frac{2R-r}{2Rr}$.

94) In ΔABC

$$\sum \frac{bc}{r_a}\frac{(b+c)^3}{b^3+c^3} \le \frac{16}{r}\left(R^2-Rr+r^2\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{bc}{r_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{bc}{r_a}\cdot 4 = 4\sum \frac{bc}{r_a} \overset{(2)}{\le} 4\cdot \frac{4}{r}\left(R^2-Rr+r^2\right)=\frac{16}{r}\left(R^2-Rr+r^2\right)==Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2)

$$\sum \frac{bc}{r_a}=\frac{p^2+r^2-8Rr}{r} \overset{Gerretsen}{\le} \frac{4R^2+4Rr+3r^2+r^2-8Rr}{r}=\frac{4\left(R^2-Rr+r^2\right)}{r}.$$

95) In ΔABC

$$\sum \frac{bc}{r_b r_c}\frac{(b+c)^3}{b^3+c^3} \le 4\left(2+\frac{R}{r}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{bc}{r_b r_c}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{bc}{r_b r_c}\cdot 4 = 4\sum \frac{bc}{r_b r_c} \overset{(2)}{\le} 4\cdot \left(2+\frac{R}{r}\right)=4\left(2+\frac{R}{r}\right)==Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow$

$$\sum \frac{bc}{r_b r_c} = 1 + \frac{(4R+r)^2}{p^2} \overset{Gerretsen}{\leq} 1 + \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}} = 1 + \frac{R+r}{r} = 2 + \frac{R}{r}.$$

96) In ΔABC

$$\sum \frac{1}{r_b r_c} \frac{(b+c)^3}{b^3+c^3} \leq \frac{4}{p^2}\left(\frac{4R}{r}+1\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{r_b r_c} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\leq} \sum \frac{1}{r_b r_c} \cdot 4 = 4\sum \frac{1}{r_b r_c} \overset{(2)}{\leq} 4 \cdot \left(\frac{4R+r}{rp^2}\right) = \frac{4}{p^2}\left(\frac{4R}{r}+1\right) Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, (2) $\Leftrightarrow \sum \frac{1}{r_b r_c} = \frac{4R+r}{rp^2}$.

97) In ΔABC

$$\sum \frac{a}{r_a^2} \frac{(b+c)^3}{b^3+c^3} \leq \frac{8}{p}\left(\frac{2R}{r}-1\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a}{r_a^2} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\leq} \sum \frac{a}{r_a^2} \cdot 4 = 4\sum \frac{a}{r_a^2} \overset{(2)}{\leq} 4 \cdot \frac{2(2R-r)}{pr} = \frac{8(2R-r)}{pr} = \frac{8}{p}\left(\frac{2R}{r}-1\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, (2) $\Leftrightarrow \sum \frac{a}{r_a^2} = \frac{2(2R-r)}{pr}$.

98) In ΔABC

$$\sum \frac{r_b r_c}{bc} \frac{(b+c)^3}{b^3+c^3} \leq 2\left(4+\frac{r}{R}\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{r_b r_c}{bc} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\leq} \sum \frac{r_b r_c}{bc} \cdot 4 = 4\sum \frac{r_b r_c}{bc} \overset{(2)}{\leq} 4 \cdot \frac{4R+r}{2R} = \frac{2(4R+r)}{R} = 2\left(4+\frac{r}{R}\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, (2) $\Leftrightarrow \sum \frac{r_b r_c}{bc} = \frac{4R+r}{2R}$.

99) In ΔABC

$$\sum \frac{a^2}{r_b r_c} \frac{(b+c)^3}{b^3+c^3} \leq 16\left(\frac{R}{r}-1\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a^2}{r_b r_c} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\leq} \sum \frac{a^2}{r_b r_c} \cdot 4 = 4\sum \frac{a^2}{r_b r_c} \overset{(2)}{\leq} 4 \cdot \frac{4(R-r)}{r} = \frac{16(R-r)}{r} = 16\left(\frac{R}{r}-1\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, (2) $\Leftrightarrow \sum \frac{a^2}{r_b r_c} = \frac{4(R-r)}{r}$.

100) In ΔABC

$$\sum \frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3} \le \frac{4p}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_b+r_c}{a}\cdot 4 = 4\sum \frac{r_b+r_c}{a} \overset{(2)}{=} 4\cdot\frac{p}{r}=\frac{4p}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{r_b+r_c}{a}=\frac{p}{r}$.

101) In ΔABC

$$\sum \frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3} \le \frac{4p}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_b+r_c}{a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_b+r_c}{a}\cdot 4 = 4\sum \frac{r_b+r_c}{a} \overset{(2)}{=} 4\cdot\frac{p}{r}=\frac{4p}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{r_b+r_c}{a}=\frac{p}{r}$.

102) In ΔABC

$$\sum \frac{r_a}{\sqrt{p^2+r_a^2}}\frac{(b+c)^3}{b^3+c^3} \le 6.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_a}{\sqrt{p^2+r_a^2}}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a}{\sqrt{p^2+r_a^2}}\cdot 4 = 4\sum \frac{r_a}{\sqrt{p^2+r_a^2}} \overset{(2)}{=} 4\cdot\frac{3}{2}=6=Md,$$

103) In ΔABC

$$\sum \frac{r_a(r+r_a)}{r^2+r_a^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{72}{5}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{r_a(r+r_a)}{r^2+r_a^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{r_a(r+r_a)}{r^2+r_a^2}\cdot 4 = 4\sum \frac{r_a(r+r_a)}{r^2+r_a^2} \overset{(2)}{\le} 4\cdot\frac{18}{5}=\frac{72}{5}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{r_a(r+r_a)}{r^2+r_a^2}\le\frac{18}{5}$.

104) In ΔABC

$$\sum \frac{m_a^2}{bc}\frac{(b+c)^3}{b^3+c^3} \le \frac{9R}{2r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{m_a^2}{bc}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{m_a^2}{bc}\cdot 4=4\sum\frac{m_a^2}{bc}\overset{(2)}{\le}4\cdot\frac{9R}{8r}=\frac{9R}{2r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{m_a^2}{bc}\le\frac{9R}{8r}$, (Bager Inequality).

105) In ΔABC

$$\sum\frac{1}{m_a^2}\frac{(b+c)^3}{b^3+c^3}\le\frac{4}{3r^2}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{m_a^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{1}{m_a^2}\cdot 4=4\sum\frac{1}{m_a^2}\overset{(2)}{\le}4\cdot\frac{1}{3r^2}=\frac{4}{3r^2}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{1}{m_a^2}\le\frac{1}{3r^2}$.

106) In ΔABC

$$\sum\frac{m_a}{m_bm_c}\frac{(b+c)^3}{b^3+c^3}\le\frac{4}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{m_a}{m_bm_c}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{m_a}{m_bm_c}\cdot 4=4\sum\frac{m_a}{m_bm_c}\overset{(2)}{\le}4\cdot\frac{1}{r}=\frac{4}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{m_a}{m_bm_c}\le\frac{1}{r}$.

107) In ΔABC

$$\sum\frac{b^2+c^2}{m_a}\frac{(b+c)^3}{b^3+c^3}\le 48R.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{b^2+c^2}{m_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{b^2+c^2}{m_a}\cdot 4=4\sum\frac{b^2+c^2}{m_a}\overset{(2)}{\le}4\cdot 12R=48R=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{b^2+c^2}{m_a}\le 12R$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

108) In ΔABC

$$\sum\frac{m_a^2}{b^2+c^2}\frac{(b+c)^3}{b^3+c^3}\le 4+\frac{r}{R}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{m_a^2}{b^2+c^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{m_a^2}{b^2+c^2}\cdot 4=4\sum\frac{m_a^2}{b^2+c^2}\overset{(2)}{\le}4\cdot\frac{4R+r}{4R}=4+\frac{r}{R}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{m_a^2}{b^2+c^2} \le \frac{4R+r}{4R}$, (Crux Math.3/2005, Mihaly Bencze)

109) In ΔABC

$$\sum \frac{1}{m_b^2 m_c^2} \frac{(b+c)^3}{b^3+c^3} \le \frac{4}{S^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{1}{m_b^2 m_c^2} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{1}{m_b^2 m_c^2} \cdot 4 = 4\sum \frac{1}{m_b^2 m_c^2} \overset{(2)}{\le} 4 \cdot \frac{1}{S^2} = \frac{4}{S^2} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{1}{m_b^2 m_c^2} \le \frac{1}{S^2}$, (Argument14/2012, Nicolae Mușuroia, Baia Mare).

110) In ΔABC

$$\sum \frac{a^2}{m_b^2 m_c^2} \frac{(b+c)^3}{b^3+c^3} \le \frac{16}{p^2}\left(1+\frac{R}{r}\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a^2}{m_b^2 m_c^2} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^2}{m_b^2 m_c^2} \cdot 4 = 4\sum \frac{a^2}{m_b^2 m_c^2} \overset{(2)}{\le} 4 \cdot \frac{4(R+r)}{rp^2} = \frac{16}{p^2}\left(1+\frac{R}{r}\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{a^2}{m_b^2 m_c^2} \le \frac{4(R+r)}{rp^2}$.

111) In ΔABC

$$\sum \frac{m_a}{a} \frac{(b+c)^3}{b^3+c^3} \le \frac{2p}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{m_a}{a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{m_a}{a} \cdot 4 = 4\sum \frac{m_a}{a} \overset{(2)}{\le} 4 \cdot \frac{p}{2r} = \frac{2p}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{m_a}{a} \le \frac{p}{2r}$, (GM 4/2004, Gh.Szollosy,Sighetu Marmației).

112) In ΔABC

$$\sum \frac{m_b m_c}{bc} \frac{(b+c)^3}{b^3+c^3} \le 1+\frac{4R}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{m_b m_c}{bc} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{m_b m_c}{bc} \cdot 4 = 4\sum \frac{m_b m_c}{bc} \overset{(2)}{\le} 4 \cdot \left(\frac{1}{4}+\frac{R}{r}\right) = 1+\frac{4R}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{m_b m_c}{bc} \le \frac{1}{4}+\frac{R}{r}$.

113) In ΔABC

$$\sum \frac{AI}{w_a}\frac{(b+c)^3}{b^3+c^3} \le 8.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{AI}{w_a}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{AI}{w_a}\cdot 4 = 4\sum \frac{AI}{w_a} \overset{(2)}{\le} 4\cdot 2 = 8 = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{AI}{w_a} = 2$.

114) In ΔABC

$$\sum AH\frac{(b+c)^3}{b^3+c^3} \le 8(R+r).$$

Marin Chirciu

Soluție.

$$Ms = \sum AH\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum AH\cdot 4 = 4\sum AH \overset{(2)}{\le} 4\cdot 2(R+r) = 8(R+r) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum AH = 2(R+r)$.

115) In ΔABC

$$\sum AI^2 \sin A\frac{(b+c)^3}{b^3+c^3} \le 8S.$$

Marin Chirciu

Soluție.

$$Ms = \sum AI^2 \sin A\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum AI^2 \sin A\cdot 4 = 4\sum AI^2 \sin A \overset{(2)}{\le} 4\cdot 2S = 8S = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum AI^2 \sin A = 2S$.

116) In ΔABC

$$\sum aAI^2\frac{(b+c)^3}{b^3+c^3} \le 16RS.$$

Marin Chirciu

Soluție.

$$Ms = \sum aAI^2\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum aAI^2\cdot 4 = 4\sum aAI^2 \overset{(2)}{\le} 4\cdot 4RS = 16RS = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum aAI^2 = 4RS$.

