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1. PROBLEMA LUNII APRILIE 2018

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If in a convex pentagon with congruent angles one of the sides is equal to the sum of adjacent sides then at least two sides have an irrational length.

Dacă într-un pentagon convex cu unghiurile congruente una din laturi este egala cu suma laturilor alăturate, atunci cel puțin două laturi au lungimea un număr irațional.

Prof. Cantemir Iliescu, Pitești

We are looking for the most interesting solutions of the problem at e-mail ***revista@mateinfo.ro*** **.**

Așteptăm rezolvări cât mai interesante pe adresa de e-mail ***revista@mateinfo.ro****.*

Deadline: 1 mai 2018

Termen: 1 mai 2018.

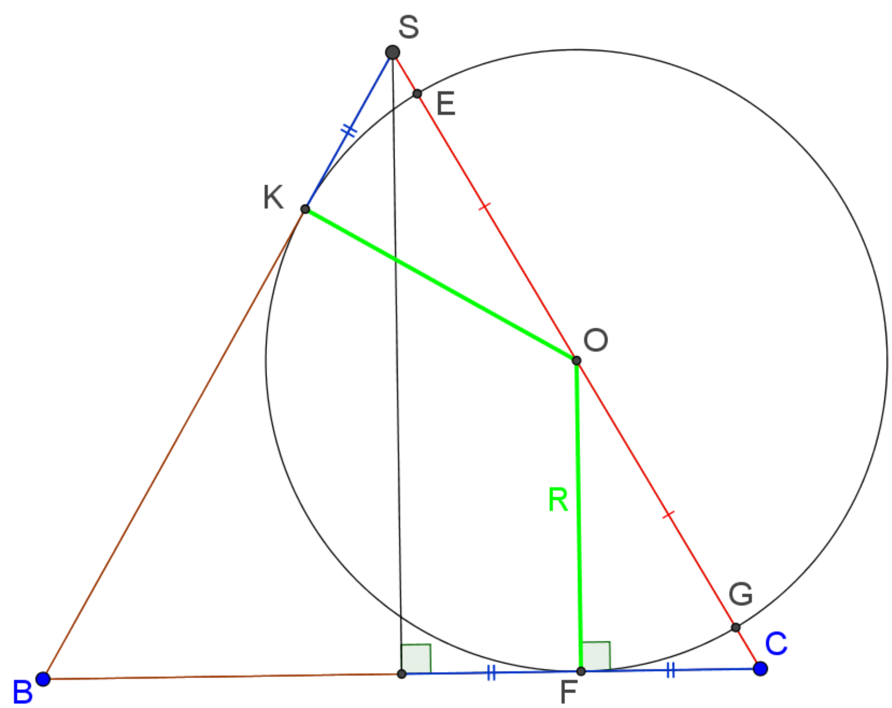
2. REZOLVARE – PROBLEMA LUNII FEBRUARIE 2018

Se dă o piramidă patrulateră regulată SABCD cu toate muchiile de lungime x și un corp sferic cu centrul în mijlocul O al muchiei $[SC]$ *, de rază* $\frac{x\sqrt{3}}{4}$ *. Notăm cu* Ω *intersecția celor două corpuri. Să se arate că* $\frac{V_{PIRAMID\breve{A}}}{V_{\Omega}} < 2,5.$

Autor: Constantin Telteu

I. Rezolvare Constantin Telteu

În figura de mai jos am desenat intersecția sferei cu fața *SBC*. Deoarece *OF* este jumătate din înălțimea triunghiului *SBC*, avem $OF = OK = \frac{x\sqrt{3}}{4} = R$.