117) In ΔABC

$$\sum \frac{h_a^2}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{p^2}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{h_a^2}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{h_a^2}{AG^2}\cdot 4=4\sum\frac{h_a^2}{AG^2}\overset{(2)}{\le}4\cdot\frac{p^2}{4r^2}=\frac{p^2}{r^2}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{h_a^2}{AG^2}\le\frac{p^2}{4r^2}$.

118) In ΔABC

$$\sum\frac{w_a}{AG}\frac{(b+c)^3}{b^3+c^3}\le\frac{9R}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{w_a}{AG}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{w_a}{AG}\cdot 4=4\sum\frac{w_a}{AG}\overset{(2)}{\le}4\cdot\frac{9R}{4r}=\frac{9R}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{w_a}{AG}\le\frac{9R}{4r}$.

119) In ΔABC

$$\sum\frac{AG}{w_a}\frac{(b+c)^3}{b^3+c^3}\le\frac{4R}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{AG}{w_a}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{AG}{w_a}\cdot 4=4\sum\frac{AG}{w_a}\overset{(2)}{\le}4\cdot\frac{R}{r}=\frac{4R}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{AG}{w_a}\le\frac{R}{r}$.

120) In ΔABC

$$\sum\frac{a^2}{AG^2}\frac{(b+c)^3}{b^3+c^3}\le 36\left(\frac{R}{r}-1\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a^2}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a^2}{AG^2}\cdot 4=4\sum\frac{a^2}{AG^2}\overset{(2)}{\le}4\cdot 9\left(\frac{R}{r}-1\right)=36\left(\frac{R}{r}-1\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{a^2}{AG^2}\le 9\left(\frac{R}{r}-1\right)$,(RMM 2018, Mehmet Șahin, Turkey).

121) In ΔABC

$$\sum\frac{bc}{AG^2}\frac{(b+c)^3}{b^3+c^3}\le 9\left(2+\frac{R}{r}\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{1}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{1}{AG^2}\cdot 4=4\sum\frac{1}{AG^2}\overset{(2)}{\le}4\cdot\frac{9}{4}\left(2+\frac{R}{r}\right)=9\left(2+\frac{R}{r}\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{bc}{AG^2} \le \frac{9}{4}\left(2+\frac{R}{r}\right)$.

122) In ΔABC

$$\sum \frac{1}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{3}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{bc}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{bc}{AG^2}\cdot 4 = 4\sum \frac{bc}{AG^2} \overset{(2)}{\le} 4\cdot\frac{3}{4r^2} = \frac{3}{r^2} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{1}{AG^2} \le \frac{3}{4r^2}$.

123) In ΔABC

$$\sum \frac{bc\cdot h_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le 54R.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{bc\cdot h_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{bc\cdot h_a}{AG^2}\cdot 4 = 4\sum \frac{bc\cdot h_a}{AG^2} \overset{(2)}{\le} 4\cdot\frac{27R}{2} = 54R = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{bc\cdot h_a}{AG^2} \le \frac{27R}{2}$.

124) In ΔABC

$$\sum \frac{(b+c)\cdot h_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{162R}{p}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{(b+c)\cdot h_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(b+c)\cdot h_a}{AG^2}\cdot 4 = 4\sum \frac{(b+c)\cdot h_a}{AG^2} \overset{(2)}{\le} 4\cdot\frac{81R}{2p} = \frac{162R}{p} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{(b+c)\cdot h_a}{AG^2} \le \frac{81R}{2p}$.

125) In ΔABC

$$\sum \frac{(b+c)\cdot h_a r_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{18Rp}{r}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{(b+c)\cdot h_a r_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(b+c)\cdot h_a r_a}{AG^2}\cdot 4 = 4\sum \frac{(b+c)\cdot h_a r_a}{AG^2} \overset{(2)}{\le} 4\cdot\frac{9Rp}{2r} = \frac{18Rp}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{(b+c)\cdot h_a r_a}{AG^2} \le \frac{9Rp}{2r}$.

126) In ΔABC

$$\sum \frac{(b+c)\cdot r_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{36}{r}\left(R^2-r^2\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{(b+c)\cdot r_a}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(b+c)\cdot r_a}{AG^2}\cdot 4 = 4\sum \frac{(b+c)\cdot r_a}{AG^2} \overset{(2)}{\le} 4\cdot\frac{9}{r}\left(R^2-r^2\right)=$$

$$=\frac{36}{r}\left(R^2-r^2\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{(b+c)\cdot r_a}{AG^2}\le \frac{9}{r}\left(R^2-r^2\right)$.

127) In ΔABC

$$\sum \frac{a}{AG^2}\frac{(b+c)^5}{b^3+c^3} \le \frac{324R^2}{S}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{a(b+c)^2}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a(b+c)^2}{AG^2}\cdot 4 = 4\sum \frac{a(b+c)^2}{AG^2} \overset{(2)}{\le} 4\cdot\frac{81R^2}{S}=\frac{324R^2}{S}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{a(b+c)^2}{AG^2}\le \frac{81R^2}{S}$.

128) In ΔABC

$$\sum \frac{a}{AG^2}\frac{(b+c)^4}{b^3+c^3} \le \frac{36R}{r}.$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{a(b+c)}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a(b+c)}{AG^2}\cdot 4 = 4\sum \frac{a(b+c)}{AG^2} \overset{(2)}{\le} 4\cdot\frac{9R}{r}=\frac{36R}{r}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{(b+c)\cdot r_a}{AG^2}\le \frac{9R}{r}$.

129) In ΔABC

$$\sum \frac{a^3}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{216}{S}\left(R^3-5r^3\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum \frac{a^3}{AG^2}\frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^3}{AG^2}\cdot 4 = 4\sum \frac{a^3}{AG^2} \overset{(2)}{\le} 4\cdot\frac{54}{S}\left(R^3-5r^3\right)=\frac{216}{S}\left(R^3-5r^3\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum \frac{a^3}{AG^2}\le \frac{54}{S}\left(R^3-5r^3\right)$.

130) In ΔABC

$$\sum \frac{a^4}{AG^2}\frac{(b+c)^3}{b^3+c^3} \le \frac{144}{r}\left(R^3-5r^3\right).$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{a^4}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{a^4}{AG^2}\cdot 4=4\sum\frac{a^4}{AG^2}\overset{(2)}{\le}4\cdot\frac{36}{r}\left(R^3-5r^3\right)=\frac{144}{r}\left(R^3-5r^3\right)=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{a^4}{AG^2}\le\frac{36}{r}\left(R^3-5r^3\right)$.

131) In ΔABC

$$\sum\frac{AI^2}{AG^2}\frac{(b+c)^3}{b^3+c^3}\le 12.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{AI^2}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{AI^2}{AG^2}\cdot 4=4\sum\frac{AI^2}{AG^2}\overset{(2)}{\le}4\cdot 3=12=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{AI^2}{AG^2}\le 3$.

132) In ΔABC

$$\sum\frac{\cos^2\frac{A}{2}}{AG^2}\frac{(b+c)^3}{b^3+c^3}\le\frac{9}{2Rr}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{\cos^2\frac{A}{2}}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{\cos^2\frac{A}{2}}{AG^2}\cdot 4=4\sum\frac{\cos^2\frac{A}{2}}{AG^2}\overset{(2)}{\le}4\cdot\frac{9}{8Rr}=\frac{9}{2Rr}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{\cos^2\frac{A}{2}}{AG^2}\le\frac{9}{8Rr}$.

133) In ΔABC

$$\sum\frac{\cot^2\frac{A}{2}}{AG^2}\frac{(b+c)^3}{b^3+c^3}\le\frac{9}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{\cot^2\frac{A}{2}}{AG^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{\cot^2\frac{A}{2}}{AG^2}\cdot 4=4\sum\frac{\cot^2\frac{A}{2}}{AG^2}\overset{(2)}{\le}4\cdot\frac{9}{4r^2}=\frac{9}{r^2}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3}\le 4 \Leftrightarrow (b+c)(b-c)^2\ge 0$, (2) $\Leftrightarrow \sum\frac{\cot^2\frac{A}{2}}{AG^2}\le\frac{9}{4r^2}$.

134) In ΔABC

$$\sum\frac{(b-c)^2}{h_a^2}\frac{(b+c)^3}{b^3+c^3}\le\frac{8OI^2}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms=\sum\frac{(b-c)^2}{h_a^2}\frac{(b+c)^3}{b^3+c^3}\overset{(1)}{\le}\sum\frac{(b-c)^2}{h_a^2}\cdot 4=4\sum\frac{(b-c)^2}{h_a^2}\overset{(2)}{\le}4\cdot\frac{2OI^2}{r^2}=\frac{8OI^2}{r^2}=Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{(b-c)^2}{h_a^2} \le \frac{2OI^2}{r^2}$.

135) In ΔABC

$$\sum \frac{(b-c)^2}{h_a r_a} \frac{(b+c)^3}{b^3+c^3} \le \frac{8OI^2}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{(b-c)^2}{h_a r_a} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(b-c)^2}{h_a r_a} \cdot 4 = 4\sum \frac{(b-c)^2}{h_a r_a} \overset{(2)}{\le} 4 \cdot \frac{1}{2r}\left(\frac{1}{r}+\frac{1}{R}\right) OI^2 =$$

$$= \frac{2}{r}\left(\frac{1}{r}+\frac{1}{R}\right) OI^2 = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{(b-c)^2}{h_a r_a} \le \frac{1}{2r}\left(\frac{1}{r}+\frac{1}{R}\right) OI^2$.

136) In ΔABC

$$\sum \frac{(b-c)^2}{h_b+h_c} \frac{(b+c)^3}{b^3+c^3} \le \frac{8OI^2}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{(b-c)^2}{h_b+h_c} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(b-c)^2}{h_b+h_c} \cdot 4 = 4\sum \frac{(b-c)^2}{h_b+h_c} \overset{(2)}{\le} 4 \cdot \frac{2OI^2}{r} = \frac{8OI^2}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{(b-c)^2}{h_b+h_c} \le \frac{2OI^2}{r}$.

137) In ΔABC

$$\sum \frac{(b-c)^2}{r_b+r_c} \frac{(b+c)^3}{b^3+c^3} \le \frac{8OI^2}{r^2}.$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{(b-c)^2}{r_b+r_c} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{(b-c)^2}{r_b+r_c} \cdot 4 = 4\sum \frac{(b-c)^2}{r_b+r_c} \overset{(2)}{\le} 4 \cdot \frac{2OI^2}{r} = \frac{8OI^2}{r} = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \le 4 \Leftrightarrow (b+c)(b-c)^2 \ge 0$, (2) $\Leftrightarrow \sum \frac{(b-c)^2}{h_b+h_c} \le \frac{2OI^2}{r}$.

138) In ΔABC

$$\sum \frac{a^2}{R_a^2} \frac{(b+c)^3}{b^3+c^3} \le 8\left(4+\frac{r}{R}\right).$$

Marin Chirciu

Soluție.

$$Ms = \sum \frac{a^2}{R_a^2} \frac{(b+c)^3}{b^3+c^3} \overset{(1)}{\le} \sum \frac{a^2}{R_a^2} \cdot 4 = 4\sum \frac{a^2}{R_a^2} \overset{(2)}{\le} 4 \cdot \left(8+\frac{2r}{R}\right) = 8\left(4+\frac{r}{R}\right) = Md,$$

unde (1) $\Leftrightarrow \frac{(b+c)^3}{b^3+c^3} \leq 4 \Leftrightarrow (b+c)(b-c)^2 \geq 0$, (2) $\Leftrightarrow \sum \frac{a^2}{R_a^2} \leq 8+\frac{2r}{R}$,(RMM 2018 Adil Abdullayev).

Remarcă.
În inegalitățile de mai sus egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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2. The Applications of the Stirling's approximation to find limits

Toyesh Prakash Sharma

Sometimes in different mathematical contests and journals, limit problems that can be solved by using Stirling's approximation have been seen, that's why through this article we shall trying to understand how Stirling's approximation works for finding limits.

1. Introduction

James Stirling was a great mathematician who discovered his approximation method to factorials in terms of (e i.e. Euler's constant- 2. 7182818..), π, and that positive number which is taken for doing this operation called factorial. basically, with reference to [1] mathematical definition of Stirling's approximation is:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ as } n \to \infty$$

Where the sign ~ means that the two quantities are asymptotic: their ratio tends to 1 as *n* tends to infinity. As far our topic is based on limits then after it, we are taking *Lalescu's sequence*, first it was published as problem 579 in 1900's Gazeta Matem˘atica [2]

1. Problem (Lalescu ; 1900-1901) Compute

$$\lim_{n\to\infty}\left(\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}\right)$$

Solution:

Using the Cauchy-D'Alembert criterium, we have that

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \frac{1}{e}$$

Let $\{u_n\}_{n\geq 2}$ be the sequence defined by

$$u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$$

Then we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} = \lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\sqrt[n]{n!}} = 1$$

$$\lim_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1$$

And

$$\lim_{n\to\infty}(u_n)^n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}} = e$$

Finally, we have

$$\lim_{n\to\infty}\left(\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}\right) = \lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1\right)$$

$$= \lim_{n\to\infty} \sqrt[n]{n!}\,(u_n - 1) = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n = \frac{1}{e}$$

Therefore

$$\lim_{n\to\infty} \left(\sqrt[n]{n!} - \sqrt[n-1]{(n-1)!}\right) = \frac{1}{e}$$

Same as

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{(n)!}\right) = \frac{1}{e}$$

Like above *Lalescu sequence*, another sequence was published as problem 2042 in 1914-1915's Vol. XIX of Gazeta Matem˘atica [3] nowadays it's well-known with *Romeo T. Ianculescu's sequence*-

$$\lim_{n\to\infty} \left((n+1)\sqrt[n+1]{(n+1)!} - n\sqrt[n]{(n)!}\right) = 1$$

Further With reference to published paper in 1998 [4]

Lemma 1. *If the positive sequence* $\{p_n\}$ *is such that* $\lim_{n\to\infty} \frac{p_{n+1}}{nP_n} = p > 0$ *then*

$$\lim_{n\to\infty} \left(\sqrt[n+1]{P_{n+1}} - \sqrt[n]{P_n}\right) = \frac{P}{e}$$

2. Example Problems

The below problem was published in SSM problem-solution section and problem was proposed by D.M. Batinetu-Giurgiu and N. Stanciu [5]

2. **Problem (2016).** If $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, then evaluate

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(n+1)!\,(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!\,(2n-1)!!}}{n}\right)$$

Solutions: Below solution is provided by Adnan Ali

With the help of lemma 1 on place of $P_n = \frac{(n!)\cdot(2n-1)!!}{n^n}$ then

$$\lim_{n\to\infty} \frac{p_{n+1}}{nP_n} = \lim_{n\to\infty} \frac{n^{n-1}(2n+1)}{(n+1)^n} = \lim_{n\to\infty} \frac{(2n+1)}{(n+1)} \frac{n^{n-1}}{(n+1)^{n-1}} = 2\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{n-1} = 2e^{-1}$$

And so from our lemma i.e. lemma 1

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(n+1)!\,(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!\,(2n-1)!!}}{n}\right) = 2e^{-2}$$

The next problem is taken from Pentagon's problem-solution section [6]

3. **Problem (2017).** Compute the following limit

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\,\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}}$$

Solution: Given solution is provided by Shafiqur Rahman

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\,\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{\sqrt[n]{\frac{\sqrt{2!}\,\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}}}{\sqrt[n]{\frac{(2n-1)!!}{n^n}}} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^{n+1}} \cdot n^n}{\frac{2n+1}{(n+1)^{n+1}} \cdot n^n}$$

$$=$$

$$\lim_{n\to\infty}\frac{n+1}{2n+1}\cdot\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}=\frac{1}{2}\lim_{n\to\infty}\frac{n+2}{(n+2)^{n+2}}(n+1)^{n+1}=\frac{1}{2}\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}=\frac{1}{2e}$$

The next limit problem was published in Romanian Mathematical Magazine and problem was proposed by D.M. Batinetu-Giurgiu and N. Stanciu [7]

4. Problem (2020). Compute

$$\lim_{n\to\infty}(\sin\gamma_n-\sin\gamma)\sqrt[n]{n!}$$

Where $\{\gamma_n\}_{n\geq 1}$ is a sequence defined by $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$ with $\lim_{n\to\infty}\gamma_n=\gamma$ (γ is the Euler-Mascheroni constant)

Solution: Below Solution is provided by Marian Ursărescu-Romania.

$$(\sin\gamma_n-\sin\gamma)\sqrt[n]{n!}=2\sqrt[n]{n!}\sin\left(\frac{\gamma_n-\gamma}{2}\right)\cos\left(\frac{\gamma_n+\gamma}{2}\right)$$

$$\frac{\sqrt[n]{n!}}{n}\frac{\sin\left(\frac{\gamma_n-\gamma}{2}\right)\cos\left(\frac{\gamma_n+\gamma}{2}\right)\cdot n\cdot(\gamma_n-\gamma)}{\frac{\gamma_n-\gamma}{2}}$$

Since we know that $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=e^{-1}$, $\lim_{n\to\infty}\cos\left(\frac{\gamma_n+\gamma}{2}\right)=\cos\gamma$ and with reference to basic properties of limit $\lim_{n\to\infty}\frac{\sin\left(\frac{\gamma_n-\gamma}{2}\right)}{\frac{\gamma_n-\gamma}{2}}=1$ so now we only have to solve $\lim_{n\to\infty}n\cdot(\gamma_n-\gamma)$ then

By using ***Stolz' result*** which is

If $l=\lim_{n\to\infty}\frac{a_n}{b_n}$ then $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=l$ by applying it here we can proceed toward solution so,

$$\lim_{n\to\infty}n\cdot(\gamma_n-\gamma)=\lim_{n\to\infty}\frac{(\gamma_{n+1}-\gamma)-(\gamma_n-\gamma)}{\frac{1}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{(\gamma_{n+1}-\gamma_n)}{\frac{1}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{(\gamma_n-\gamma_{n+1})}{\frac{1}{n}-\frac{1}{n+1}}$$

And by putting the given value of γ_n we can say that

$$(\gamma_n-\gamma_{n+1})=\ln\frac{n+1}{n}-\frac{1}{n+1}$$

With it our limit becomes as

$$\lim_{n\to\infty}n(n+1)\left(\ln\frac{n+1}{n}-\frac{1}{n+1}\right)$$

As

$$\lim_{n\to\infty}n(n+1)=\lim_{n\to\infty}n^2\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}n^2$$

So,

$$\lim_{n\to\infty}n^2\left(\ln\frac{n+1}{n}-\frac{1}{n+1}\right)=\lim_{\substack{x\to 0\\ x>0}}\frac{\ln\left(1+\frac{1}{x}\right)+\ln x-\frac{x}{x+1}}{x^2}=\frac{1}{2}$$

Hence

$$\lim_{n\to\infty}(\sin\gamma_n-\sin\gamma)\sqrt[n]{n!}=\frac{\cos\gamma}{2e}$$

The next problem was proposed by D.M. Batinetu-Giurgiu and N. Stanciu, taken from SSM problem-solution corner [8]

5. Problem (2020) . Compute

$$\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2}\sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2\right]$$

Where $[x]$ denotes the integer part of x.

Solution: Below solution is provided by Brian Bradie.

By the Stolz' result given limit can be written as-

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\left[\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2\right]-\sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2\right]}{\left(\sqrt[n+1]{(2n+1)!!}\right)^2-\left(\sqrt[n]{(2n-1)!!}\right)^2}$$

$$\lim_{n\to\infty}\frac{\left[\left(\sqrt[2(n+1)]{(n+1)!}+\sqrt[2(n+2)]{(n+2)!}\right)^2\right]}{\left(\sqrt[n+1]{(2n+1)!!}\right)^2-\left(\sqrt[n]{(2n-1)!!}\right)^2}$$

After it by using Stirling's approximation $n!\sim\left(\frac{n}{e}\right)^n$ so,

$$\sqrt[2(n+1)]{(n+1)!}\sim\sqrt{\frac{n+1}{e}},\ \sqrt[2(n+2)]{(n+2)!}\sim\sqrt{\frac{n+2}{e}}$$

And

$$\left(\sqrt[2(n+1)]{(n+1)!}+\sqrt[2(n+2)]{(n+2)!}\right)^2\sim\frac{4n}{e}$$

Moreover

$$\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e},\ \sqrt[n+1]{(2n+1)!!}\sim\frac{2n+2}{e}$$

$$\left(\sqrt[n+1]{(2n+1)!!}\right)^2-\left(\sqrt[n]{(2n-1)!!}\right)^2\sim\frac{8n}{e^2}$$

Finally,

$$\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2}\sum_{k=1}^{n}\left[\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2\right]=\frac{e^2}{8}\lim_{n\to\infty}\frac{[4n/e]}{n}=\frac{e}{2}$$

The next problem is taken from SSM problem-solution section and problem proposed by D.M. Batinetu-Giurgiu and N. Stanciu [9]

6. Problem (2019) . If $a\epsilon(0,1)$ then calculate

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\left(\frac{a\cdot\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-\sin a\right)$$

Solution: Below solution is provided by Brian Bradie's solution

With reference to Stirling's approximation $\sqrt[n+1]{(n+1)!}\sim\frac{n+1}{e}$, $\sqrt[n]{n!}\sim\frac{n}{e}$ and $\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$ so,

$$\frac{a\cdot\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\sim a\left(\frac{n+1}{n}\right)$$

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\left(\frac{a\cdot\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-\sin a\right)\sim\lim_{n\to\infty}\frac{2n}{e}\left(\sin a\left(1+\frac{1}{n}\right)-\sin a\right)$$

By using trigonometry identity

$$\left(\sin a\left(1+\frac{1}{n}\right)-\sin a\right)=2\sin\left(\frac{a}{2n}\right)\cos\left(a+\frac{1}{2n}\right)$$

So given limit is-

$$\lim_{n\to\infty}\frac{2n}{e}\cdot 2\sin\left(\frac{a}{2n}\right)\cos\left(a+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{2a}{e}\cdot\frac{\sin\left(\frac{a}{2n}\right)\cos\left(a+\frac{1}{2n}\right)}{\left(\frac{a}{2n}\right)}=\frac{2a\cos a}{e}$$

3. Problems for Independent Study

After several example problems, we provide the following list of problems for the readers to attempt. The requirement to find the limit $\to\infty$.