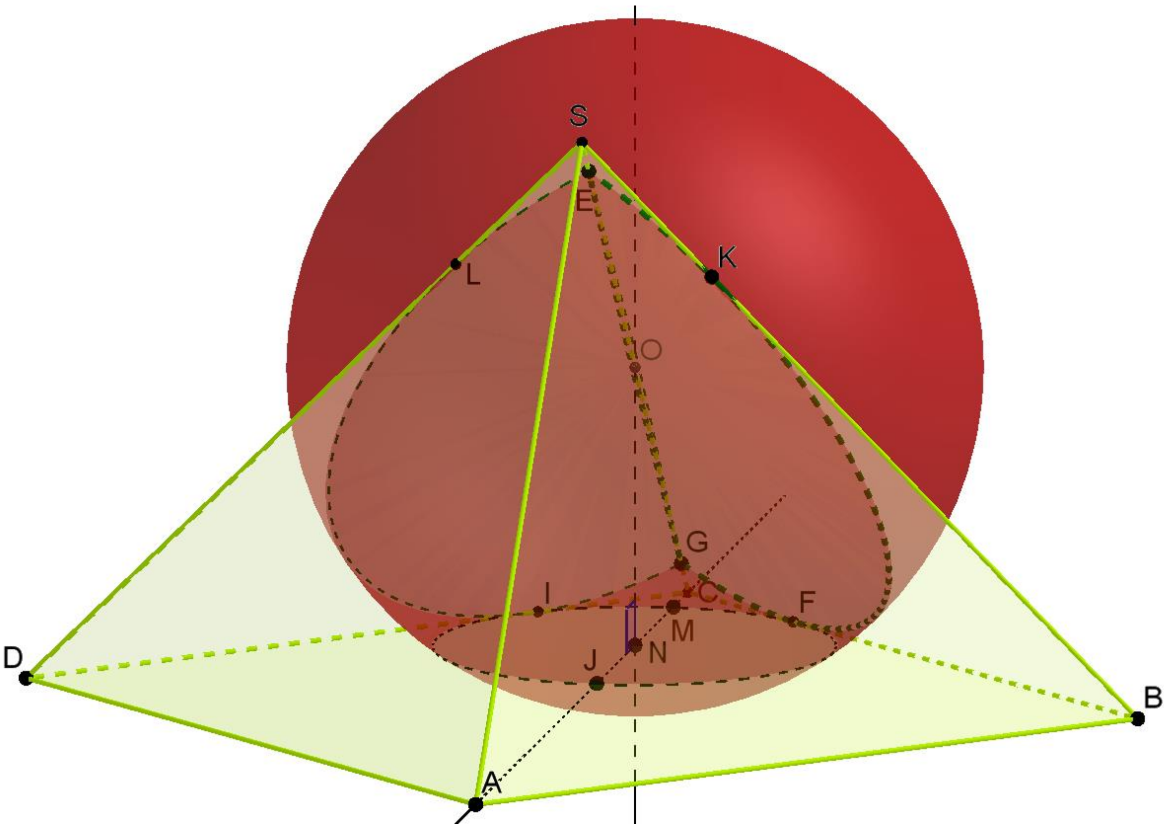


În figura următoare sunt desenate cele două corpuri.
Din figură se observă că intersecția celor două corpuri este formată dintr-o „felie” (mărginită de un biunghi, sau fus sferic) de corp sferic din care lipsește porțiunea mărginită de o calotă a sferei și baza piramidei (aceasta este un sector sferic din care s-a scăzut un con circular drept).
Din desenul precedent și cel următor, deducem că sfera este tangentă la muchiile $[SD],[SB],[BC],[DC]$ și intersectează baza piramidei după un cerc tangent muchiilor

$[BC],[DC]$. Sfera are doar câte un punct comun cu fețele $[SBC]$ și $[SDC]$, deoarece unghiul diedru a două fețe laterale ale piramidei este mai mare de $\frac{\pi}{2}$. Demonstrăm ultima afirmație:

$$BO^2+DO^2=2\cdot\left(\frac{x\sqrt{3}}{2}\right)^2=\frac{6x^2}{4}<2x^2=BD^2\Rightarrow m\left(BOD\right)>\frac{\pi}{2}\ .$$

Volumul corpului mărginit de fusul sferic și semidiscurile ce-l mărginesc are formula (care se obține imediat cu regula de trei simplă) $V_{felie}=\frac{2R^3}{3}\cdot\alpha$, când α este măsura în radiani a unghiului diedru determinat de planele celor două semidiscuri ce mărginesc „felia”.



Determinăm acum pe $\alpha=m\left(\left(SDC\right),\left(SBC\right)\right)=m\left(BOD\right)$.