7. Problem.

Calculate-

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{(2k-1)!!}}$$

8. Problem (2014). [10]

Find

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\tan\left(\frac{\pi\cdot\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}\right)-1\right)$$

9. Problem (2020). [11]

Find

$$\lim_{n\to\infty}\sqrt[3]{n^2}\left(\frac{\sqrt[3]{(n+1)^2}}{\sqrt[3n+3]{(n+1)!}}-\frac{\sqrt[3]{n^2}}{\sqrt[3n]{(n)!}}\right)$$

10. Problem. [12]

Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}$$

11. Problem. [13]

Let $f:\Re_+\to\Re_+$ be a continuous function such that $\lim_{n\to\infty}\frac{f(x)}{x^2}=a\epsilon\Re_+$ ($\Re_+$stands for the positive real numbers) **calculate:**

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1}\frac{f(k)}{k}}-\sqrt[n]{\prod_{k=1}^{n}\frac{f(k)}{k}}\right)$$

12. Problem (2015). [14]

Let a>0, b>0 and $c\geq 0$. **Find:**

(i)

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!\,aF_{n+1}^{c}}}-\frac{n^2}{\sqrt[n]{(n)!\,bF_{n}^{c}}}\right)$$

(ii)

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!\,aL_{n+1}^c}}-\frac{n^2}{\sqrt[n]{(n)!\,bL_n^c}}\right)$$

Note- in above problem F denotes for Fibonacci number and L for Lucas number.

13. Problem. [15]

Compute

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt[2(n+1)]{(n+1)!}-\sqrt[2n]{n!}\right)$$

14. Problem (2015). [16]

Let $m>0$ and $\Gamma:(0,\infty)\to(0,\infty)$ be gamma function. **Calculate**

$$\lim_{n\to\infty}\int_{\sqrt[n]{n!}}^{\sqrt[(n+1)]{(n+1)!}}\Gamma\left(\frac{x}{n}\sqrt[n]{F_n^{\,m}}\right)$$

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3. Importanța cazului de egalitate în inegalități

Ioan Viorel Codreanu, Școala Gimnazială Satulung, Maramureș

În demonstrarea următoarelor inegalități vom evidenția importanța cazului de egalitate în inegalități. Pentru inegalitățile între medii, egalitatea are loc atunci când toate variabilele sunt egale. Având în vedere cazul de egalitate putem obține soluții surprinzătoare și elegante.

Problema 1. Fie a,b,c **numere reale pozitive astfel încât** $a+b+c=3$**. Demonstrați că**

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq a^2+b^2+c^2.$$

Dragojlub Milosevici, Mathematical Reflections

Soluție. Pentru $a=b=c=1$ obținem egalitate. Folosind egalitatea $a+b+c=3$ obținem $a^2+b^2+c^2=9-2(ab+bc+ca)$. Inegalitatea din enunț este echivalentă cu

$$\frac{a^3}{b^2}+2ab+\frac{b^3}{c^2}+2bc+\frac{c^3}{a^2}+2ca\geq 9.$$

Folosind de două ori **Inegalitatea MA-MG**, având în vedere cazul de egalitate, obținem

$$\frac{a^3}{b^2}+2ab\geq 3\sqrt[3]{a^5}=\frac{3a^2}{\sqrt[3]{a}}\geq\frac{3a^2}{\frac{a+2}{3}}=\frac{9a^2}{a+2}.$$

Similar, $\frac{b^3}{c^2}+2bc\geq\frac{9b^2}{b+2}$ și $\frac{c^3}{a^2}+2ca\geq\frac{9c^2}{c+2}$. Atunci, adunând aceste inegalități și folosind **Inegalitatea Bergstöm** avem

$$\sum\left(\frac{a^3}{b^2}+2ab\right)\geq 9\sum\frac{a^2}{a+2}\geq 9\frac{\left(\sum a\right)^2}{\sum a+6}=9.$$

Problema 2. Dacă a,b,c **sunt numere reale pozitive, arătați că**

$$\frac{9}{2(ab+bc+ca)}+\sum\frac{a^3}{b^2-bc+c^2}\geq\frac{9}{2}.$$

Nguyen Viet Hung, Recreații Matematice

Soluție. Pentru $a=b=c=1$ obținem egalitate. Arătăm că are loc inegalitatea

$$\sum\frac{a^3}{b^2-bc+c^2}\geq\sum a.$$

Folosind **Inegalitatea Bergström**, obținem

$$\sum\frac{a^3}{b^2-bc+c^2}=\sum\frac{a^4}{a\left(b^2-bc+c^2\right)}\geq\frac{\left(\sum a^2\right)^2}{\sum a\left(b^2-bc+c^2\right)}.$$

Așadar trebuie să demonstrăm că

$$\left(\sum a^2\right)^2\geq\left(\sum a\right)\left(\sum\left(ab^2+ac^2-abc\right)\right).$$

Această inegalitate este echivalentă cu

$$\sum a^4+abc\sum a\geq\sum ab\left(a^2+b^2\right)$$

care este **Inegalitatea Schur**.

Folosind binecunoscuta inegalitate $\left(\sum a\right)^2 \geq 3\sum ab$ și **Inegalitatea MA-MG**, ținînd seama de cazul de egalitate, obținem

$$\frac{9}{2\sum ab}+\sum\frac{a^3}{b^2-bc+c^2}\geq\frac{9}{2\sum ab}+\sum a=\frac{9}{2\sum ab}+\frac{\sum a}{2}+\frac{\sum a}{2}\geq 3\sqrt[3]{\frac{9\left(\sum a\right)^2}{8\sum ab}}\geq\frac{9}{2}.$$

Problema 3. Demonstrați că dacă $a,b,c\in(0,2]$ atunci

$$3\sqrt{2}\leq\sum\frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c}\leq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Daniel Sitaru, Kappa Mu Epsilon

Soluție. Pentru $a=b=c=1$ obținem egalitate în inegalitatea din dreapta. Aplicând **Inegalitatea MA-MG**, ținând cont de cazul de egalitate, avem

$$\frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c}\leq\frac{b\left(\frac{a+1}{2}+\frac{2-a+1}{2}\right)}{c}=\frac{2b}{c}.$$

Adunînd această inegalitate cu alte două inegalități analoage, obținem

$$\sum\frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c}\leq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Observăm că egalitatea are loc dacă și numai dacă $a=b=c=1$. Pentru inegalitatea din stânga, avem

$$\sqrt{a}+\sqrt{2-a}\geq\sqrt{2}\Leftrightarrow a+2-a+2\sqrt{a(2-a)}\geq 2\Leftrightarrow\sqrt{a(2-a)}\geq 0,$$

evident adevărată. Adunînd această inegalitate cu alte două inegalități analoage, avem

$$\sum\frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c}\geq\sqrt{2}\sum\frac{b}{c}.$$

Folosind **Inegalitatea MA-MG**, obținem

$$\sum\frac{b}{c}\geq 3\sqrt[3]{\prod\frac{b}{c}}=3,$$

deci,

$$\sum\frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c}\geq 3\sqrt{2}.$$

Observăm că egalitatea are loc dacă și numai dacă $a=b=c=2$.

Problema 4. Fie a,b,c numere reale pozitive astfel încât

$$\frac{1}{\sqrt{1+a^3}}+\frac{1}{\sqrt{1+b^3}}+\frac{1}{\sqrt{1+c^3}}\leq 1.$$

Demonstrați că $a^2+b^2+c^2\geq 12$.

Nguyen Viet Hung, Mathematical Reflections

Soluție. Pentru $a=b=c=2$ obținem egalitate. Folosind **Inegalitatea MA-MG**, ținînd seama de cazul de egalitate, obținem

$$\sqrt{1+a^3}=\sqrt{(1+a)\left(1-a+a^2\right)}\leq\frac{a^2+2}{2}$$

și

$$\frac{1}{\sqrt{1+a^3}} \geq \frac{2}{a^2+2}.$$

Similar, $\frac{1}{\sqrt{1+b^3}} \geq \frac{2}{b^2+2}$ și $\frac{1}{\sqrt{1+c^3}} \geq \frac{2}{c^2+2}$. Adunând aceste trei inegalități, obținem

$$\sum \frac{1}{\sqrt{1+a^3}} \geq 2\sum \frac{1}{a^2+2}.$$

Deducem că

$$\sum \frac{1}{a^2+2} \leq \frac{1}{2}.$$

Folosind **Inegalitatea MA-MH**, obținem

$$\left(\sum (a^2+2)\right)\left(\sum \frac{1}{a^2+2}\right) \geq 9.$$

Deducem că

$$\sum (a^2+2) \geq 18,$$

echivalent cu

$$\sum a^2 \geq 12.$$

Problema 5. Fie a,b,c numere reale pozitive astfel încât $abc=1$. Demonstrați că

$$a^3+b^3+c^3+\frac{8}{(a+b)(b+c)(c+a)} \geq 4.$$

Alessandro Ventulo, Mathematical Reflections

Soluție. Pentru $a=b=c=1$ obținem egalitate. Observăm că are loc inegalitatea $a^3+b^3 \geq ab(a+b)$, deoarece este echivalentă cu $(a-b)^2(a+b) \geq 0$. Similar $b^3+c^3 \geq bc(b+c)$ și $c^3+a^3 \geq ca(c+a)$. Atunci

$$\sum a^3 \geq \sum \frac{ab(a+b)}{2}.$$

Deducem că

$$\sum a^3 + \frac{8}{\prod (a+b)} \geq \sum \frac{ab(a+b)}{2} + \frac{8}{\prod (a+b)}.$$

Folosind **Inegalitatea MA-MG**, ținând seama de cazul de egalitate, obținem

$$\sum \frac{ab(a+b)}{2} + \frac{8}{\prod (a+b)} \geq 4\sqrt[4]{\left(\prod \frac{ab(a+b)}{2}\right)\cdot \frac{8}{\prod (a+b)}} = 4,$$

și rezultă concluzia.

Problema 6. Fie a și b numere reale pozitive astfel încât $ab=1$. Demonstrați că

$$\frac{2}{a^2+b^2+1} \leq \frac{1}{a^2+b+1} + \frac{1}{a+b^2+1} \leq \frac{2}{a+b+1}.$$

An Zhenping, Mathematical Reflections

Soluție. Pentru $a = b = 1$ obținem egalitate. Pentru început să dovedim inegalitatea din stânga. Inegalitatea se scrie

$$\frac{2}{a^2+\frac{1}{a^2}+1} \le \frac{1}{a^2+\frac{1}{a}+1}+\frac{1}{a+\frac{1}{a^2}+1},$$

echivalentă cu

$$2a\left(a^3+a+1\right)\left(a^3+a^2+1\right) \le \left(a^4+a^2+1\right)\left(a^4+a^3+2a^2+a+1\right).$$

Ultima inegalitate se transformă în

$$a^8-a^7+a^6-2a^4+a^2-a+1 \ge 0,$$

care se scrie echivalent, ținând seama de cazul de egalitate, în forma

$$(a-1)^2\left(a^6+a^5+2a^4+3a^3+2a^2+a+1\right) \ge 0,$$

evident adevărată.

Pentru inegalitatea din dreaptă procedăm asemănător scriind-o în forma

$$\frac{1}{a^2+\frac{1}{a}+1}+\frac{1}{a+\frac{1}{a^2}+1} \le \frac{2}{a+\frac{1}{a}+1}$$

echivalentă cu

$$\left(a^4+a^3+2a^2+a+1\right)\left(a^2+a+1\right) \le 2\left(a^3+a+1\right)\left(a^3+a^2+1\right).$$

După efectuarea calculelor, ultima inegalitate devine

$$a^6-2a^4+2a^3-2a^2+1 \ge 0,$$

care este echivalentă cu

$$(a-1)^2\left(a^4+2a^3+a^2+2a+1\right) \ge 0,$$

Evident adevărată.

Avem egalitate pentru $a = b = 1$.

Problema 7. Fie numerele reale nenegative a și b. Să se demonstreze că

$$\left(3a+\frac{4}{a+1}+\frac{8}{\sqrt{2\left(1+b^2\right)}}\right)\left(3b+\frac{4}{b+1}+\frac{8}{\sqrt{2\left(1+a^2\right)}}\right) \ge 81$$

Marius Stănean, Gazeta Matematică

Soluție. Pentru $a = b = 1$ obținem egalitate. Notăm cu α, β expresia din prima paranteză, respectiv, a doua paranteză. Fie $c = \frac{\sqrt{2\left(a^2+1\right)}}{2}, d = \frac{\sqrt{2\left(b^2+1\right)}}{2}$.

Avem

$$\alpha+1 = a+1+2a+\frac{4}{a+1}+\frac{8}{\sqrt{2\left(b^2+1\right)}} =$$

$$= a+1+\frac{2a^2+2a+4}{a+1}+\frac{8}{\sqrt{2\left(b^2+1\right)}} =$$

$$= a+1+2\cdot\frac{a^2+1}{a+1}+2+\frac{8}{\sqrt{2\left(b^2+1\right)}}.$$

Folosind **Inegalitatea MA-GM**, ținînd seama de cazul de egalitate, obținem

$$a+1+2\cdot\frac{a^2+1}{a+1}\geq 2\sqrt{2\left(a^2+1\right)}.$$

Atunci

$$\alpha\geq 4c+\frac{4}{d}+1$$

și folosind **Inegalitatea MA-MG**, ținînd cont de cazul de egalitate, obținem

$$\alpha\geq 9\sqrt[9]{\frac{c^4}{d^4}}.$$

Deoarece expresiile din paranteze sunt simetrice în raport cu a și b, obținem

$$\beta\geq 9\sqrt[9]{\frac{d^4}{c^4}}.$$

Din ultimele două inegalități rezultă $\alpha\cdot\beta\geq 81$.

Problema 8. Fie numerele reale nenegative a,b,c, astfel încât $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$.

Demonstrați că

$$\sqrt{(a+b+1)(c+2)}+\sqrt{(b+c+1)(a+2)}+\sqrt{(c+a+1)(b+2)}\geq 9.$$

Titu Andreescu, Mathematical Reflections

Soluție. Pentru $a=b=c=1$ obținem egalitate. Folosind **Inegalitatea Cauchy-Schwarz**, având în vedere cazul de egalitate, obținem

$$(a+b+1)(c+2)=\left(\sqrt{a}^2+\sqrt{b}^2+1^2\right)\left(1^2+1^2+\sqrt{c}^2\right)\geq\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2=9,$$

adică $\sqrt{(a+b+1)(c+2)}\geq 3$. Analog, $\sqrt{(b+c+1)(a+2)}\geq 3$ și $\sqrt{(c+a+1)(b+2)}\geq 3$.

Adunând aceste trei inegalități, rezultă imediat concluzia

Problema 9. Demonstrați că dacă a,b,c sunt numere reale pozitive, are loc inegalitatea

$$\frac{a^2+b^2+c^2-ab-bc-ca}{(a+b+c)^2}+\frac{2a}{2a+b+c}+\frac{2b}{a+2b+c}+\frac{2c}{a+b+2c}\geq\frac{3}{2}.$$

Titu Zvonaru, Recreații Matematice

Soluție. Fără a restrânge generalitatea, presupunem că $\sum a=1$.Pentru $a=b=c=\frac{1}{3}$ obținem egalitate. Folosind **Inegalitatea Bergström** obținem

$$\sum\frac{a}{a+1}=\sum\frac{a^2}{a^2+a}\geq\frac{\left(\sum a\right)^2}{\sum a^2+\sum a}=\frac{1}{2-2\sum ab}$$

și atunci

$$\sum \frac{2a}{2a+b+c} = 2\sum \frac{a}{a+1} \geq \frac{1}{1-\sum ab}.$$

Este suficient să demonstrăm că

$$1-3\sum ab + \frac{1}{1-\sum ab} \geq \frac{3}{2}.$$

Folosind binecunoscuta inegalitate $\sum ab \leq \frac{\left(\sum a\right)^2}{3} = \frac{1}{3}$, avem

$$1-3\sum ab + \frac{1}{1-\sum ab} = \frac{1}{1-\sum ab} + \frac{9}{4}\left(1-\sum ab\right) - \frac{3}{4}\sum ab - \frac{5}{4} \geq$$

$$\geq 2\sqrt{\frac{9}{4}\left(1-\sum ab\right)\cdot\frac{1}{1-\sum ab}} - \frac{3}{4}\cdot\frac{1}{3} - \frac{5}{4} = 2\cdot\frac{3}{2} - \frac{1}{4} - \frac{5}{4} = \frac{3}{2},$$

unde am folosit **Inegalitatea MA-MG**, ținînd seama de cazul de egalitate.

Problema 10. Fie a,b,c,d **numere reale pozitive astfel încât** $a+b+c+d=4$**.**

Demonstrați că

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} \leq \frac{1}{abcd}$$

An Zhenping, Mathematical Reflections

Soluție. Pentru $a=b=c=d=1$ obținem egalitate. Folosind **Inegalitatea MA-MG**, ținând cont de cazul de egalitate, obținem

$$a+3 \geq 4\sqrt[4]{a}.$$

Avem

$$\sum \frac{1}{a+3} \leq \frac{1}{4}\sum \frac{1}{\sqrt[4]{a}} = \frac{1}{4\sqrt[4]{abcd}}\sum \sqrt[4]{bcd}.$$

Folosind **Inegalitatea MA-MG**, ținînd cont de cazul de egalitate, obținem

$$\sqrt[4]{bcd} \leq \frac{b+c+d+1}{4}.$$

Atunci, avem

$$\sum \frac{1}{a+3} \leq \frac{1}{4\sqrt[4]{abcd}}\sum \frac{b+c+d+1}{4} = \frac{1}{4\sqrt[4]{abcd}}\cdot\frac{4+3\sum a}{4} = \frac{1}{\sqrt[4]{abcd}} \leq \frac{1}{abcd}$$

deoarece, din **Inegalitatea MA-MG**, obținem

$$\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4} = 1$$

și atunci $abcd \le 1$ și $\frac{1}{\sqrt[4]{abcd}} \le \frac{1}{abcd}$, această ultimă inegalitate fiind echivalentă cu $abcd \le 1$.

Avem egalitate dacă $a = b = c = d = 1$.

Problema 11. Fie $x, y, z, t > 0$ **astfel încât** $x \le 2, x + y \le 6, x + y + z \le 12,$ **, și** $x + y + z + t \le 24$ **. Demonstrați că**

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge 1$$

Mircea Lascu și Marius Stănean, Mathematical Reflections

Soluție. Pentru $x = 2, y = 4, z = 6, t = 12$ obținem egalitate. Folosind **Inegalitatea MA-MG**, ținând cont de cazul de egalitate, obținem

$$\frac{1}{x} + \frac{x}{4} \ge 2\sqrt{\frac{1}{x} \cdot \frac{x}{4}} = 1$$

$$\frac{1}{y} + \frac{y}{16} \ge 2\sqrt{\frac{1}{y} \cdot \frac{y}{16}} = \frac{1}{2}$$

$$\frac{1}{z} + \frac{z}{36} \ge 2\sqrt{\frac{1}{z} \cdot \frac{z}{36}} = \frac{1}{3}$$

și

$$\frac{1}{t} + \frac{t}{144} \ge 2\sqrt{\frac{1}{t} \cdot \frac{t}{144}} = \frac{1}{6}.$$

Atunci avem

$$\sum \frac{1}{x} \ge 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - \left(\frac{x}{4} + \frac{y}{16} + \frac{z}{36} + \frac{t}{144}\right) = 2 - \frac{1}{144}(36x + 9y + 4z + t).$$

Deoarece $x \le 2, x + y \le 6, x + y + z \le 12,$ și $x + y + z + t \le 24$ avem

$$36x + 9y + 4z + t = (x + y + z + t) + 3(x + y + z) + 5(x + y) + 27x \le$$
$$24 + 3 \cdot 12 + 5 \cdot 6 + 27 \cdot 2 = 144.$$

În final,

$$\sum \frac{1}{x} \ge 2 - \frac{1}{144} \cdot 144 = 1.$$

Avem egalitate dacă $x = 2, y = 4, z = 6$ și $t = 12$.

Problema 12. Fie a, b, c **numere reale pozitive. Demonstrați că**

$$a^2 + b^2 + c^2 \ge a\sqrt[3]{\frac{b^3 + c^3}{2}} + b\sqrt[3]{\frac{c^3 + a^3}{2}} + c\sqrt[3]{\frac{a^3 + b^3}{2}}.$$

Nguyen Viet Hung, Mathematical Reflections

Soluție. Pentru $a=b=c$ obținem egalitate. Folosind **Inegalitatea MA-MG**, ținând seama de cazul de egalitate, obținem

$$a\sqrt[3]{\frac{b^3+c^3}{2}}=\sqrt[3]{\frac{(ab)^3+(ac)^3}{2}}=\sqrt[3]{\frac{ab+ac}{2}\cdot a^2\left(b^2+c^2-bc\right)}\le$$

$$\le\frac{\frac{ab+ac}{2}+a^2+b^2+c^2-bc}{3}=\frac{\sum a^2}{3}+\frac{ab+ac-2bc}{6}.$$

Analog,

$$b\sqrt[3]{\frac{c^3+a^3}{2}}\le\frac{\sum a^2}{3}+\frac{bc+ba-2ca}{6}\quad \text{și}\quad c\sqrt[3]{\frac{a^3+b^3}{2}}\le\frac{\sum a^2}{3}+\frac{ca+cb-2ab}{6}.$$

Adunând aceste inegalități, obținem

$$\sum a\sqrt[3]{\frac{b^3+c^3}{2}}\le\sum a^2.$$

Problema 13. Fie a,b,c,d numere reale pozitive astfel încât $a+b+c+d=3$. Demonstrați că

$$a^2+b^2+c^2+d^2+\frac{64}{27}abcd\ge 3.$$

An Zhenping, Mathematical Reflections

Soluție. Pentru $a=b=c=d=\frac{3}{4}$ obținem egalitate. Inegalitatea este echivalentă cu

$$a^2+b^2+c^2+d^2+\frac{16}{9}abcd+\frac{16}{9}abcd-\frac{32}{27}abcd\ge 3.$$

Folosind **Inegalitatea MA-MG**, având în vedere cazul de egalitate, obținem

$$a^2+b^2+c^2+d^2+\frac{16}{9}abcd+\frac{16}{9}abcd\ge 6\sqrt[6]{\frac{16^2}{9^2}\cdot a^4b^4c^4d^4}=4\sqrt[3]{6a^2b^2c^2d^2}.$$

Așadar, este suficient să demonstrăm că

$$\frac{32}{27}abcd-4\sqrt[3]{6a^2b^2c^2d^2}+3\le 0$$

sau

$$4\sqrt[3]{a^2b^2c^2d^2}\left(\frac{8}{27}\sqrt[3]{abcd}-\sqrt[3]{6}\right)+3\le 0.$$

Folosind **Inegalitatea MA-MG**, avem

$$abcd\le\left(\frac{a+b+c+d}{4}\right)^4=\left(\frac{3}{4}\right)^4.$$

Atunci

$$4\sqrt[3]{a^2b^2c^2d^2}\left(\frac{8}{27}\sqrt[3]{abcd}-\sqrt[3]{6}\right)+3\le 4\sqrt[3]{\left(\frac{3}{4}\right)^8}\left(\frac{8}{27}\sqrt[3]{\left(\frac{3}{4}\right)^4}-\sqrt[3]{6}\right)+3=0.$$

Egalitatea are loc dacă și numai dacă $a=b=c=d=\frac{3}{4}$.