Pentru aceasta scriem în două feluri aria triunghiului *BOD*, ținând cont că înălțimea sa din O este $\frac{x}{2}$, fiind linie mijlocie în triunghiul SAC, obținem:

$$BD\cdot\frac{x}{2}\cdot\frac{1}{2}=OD\cdot OB\cdot\sin BOD\cdot\frac{1}{2}\Rightarrow\frac{x^2\sqrt{2}}{4}=\left(\frac{x\sqrt{3}}{2}\right)^2\cdot\sin BOD\cdot\frac{1}{2}\Rightarrow\sin BOD=\frac{2\sqrt{2}}{3}\Rightarrow$$

$$\Rightarrow m\left(BOD\right)=\pi-\arcsin\frac{2\sqrt{2}}{3}\Rightarrow V_{felie}=\frac{2}{3}\left(\frac{x\cdot\sqrt{3}}{4}\right)^{3}\cdot\left(\pi-\arcsin\frac{2\sqrt{2}}{3}\right)=\frac{x^{3}\sqrt{3}}{32}\cdot\left(\pi-\arcsin\frac{2\sqrt{2}}{3}\right).$$

Deci unghiul dintre două fețe laterale ale piramidei este de

$$\pi-\arcsin\frac{2\sqrt{2}}{3}\approx 109,5^{\circ}\approx\left(\pi-1,23\right)rad\approx 1,9116\,rad$$

Pentru calculul volumului „calotei pline”, calculăm mai întâi înălțimea ei. Din figura precedentă (ON este jumătate din înălțimea piramidei, ΔSAC este dreptunghic isoscel) avem:

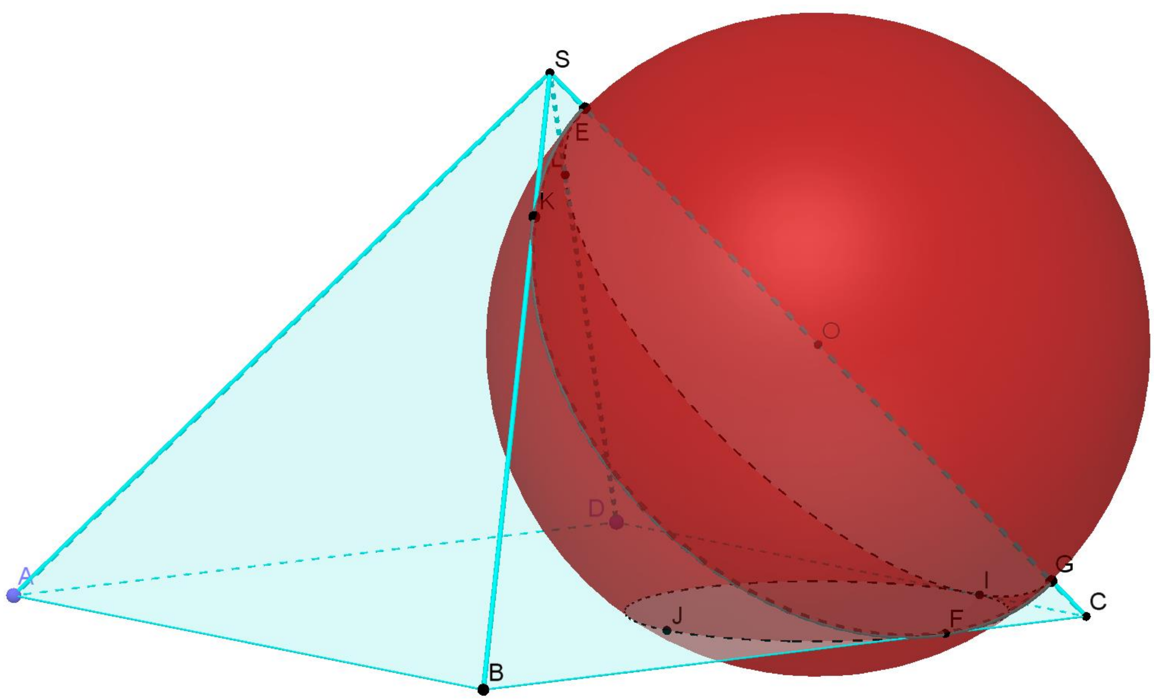
$$h_{calotă}=R-ON=\frac{x\sqrt{3}}{4}-\frac{x\sqrt{2}}{4}=\frac{x}{4}\left(\sqrt{3}-\sqrt{2}\right).$$