4. Inegalitatea Bergström

Ioan Viorel Codreanu, Școala Gimnazială Satulung, Maramureș

Una dintre problemele dificile ale Olimpiadei Internaționale de Matematică 1995, Canada, a fost următoarea inegalitate algebrică:

Problema 1. Fie a,b,c numere reale pozitive, astfel încât $abc=1$. Să se arate că:

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Rusia

Mulți elevi buni au observat că substituția $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$ păstrează condiția $xyz=1$ și transformă inegalitatea dată într-una mai simplă:

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2}.$$

Soluția oficială făcea apel la inegalitatea Cauchy-Schwarz:

$$[(y+z)+(z+x)+(x+y)]\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)\geq(x+y+z)^2,$$

iar prin simplificare s-ar obține:

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{x+y+z}{2}\geq\frac{3}{2},$$

ultima rezultând din inegalitatea mediilor.
Se poate spune că o serie de elevi bine pregătiți nu au observat acest lucru. Alte metode se arată greoaie, conducând fie la calcule complicate, fie la situații fără ieșire.
Propunem în continuare o metodă simplă, cu spectru larg de aplicații:

***Lema 1*. (Titu Andreescu, RMT, 1979).**
Fie $\alpha,\beta>0$ două numere reale. Atunci

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}\geq\frac{(x+y)^2}{\alpha+\beta},$$

oricare ar fi numerele reale x și y.

Demonstrație. După efectuarea calculelor se ajunge la $(\beta x-\alpha y)^2\geq 0$. Egalitatea are loc dacă și numai dacă $\frac{x}{\alpha}=\frac{y}{\beta}$.

O generalizare simplă este următoarea:

***Lema 2*. (Inegalitatea Bergström).**
Fie $\alpha_1,\alpha_2,...,\alpha_n>0$ numere reale. Atunci:

$$\frac{x_1^2}{\alpha_1}+\frac{x_2^2}{\alpha_2}+...+\frac{x_n^2}{\alpha_n}\geq\frac{(x_1+x_2+...+x_n)^2}{\alpha_1+\alpha_2+...+\alpha_n},$$

oricare ar fi numerele reale $x_1,x_2,...,x_n$.

Demonstrație. Aplicând succesiv lema 1, avem

$$\frac{x_1^2}{\alpha_1}+\frac{x_2^2}{\alpha_2}+\frac{x_3^2}{\alpha_3}+...+\frac{x_n^2}{\alpha_n}\geq\frac{(x_1+x_2)^2}{\alpha_1+\alpha_2}+\frac{x_3^2}{\alpha_3}+...+\frac{x_n^2}{\alpha_n}\geq...\geq\frac{(x_1+x_2+...+x_n)^2}{\alpha_1+\alpha_2+...+\alpha_n},$$

ceea ce era de demonstrat.

O aplicație imediată este chiar problema sus menționată, inegalitatea $\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2}$ se demonstrează utilizând lema 2 astfel:

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{(x+y+z)^2}{y+z+z+x+x+y}=\frac{x+y+z}{2}\geq\frac{3}{2},$$

pentru ultima parte folosind din nou inegalitatea mediilor.
Iată o soluție a inegălității 1 fără a apela la substituții . Folosind Inegalitatea Bergström, obținem

$$\sum\frac{1}{a^3(b+c)}=\sum\frac{a^3b^3c^3}{a^3(b+c)}=\sum\frac{b^3c^3}{b+c}=\sum\frac{(bc)^2}{\frac{1}{b}+\frac{1}{c}}\geq\frac{\left(\sum ab\right)^2}{2\sum\frac{1}{a}}=\frac{\left(\sum ab\right)^2}{2\sum ab}=\frac{\sum ab}{2}\geq\frac{3}{2},$$

unde la ultima inegalitate am aplicat din nou inegalitatea mediilor.
O altă soluție folosind inegalitatea Bergström

$$\sum\frac{1}{a^3(b+c)}=\sum\frac{a^2b^2c^2}{a^3(b+c)}=\sum\frac{(bc)^2}{ab+bc}\geq\frac{\left(\sum bc\right)^2}{2\sum bc}=\frac{\sum bc}{2}\geq\frac{3}{2}.$$

Prezentăm în continuare alte aplicații ale acestei metode:

Problema 2. Dacă $a,b,c>0$ **și** $a+b+c=1$**, să se arate că:**

$$\frac{(a+b)^2}{c+1}+\frac{(b+c)^2}{a+1}+\frac{(c+a)^2}{b+1}\geq 1.$$

Mihai Opincariu, GM 10/2000

Soluție. Folosind Inegalitatea Bergström, obținem

$$\sum\frac{(a+b)^2}{c+1}\geq\frac{\left(2\sum a\right)^2}{\sum a+3}=1.$$

Problema 3. Dacă $a,b,c,d>0$ **și** $a+b+c+d=1$**, să se arate că:**

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{1}{2}.$$

OM Irlanda, 1999

Soluție. Folosind Inegalitatea Bergström, obținem

$$\sum\frac{a^2}{a+b}\geq\frac{\left(\sum a\right)^2}{2\sum a}=\frac{1}{2}.$$

Problema 4. Dacă $a,b>1$**, atunci** $\frac{a^2}{b-1}+\frac{b^2}{a-1}\geq 8$**.**

OM Rusia, 1992

Soluție. Folosind Inegalitatea Titu Andreescu, obținem

$$\frac{a^2}{b-1}+\frac{b^2}{a-1}\geq\frac{(a+b)^2}{a+b-2}.$$

Notând $a+b=x$, avem de demonstrat că $\frac{x^2}{x-2}\geq 8$, știind că $x>2$. Inegalitatea se reduce la $(x-4)^2\geq 0$.

Problema 5. Dacă $a,b,c>0$ **sunt numere reale cu proprietatea că** $a^2+b^2+c^2=3$**, să se arate că:**

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\geq\frac{3}{2}.$$

OM Belarus, 1999

Soluție. Folosind Inegalitatea Bergström. obținem

$$\sum \frac{1}{1+ab} = \sum \frac{1^2}{1+ab} \geq \frac{9}{\sum(1+ab)} \geq \frac{9}{3+\sum a^2} = \frac{3}{2},$$

unde în final am folosit binecunoscuta inegalitate $\sum a^2 \geq \sum ab$.

Problema 6. Fie a,b,c numere reale pozitive. Demonstrați că

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} \geq 4b.$$

Titu Andreescu, Mathematical Reflections

Soluție. Folosind Inegalitatea Titu Andreescu, avem

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)^2}{a+c},$$

deci este suficient să demonstrăm că

$$\frac{(a+b+c)^2}{a+c} \geq 4b.$$

Această inegalitate este echivalentă cu $(a-b+c)^2 \geq 0$, și problema este rezolvată.

Problema 7. (*Inegalitatea Nesbitt*). Fie a,b,c numere reale pozitive. Demonstrați că

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Soluția 1. Folosind Inegalitatea Bergström, obținem

$$\sum \frac{a}{b+c} = \sum \frac{a^2}{ab+ac} \geq \frac{\left(\sum a\right)^2}{2\sum ab} \geq \frac{3\sum ab}{2\sum ab} = \frac{3}{2},$$

unde în final am folosit binecunoscuta inegalitate $\left(\sum a\right)^2 \geq 3\sum ab$.

Soluția 2. Folosind metoda deligamentării, avem

$$\frac{a}{b+c} - \frac{1}{2} = \frac{2a-b-c}{2(b+c)} = \frac{a-b}{2(b+c)} + \frac{a-c}{2(b+c)},$$

și alte două relații analoage. Atunci, grupând fracțiile în funcție de numărătorii lor, obținem

$$\sum\left(\frac{a}{b+c} - \frac{1}{2}\right) = \sum\left(\frac{a-b}{2(b+c)} + \frac{b-a}{2(c+a)}\right) = \sum \frac{(a-b)^2}{2(b+c)(c+a)} \geq 0.$$

Egalitatea are loc dacă și numai dacă $a=b=c$.
Metoda deligamentării, folosită în demonstrația inegalităților, duce și la obținerea unor rafinări ale acestora. Chiar dacă sunt necesare unele calcule, acestea sunt în multe situații ușor de condus către rezultatul dorit.

Soluția 3. Inegalitatea fiind omogenă, putem presupune $a+b+c=1$.
Avem $\frac{x}{1-x} \geq \frac{9x-1}{4}$, pentru orice $x \in (0,1)$, inegalitate echivalentă cu $(3x-1)^2 \geq 0$.
Ca urmare

$$\sum \frac{a}{b+c} = \sum \frac{a}{1-a} \geq \sum \frac{9a-1}{4} = \frac{3}{2},$$

și inegalitatea este demonstrată.

Această soluție este de tip ”iepurașul scos din joben” și cititorul se întreabă de unde a apărut inegalitatea ajutătoare. Inegalitatea ajutătoare a fost construită ținând cont de cazul de egalitate în Inegalitatea Bergström și condiția $a+b+c=1$.

Soluția 4. Putem să presupunem că $a \le b \le c$. Ca urmare, avem $\frac{1}{b+c} \le \frac{1}{c+a} \le \frac{1}{a+b}$.

Folosind Inegalitatea Cebîșev, avem

$$\sum \frac{a}{b+c} \ge \frac{1}{3}\left(\sum a\right)\left(\sum \frac{1}{b+c}\right) \ge \frac{1}{3}\left(\sum a\right)\cdot \frac{9}{\sum (b+c)} = \frac{3}{2}.$$

Soluția 5. Folosind ordonarea din soluția precedentă și inegalitatea aranjamentelor sau rearanjărilor, avem

$$\sum \frac{a}{b+c} \ge \frac{a}{c+a} + \frac{b}{a+b} + \frac{c}{b+c}$$

și

$$\sum \frac{a}{b+c} \ge \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$$

Adunând aceste două inegalități și grupând termenii care au același numitor, obținem

$$2\sum \frac{a}{b+c} \ge \sum \frac{a+b}{a+b} = 3,$$

de unde rezultă concluzia.

Soluția 6. Este o soluție calculatorie unde folosim spargerea ultimei ingalități și bincunoscuta inegalitate $x^3+y^3 \ge x^2y+xy^2$, echivalentă cu $(x-y)^2 \ge 0$, pentru orice numere pozitive x și y. După aducere la același numitor și efectuarea reducerilor termenilor asemenea, inegalitatea devine

$$\left(a^3+b^3\right)+\left(b^3+c^3\right)+\left(c^3+a^3\right) \ge \left(a^2b+b^2a\right)+\left(b^2c+c^2b\right)+\left(c^2a+a^2c\right),$$

evident adevărată ținând cont de inegalitatea amintită mai sus.

Soluția 7. Vom folosi metoda substituției. Notând $x=b+c, y=c+a, z=a+b$, avem

$$a=\frac{y+z-x}{2}, b=\frac{z+x-y}{2}, c=\frac{x+y-z}{2}.$$

Atunci

$$\sum \frac{a}{b+c} = \sum \frac{y+z-x}{2x} = \frac{1}{2}\left(\sum\left(\frac{x}{y}+\frac{y}{x}\right)-3\right) \ge \frac{1}{2}(6-3) = \frac{3}{2}.$$

Soluția 8. Adunând 3 în ambii membri ai inegalității, aceasta se scrie

$$\sum\left(\frac{a}{b+c}+1\right) \ge \frac{9}{2}$$

echivalentă cu

$$\left(\sum a\right)\left(\sum \frac{1}{b+c}\right) \ge \frac{9}{2},$$

care rezultă ușor folosind Inegalitatea MA-MH.

Soluția 9. Vom folosi procedeul de reducere a numărului de variabile-mixing variables.

Fie $E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}$. Vom arăta că $E(a,b,c) \ge E\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)$.