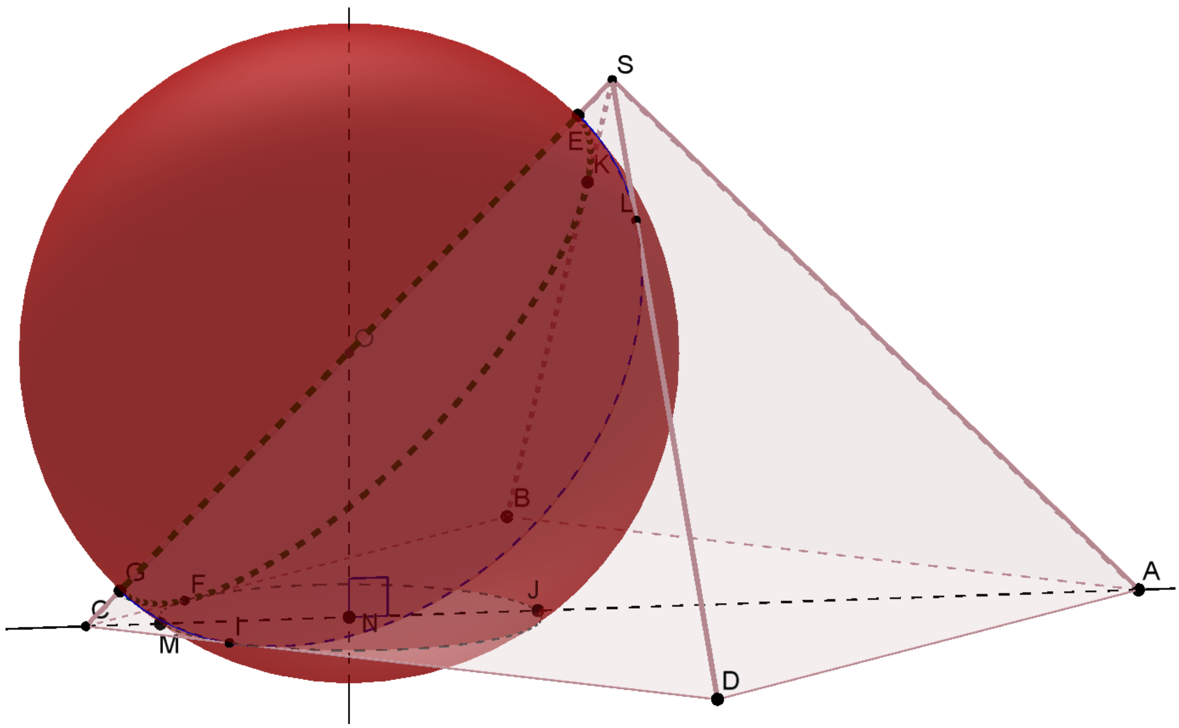
$$V_{calotă}=\frac{\pi h^{2}\left(3R-h\right)}{3}=\frac{\pi x^{2}}{16}\left(\sqrt{3}-\sqrt{2}\right)^{2}\left[\frac{3x\sqrt{3}}{4}-\frac{x}{4}\left(\sqrt{3}-\sqrt{2}\right)\right]\cdot\frac{1}{3}=...=\frac{\pi x^{3}}{192}\left(6\sqrt{3}-7\sqrt{2}\right).$$

Intersecția celor două corpuri are volumul:

$$V_{\Omega}=V_{felie}-V_{calotă}=\frac{x^{3}\sqrt{3}}{32}\cdot\left(\pi-\arcsin\frac{2\sqrt{2}}{3}\right)-\frac{\pi x^{3}}{192}\cdot\left(6\sqrt{3}-7\sqrt{2}\right)\approx x^{3}\cdot 0,095352474.$$

$$V_{PIRAMIDA}=\frac{x^{3}\sqrt{2}}{6}\approx x^{3}\cdot 0,2357\Rightarrow\frac{V_{PIRAMIDA}}{V_{\Omega}}\approx 2,47\ .$$

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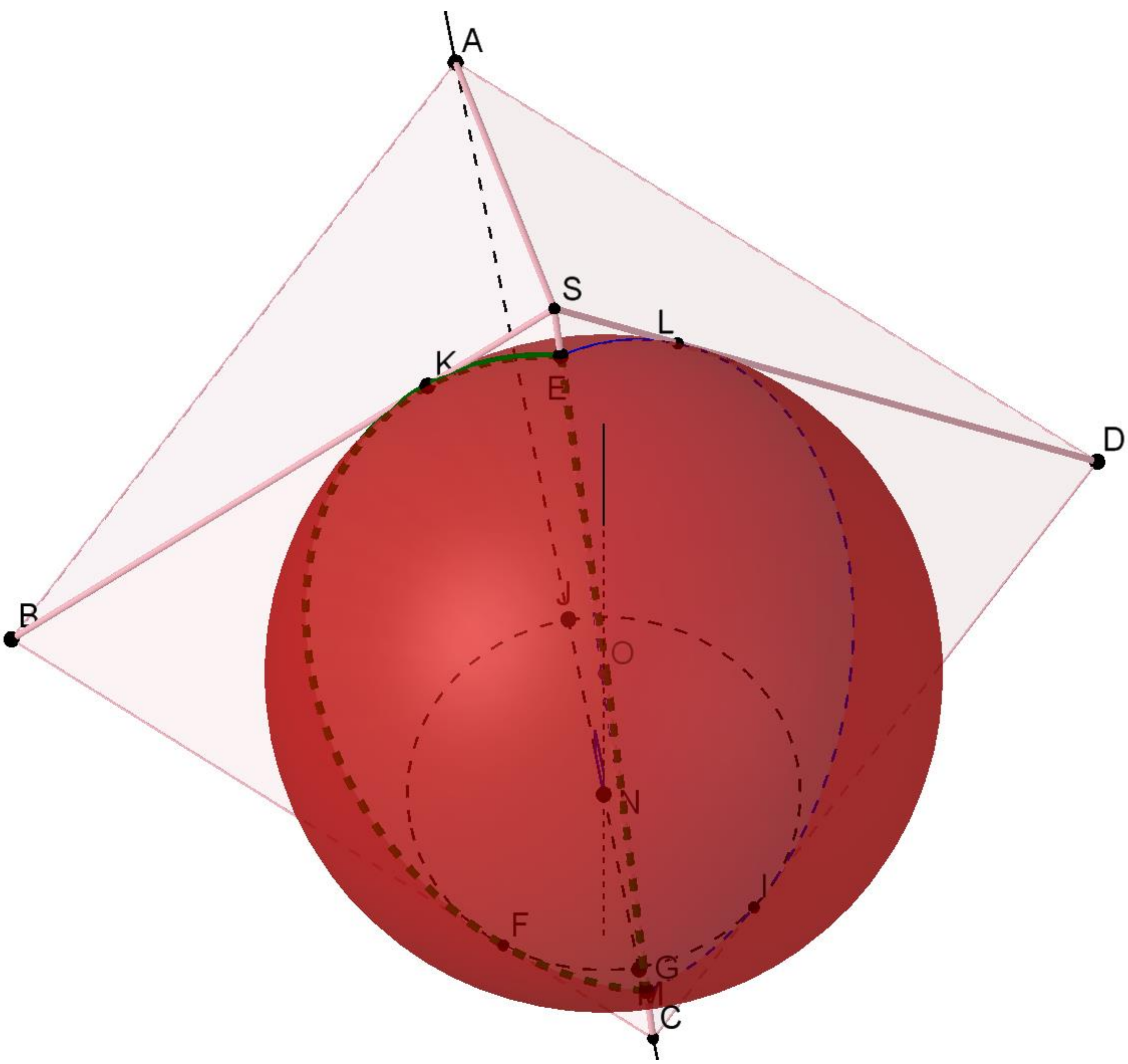
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II. Rezolvare Biro Istvan

Intersecția dintre (SBC), (SDC) și sferă determină un fus sferic (biunghi) iar intersecția sferei cu baza piramidei determină o calotă sferică. Prin urmare volumul corpului sferic Ω va fi $V_\Omega = V_{fus\ sferic} - V_{calotă}$. În continuare folosim notațiile:

$R = \frac{x\sqrt{3}}{4}$, raza sferei

r, raza calotei sferice ($O'P$)

h, înălțimea calotei sferice ($O'M$)