Explicitând și apoi desfăcând parantezele, se obține inegalitatea echivalentă

$$b^3+c^3+ab^2+ac^2\geq 2abc+b^2c+bc^2.$$

inegalitate imediată datorită inegalității mediilor $a(b^2+c^2)\geq 2abc$ și $b^3+c^3\geq b^2c+c^2b$. Prin urmare, rămâne să arătăm că $E(a,t,t)\geq 0$, unde $t=\frac{b+c}{2}$. Această relație este echivalentă cu

$$\frac{a}{2t}+\frac{2t}{a+t}\geq\frac{3}{2},$$

care este echivalentă cu $(a-t)^2\geq 0$.

Soluția 10. Este tot o soluție calculatorie. Putem presupune că $\sum a=1$, deoarece Inegalitatea Nesbitt este omogenă. Atunci trebuie să dovedim că

$$\sum\frac{a}{1-a}\geq\frac{3}{2},$$

care după efectuarea calculelor se reduce la

$$2+9\prod a\geq 7\sum ab.$$

Două dintre numerele a,b,c sunt la fel așezate față de $\frac{1}{3}$. Fie acestea a și b. Atunci

$$(3a-1)(3b-1)\geq 0,$$

de unde rezultă că

$$9ab+1\geq 3a+3b,$$

și după multiplicare cu c

$$9abc+c\geq 3ac+3bc.$$

Deci este suficient să dovedim că

$$2-c\geq 4ac+4bc+7ab,$$

ca după adunare cu ultima inegalitate să rezulte concluzia.
Înlocuind $c=1-a-b$ în inegalitatea de demonstrat, rezultă după efectuarea calculelor că trebuie să dovedim inegalitatea

$$4a^2+4b^2+ab-3a-3b+1\geq 0.$$

Folosind inegalitatea mediilor $a^2+b^2\geq 4ab$, mai rămâne să dovedim că

$$9ab-3a-3b+1\geq 0,$$

care este de fapt

$$(3a-1)(3b-1)\geq 0,$$

evident adevărată din presupunerea făcută mai sus.

Soluția 11. Este de fapt o altă soluție pentru ingalitatea

$$2+9\prod a\geq 7\sum ab,$$

la care am ajuns în soluția 10. Forma ei sugerează folosirea inegalității Schur în cazul particular $\sum a=1$. Aceasta este

$$\sum ab\leq\frac{1+9\prod a}{4}.$$

Folosind Inegalitatea Schur este suficient să arătăm că

$$27\prod a\leq 1,$$

care rezultă din aplicarea inegalității mediilor.

Problema 8. Pentru $n\geq 2$, fie $a_1,a_2,...,a_{n-1}$ numere reale pozitive cu suma 1 și $b_1,b_2,...,b_n$ numere reale arbitrare. Demonstrați inegalitatea:

$$b_1^2+\frac{b_2^2}{a_1}+\frac{b_3^2}{a_2}+\ldots+\frac{b_n^2}{a_{n-1}}\geq 2b_1(b_1+b_2+\ldots+b_n).$$

Dumitru Acu, Baraj OBMJ, 2000

Soluție. Folosind Inegalitatea Bergström, avem

$$b_1^2+\frac{b_2^2}{a_1}+\frac{b_3^2}{a_2}+\ldots+\frac{b_n^2}{a_{n-1}}\geq b_1^2+\frac{(b_2+b_3+\ldots+b_n)^2}{a_1+a_2+\ldots+a_{n-1}},$$

și folosind condiția din enunț și inegalitatea mediilor rezultă concluzia.

Problema 9. Se dau numerele reale $x_0>x_1>x_2>\ldots>x_n$. Să se arate că:

$$x_0+\frac{1}{x_0-x_1}+\frac{1}{x_1-x_2}+\ldots+\frac{1}{x_{n-1}-x_n}\geq x_n+2n.$$

Concursul orașului Sankt Petersburg, 1999

Soluție. Folosind Inegalitatea Bergström, obținem

$$x_0+\frac{1}{x_0-x_1}+\frac{1}{x_1-x_2}+\ldots+\frac{1}{x_{n-1}-x_n}\geq x_0+\frac{n^2}{x_0-x_n},$$

și folosind inegalitatea mediilor

$$x_0-x_n+\frac{n^2}{x_0-x_n}\geq 2n,$$

rezultă concluzia.

Problema 10. Fie a,b,c numere pozitive. Demonstrați că

$$\frac{bc}{(2a+b)(2a+c)}+\frac{ca}{(2b+c)(2b+a)}+\frac{ab}{(2c+a)(2c+b)}\geq\frac{1}{3}$$

An Zhenping, Mathematical Reflections

Soluție. Din Inegalitatea Bergström, avem

$$\sum\frac{bc}{(2a+b)(2a+c)}=\sum\frac{(bc)^2}{4a^2bc+2ab^2c+2abc^2+b^2c^2}\geq\frac{\left(\sum bc\right)^2}{8abc\sum a+\sum b^2c^2}.$$

Este suficient să demonstrăm că

$$3\left(\sum b^2c^2+2abc\sum a\right)\geq 8abc\sum a+\sum b^2c^2,$$

echivalent cu

$$\sum b^2c^2\geq abc\sum a.$$

Avem

$$\sum b^2c^2\geq\sum abbc=abc\sum a.$$

Aceasta încheie demonstrația.

Problema 11. Fie a,b,c numere pozitive. Demonstrați că

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+4\left(\frac{a}{2a+b+c}+\frac{b}{a+2b+c}+\frac{c}{a+b+2c}\right)\geq\frac{9}{2}$$

Titu Zvonaru, Mathematical Reflections

Soluție. Folosind Inegalitatea Bergström, obținem

$$\sum\frac{a}{b+c}+4\sum\frac{a}{2a+b+c}=\sum\frac{a^2}{ab+ac}+\sum\frac{(2a)^2}{2a^2+ab+ac}\geq$$
$$\geq\frac{\left(\sum 3a\right)^2}{2\sum a^2+4\sum ab}=\frac{9\left(\sum a\right)^2}{2\left(\sum a\right)^2}=\frac{9}{2}.$$

Problema 12. Pentru a,b,c numere reale pozitive, demonstrați inegalitatea

$$a\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+c\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\frac{18}{a+b+c}.$$

Florin Stănescu, Recreații Matematice 1/2012

Soluția 1. Aplicând Inegalitatea Bergström, obținem

$$\sum a\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\sum\left(\frac{\left(\frac{1}{b}\right)^2}{\frac{1}{a}}+\frac{\left(\frac{1}{c}\right)^2}{\frac{1}{a}}\right)\geq\frac{\left(2\sum\frac{1}{a}\right)^2}{2\sum\frac{1}{a}}=2\sum\frac{1}{a}$$

și folosind Inegalitatea MA-MH, $\left(\sum\frac{1}{a}\right)\left(\sum a\right)\geq 9$, obținem inegalitatea din enunț.

Soluția 2. Putem presupune că $a\leq b\leq c$ fără a restrânge generalitatea. Ca urmare, $\frac{1}{b^2}+\frac{1}{c^2}\leq\frac{1}{c^2}+\frac{1}{a^2}\leq\frac{1}{a^2}+\frac{1}{b^2}$. Aplicând Inegalitatea Cebîșev, obținem

$$\sum a\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\frac{1}{3}\left(\sum a\right)\left(\sum\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\right)=\frac{2}{3}\left(\sum a\right)\left(\sum\frac{1}{a^2}\right)\geq$$
$$\geq\frac{2}{9}\left(\sum a\right)\left(\sum\frac{1}{a}\right)^2\geq 2\sum\frac{1}{a}\geq\frac{18}{\sum a},$$

unde am folosit Inegalitatea Cauchy-Schwarz $3\sum\frac{1}{a^2}\geq\left(\sum\frac{1}{a}\right)^2$ și Inegalitatea MA-MH $\left(\sum a\right)\left(\sum\frac{1}{a}\right)\geq 9$.

Soluția 3. Aplicând Inegalitatea MA-MG, obținem $\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{2}{bc}$ și analoagele ei. Avem

$$\sum a\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\geq 2\sum\frac{a}{bc}=2\sum\frac{a^2}{abc}.$$

Folosind Inegalitatea Bergström, obținem

$$\sum\frac{a^2}{abc}\geq\frac{\left(\sum a\right)^2}{3\prod a},$$

și folosind Inegalitatea MA-MG în forma $\left(\sum a\right)^3\geq 27\prod a$, obținem concluzia.

Problema 13. Fie $a,b,c\in(0,\infty)$. Demonstrați că

$$\frac{a}{a+2b+2c}+\frac{b}{2a+b+2c}+\frac{c}{2a+2b+c}\geq\frac{3}{5}.$$

Theodorus Sampas, Mathematical Inequalities

Soluția 1. Folosind Inegalitatea Bergström, avem

$$\sum \frac{a}{a+2b+2c} = \sum \frac{a^2}{a^2+2ab+2ac} \geq \frac{\left(\sum a\right)^2}{\sum a^2 + 4\sum ab} = \frac{\sum a^2 + 2\sum ab}{\sum a^2 + 4\sum ab} \geq \frac{3}{5},$$

deoarece ultima inegalitate este echivalentă cu $\sum a^2 \geq \sum ab$.

Soluția 2. Notăm $a+2b+2c = x, 2a+b+2c = y, 2a+2b+c = z; x, y, z > 0$. Rezultă

$$a = \frac{-3x+2y+2z}{5}, b = \frac{2x-3y+2z}{5}, c = \frac{2x+2y-3z}{5}.$$

Inegalitatea de demonstrat devine

$$-\frac{9}{2} + \frac{2}{5} \cdot \sum \left(\frac{x}{y} + \frac{y}{x}\right) \geq \frac{3}{5},$$

echivalentă cu

$$\sum \left(\frac{x}{y} + \frac{y}{x}\right) \geq 6,$$

care este adevărată din inegalitatea mediilor.

Problema 14. Fie $n \in N, n \geq 2$ și $a_1, a_2, ..., a_n$ numere reale strict pozitive astfel încât $a_1 + a_2 + ... + a_n = 1$. Demonstrați că

$$\frac{a_1}{2-a_1} + \frac{a_2}{2-a_2} + ... + \frac{a_n}{2-a_n} \geq \frac{n}{2n-1}.$$

Olimpiada Balcanică de Matematică

Soluția 1. Notăm $b_1 = 2-a_1, b_2 = 2-a_2, ..., b_n = 2-a_n$. Rezultă $b_1 + b_2 + ... + b_n = 2n-1$.

Inegalitatea de demonstrat devine

$$\frac{2-b_1}{b_1} + \frac{2-b_2}{b_2} + ... + \frac{2-b_n}{b_n} \geq \frac{n}{2n-1},$$

echivalentă cu

$$\frac{1}{b_1} + \frac{1}{b_2} + ... + \frac{1}{b_n} \geq \frac{n^2}{2n-1},$$

care rezultă ușor din Inegalitatea Bergström

$$\frac{1}{b_1} + \frac{1}{b_2} + ... + \frac{1}{b_n} \geq \frac{(1+1+...+1)^2}{b_1+b_2+...+b_n} = \frac{n^2}{2n-1}.$$

Soluția 2. Folosind Inegalitatea Bergström, avem

$$\sum_{i=1}^{n} \frac{a_i}{2-a_i} = \sum_{i-1}^{n} \frac{a_i^2}{2a_i - a_i^2} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{2\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i^2} = \frac{1}{2 - \sum_{i=1}^{n} a_i^2} \geq \frac{n}{2n-1},$$

ultima inegalitate rezultă din aplicarea inegălității Cauchy-Schwarz. Într-adevăr

$$n\sum_{i=1}^{n} a_i^2 \geq \left(\sum_{i=1}^{n} a_i\right)^2 = 1,$$

echivalentă cu ultima inegalitate ce trebuia dovedită.

Problema 15. Fie a,b,c numere reale pozitive astfel încât $a+b+c=3$. Demonstrați că

$$\frac{a}{a^2+bc+1}+\frac{b}{b^2+ca+1}+\frac{c}{c^2+ab+1}\le 1.$$

An Zhenping, Mathematical Reflections

Soluție. Avem $a^2+1\ge 2a, b^2+1\ge 2b$ și $c^2+1\ge 2c$. Atunci

$$\sum\frac{a}{a^2+bc+1}\le\sum\frac{a}{2a+bc}.$$

Deci este suficient să demonstrăm că

$$\sum\frac{a}{2a+bc}\le 1.$$

Scriem această inegalitate în forma

$$\sum\left(\frac{1}{2}-\frac{a}{2a+bc}\right)\ge\frac{1}{2}$$

echivalent cu

$$\sum\frac{bc}{2a+bc}\ge 1.$$

Din Inegalitatea Bergström, obținem

$$\sum\frac{bc}{2a+bc}=\sum\frac{(bc)^2}{2abc+(bc)^2}\ge\frac{\left(\sum bc\right)^2}{6abc+\sum(bc)^2}=\frac{\sum(bc)^2+2abc\sum a}{\sum(bc)^2+6abc}=1,$$

și inegalitatea este demonstrată.

Problema 16. Fie a,b,c numere pozitive. Demonstrați că

$$\frac{bc}{(2a+b)(2a+c)}+\frac{ca}{(2b+c)(2b+a)}+\frac{ab}{(2c+a)(2c+b)}\ge\frac{1}{3}.$$

An Zhenping, Mathematical Reflections

Soluție. Din Inegalitatea Bergström , avem

$$\sum\frac{bc}{(2a+b)(2a+c)}=\sum\frac{(bc)^2}{4a^2bc+2abc^2+2ab^2c+b^2c^2}\ge\frac{\left(\sum bc\right)^2}{8abc\sum a+\sum b^2c^2}.$$

Este suficient să demonstrăm că

$$3\left(\sum b^2c^2+2abc\sum a\right)\ge 8abc\sum a+\sum b^2c^2,$$

echivalent cu

$$\sum b^2c^2\ge abc\sum a.$$

Avem

$$\sum b^2c^2\ge\sum abbc=abc\sum a.$$

Aceasta încheie demonstrația problemei.

5. Teorema lui Kiril Docev

Ioan Viorel Codreanu, Școala Gimnazială Satulung, Maramureș

În cele ce urmează vom prezenta o formulă pentru calcularea ariei unui poligon convex, în funcție de afixele vîrfurilor sale.

Teoremă (Kiril Docev). Fie poligonul convex $A_1A_2\cdots A_n$ orientat în sens trigonometric, cu $a_1, a_2, \cdots, a_n \in \mathbb{C}$ afixele vârfurilor sale. Atunci aria sa este:

$$S_{A_1A_2\cdots A_n} = \frac{1}{2} Im(\overline{a_1}a_2 + \overline{a_2}a_3 + \cdots + \overline{a_n}a_1)$$

Demonstrație. Observăm că dacă ABC este un triunghi orientat în sens trigonometric și vârfurile sale au afixele a, b, respectiv c, atunci

$$2A_{ABC} = AB \cdot AC \cdot \sin A = |b-a| \cdot |c-a| \cdot \sin A = Im\left((c-a)\overline{(b-a)}\right).$$

Fie $P(p)$ un punct în interiorul poligonului $A_1A_2\cdots A_n$. Atunci triunghiurile PA_kA_{k+1} sunt orientate trigonometric, deci

$$2S_{A_1A_2\ldots A_n} = 2\sum_{k=1}^{n} S_{PA_kA_{k+1}} = Im\sum_{k=1}^{n}\overline{(a_k-p)}\,(a_{k+1}-p) =$$

$$= Im\sum_{k=1}^{n}\overline{a_k}\,a_{k+1} - Im\sum_{k=1}^{n}(p\overline{a_k} + \bar{p}a_{k+1}) + nIm|p|^2 =$$

$$= Im(\overline{a_1}a_2 + \overline{a_2}a_3 + \cdots + \overline{a_n}a_1).$$

Vom aplica teorema lui Kiril Docev la rezolvarea a două problem de geometrie

Problema 1. Fie P un punct interior triunghiului ABC și H_A, H_B, H_C ortocentrele triunghiurilor PBC, PAC, respectiv PAB. Demonstrați că triunghiurile $H_AH_BH_C$ și ABC au aceeași arie.

Mathproblems, Kosovo

***Soluție*.** Fie O central cercului circumscris triunghiului ABC. Alegem originea axelor în O și considerăm: $A(a), B(b), C(c), P(p), H_A(h_1), H_B(h_2), H(h_3)$. Deoarece H_A, H_B, H_C sunt ortocentrele triunghiurilor PBC, PAC, respectiv PAB, avem

$$h_1 = p+b+c, h_2 = p+a+c, h_3 = p+a+b$$

Folosind **Teorema lui Kiril Docev** pentru triunghiul $H_AH_BH_C$ obținem

$$S_{H_AH_BH_C} = \frac{1}{2}\mathrm{Im}\left(h_2\overline{h_1} + h_3\overline{h_2} + h_1\overline{h_3}\right)$$

$$= \frac{1}{2}\mathrm{Im}\left((p+a+b)(\overline{p}+\overline{b}+\overline{c}) + (p+a+b)(\overline{p}+\overline{a}+\overline{c}) + (p+b+c)(\overline{p}+\overline{a}+\overline{b})\right)$$

După efectuarea calculelor, folosind proprietățile părții imaginare $\mathrm{Im}(z\overline{z}) = \mathrm{Im}(|z|^2) = 0$ și $\mathrm{Im}(z_1\overline{z_2}) = -\mathrm{Im}(\overline{z_1}z_2)$, obținem

$$S_{H_AH_BH_C}=\frac{1}{2}\mathrm{Im}\left(3p\bar{p}+\sum a\bar{a}+2\sum\left(p\bar{a}+\bar{p}a\right)+\sum\left(a\bar{b}+\bar{a}b\right)+b\bar{a}+c\bar{b}+a\bar{c}\right)$$
$$=\frac{1}{2}\mathrm{Im}\left(b\bar{a}+c\bar{b}+a\bar{c}\right)=S_{ABC}$$

unde în final am folosit **Teorema lui Kiril Docev** pentru triughiul ABC.

Problema 2. Se consideră triunghiul ABC și punctele M, N, P, Q, R, S definite prin $\vec{BM}=k\vec{MC},\ \vec{CN}=k\vec{NA},\ \vec{AP}=k\vec{PB},\ \vec{AM}=p\vec{MQ},\ \vec{BN}=p\vec{NR},\ \vec{CP}=p\vec{PS}$ **, unde k, p** $\in\mathbb{R}^*\setminus\{-1\}$**. Demonstrați că** $S_{MNP}\geq\frac{1}{4}\cdot S_{ABC}$**, iar** $S_{QRS}\geq\left(\frac{p+3}{2p}\right)^2\cdot S_{ABC}$ **.**

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Soluție. Fie în planul complex punctele A(a), B(b), C(c), M(m), N(n), P(z), Q(q), R(r) și S(s). Ținând cont de felul cum au fost definite punctele M, N, P, Q, R și S avem:

$$m=\frac{b+kc}{1+k},\ n=\frac{c+ka}{1+k},z=\frac{a+kb}{1+k},q=\frac{m(1+p)-a}{p},r=\frac{n(1+p)-b}{p}\ \text{și}\ s=\frac{z(1+p)-c}{p}.$$

Aplicând **Teorema lui Kiril Docev**, obținem:

$$S_{MNP}=\frac{1}{2}\mathrm{Im}\left(n\bar{m}+z\bar{n}+m\bar{z}\right)=\frac{1}{2(k+1)^2}\mathrm{Im}\left[(c+ka)(\bar{b}+k\bar{c})+(a+kb)(\bar{c}+k\bar{a})+(b+kc)(\bar{a}+k\bar{b})\right]=$$
$$=\frac{1}{2(k+1)^2}\left[\mathrm{Im}\left(b\bar{a}+c\bar{b}+a\bar{c}\right)+k\mathrm{Im}\left(\overline{b\bar{a}+c\bar{b}+a\bar{c}}\right)+k\mathrm{Im}\left(|a|^2+|b|^2+|c|^2\right)+k^2\mathrm{Im}\left(b\bar{a}+c\bar{b}+a\bar{c}\right)\right]$$

după care folosind din nou **Teorema lui Kiril Docev** $S_{ABC}=\frac{1}{2}\mathrm{Im}\left(b\bar{a}+c\bar{b}+a\bar{c}\right)$ și faptul că

$\mathrm{Im}\left(|a|^2+|b|^2+|c|^2\right)=0$, obținem $S_{MNP}=\frac{k^2-k+1}{(k+1)^2}S_{ABC}$.

Avem

$$\frac{k^2-k+1}{(k+1)^2}\geq\frac{1}{4}\Leftrightarrow 4k^2-4k+4\geq k^2+2k+1\Leftrightarrow 3(k-1)^2\geq 0.$$

Așadar $S_{MNP}\geq\frac{1}{4}S_{ABC}$. Egalitatea are loc pentru k = 1, adică pentru triunghiul MNP median.

Folosind **Teorema lui Kiril Docev**, obținem:

$$S_{QRS}=\frac{1}{2}\mathrm{Im}\left(r\bar{q}+s\bar{r}+q\bar{s}\right)=$$
$$=\frac{1}{2p^2}\mathrm{Im}\left[(n(1+p)-b)(\bar{m}(1+p)-\bar{a})+(z(1+p)-c)(\bar{n}(1+p)-\bar{b})+(m(1+p)-a)(\bar{z}(1+p)-\bar{c})\right]$$

$$=\frac{1}{2p^2}\left[(1+p)^2\,\mathrm{Im}\left(n\overline{m}+z\overline{n}+m\overline{z}\right)+\mathrm{Im}\left(b\overline{a}+c\overline{b}+a\overline{c}\right)-(1+p)\mathrm{Im}\left(b\overline{m}+n\overline{a}+z\overline{b}+c\overline{n}+m\overline{c}+a\overline{z}\right)\right]$$

Expresia din ultima paranteză o scriem

$$b\overline{m}+n\overline{a}+z\overline{b}+c\overline{n}+m\overline{c}+a\overline{z}=\overline{c\overline{m}+n\overline{c}+m\overline{n}}+\overline{a\overline{n}+z\overline{a}+n\overline{z}}+\overline{b\overline{z}+m\overline{b}+z\overline{m}}$$

$-\left(n\overline{m}+z\overline{n}+m\overline{z}\right)$ și prin trecere la părți imaginare, obținem:

$$\mathrm{Im}\left(b\overline{m}+n\overline{a}+z\overline{b}+c\overline{n}+m\overline{c}+a\overline{z}\right)=-2\left(S_{MCN}+S_{NAP}+S_{PBM}+S_{MNP}\right)=-2S_{ABC}$$

Atunci

$$S_{QRS}=\frac{1}{p^2}\left[(p+1)^2S_{MNP}+(p+2)S_{ABC}\right]\geq\frac{1}{p^2}\left[\frac{(p+1)^2S_{ABC}}{4}+(p+2)S_{ABC}\right]=\left(\frac{p+3}{2p}\right)^2S_{ABC}.$$