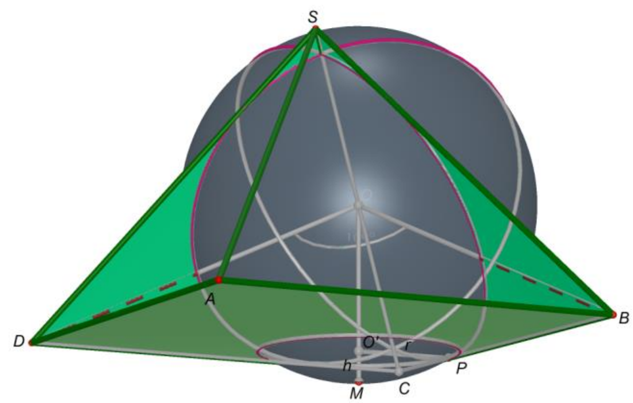
h_p, înălțimea piramidei (SS')

α, măsura unghiului diedru dintre (SBC) și (SDC).

Fețele latelare fiind triunghiuri echilaterale rezultă că $BO = DO,\ BO \perp SC,\ DO \perp SC$ și din teorema cosinusurilor obținem:

$$\alpha = \arccos\left(-\frac{1}{3}\right) = \pi - \arccos\left(\frac{1}{3}\right) \approx 109{,}47^\circ$$

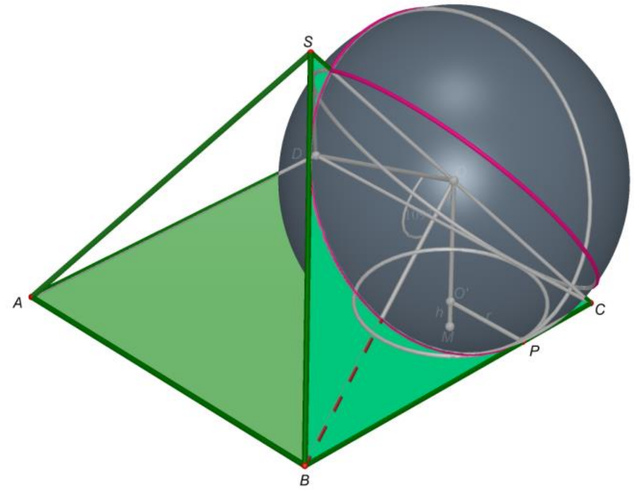
$$V_{fus\ sferic} = \frac{2\alpha R^3}{3} = \frac{\alpha x^3\sqrt{3}}{32}.$$



Evident avem $h_p = \frac{x\sqrt{2}}{2}$ și în triunghiul SS'C, OO' este linie mijlocie, deci $OO' = \frac{SS'}{2} = \frac{x\sqrt{2}}{4}$. Pe de altă parte patrulaterul O'S'BP este inscriptibil și $\beta = m(S'BP) = m(CO'P)$, de unde

$\cos\beta = \frac{S'B}{BC} = \frac{r}{O'C} \Rightarrow r = \frac{x}{4}$. Observăm că în sferă $OM = R$ și

$$O'P \perp OM \Rightarrow r^2 = h(2R - h) \Rightarrow h = \frac{x(\sqrt{3}-\sqrt{2})}{4}.$$



În consecință avem :

$$V_{piramidă} = \frac{x^2 h_p}{3} = \frac{\sqrt{2}x^3}{6}$$

$$V_{calotă} = \frac{\pi h(h^2 + 3r^2)}{6} = \frac{\pi(6\sqrt{3} - 7\sqrt{2})x^3}{192}$$

$$V_\Omega = V_{fus\ sferic} - V_{calotă} = \frac{(7\sqrt{6}\pi + 18\alpha - 18\pi)\sqrt{3}x^3}{576} = \frac{(7\sqrt{6}\pi - 18\arccos\frac{1}{3})\sqrt{3}x^3}{576}$$

$$\frac{V_{piramidă}}{V_{\Omega}}=\frac{32\sqrt{6}}{7\sqrt{6}\pi-18\arccos\frac{1}{3}}=\frac{32}{7\pi-3\sqrt{6}\arccos\frac{1}{3}}.$$

În continuare putem folosi următoarele relații:

(1) $\arcsin x+\arccos x=\frac{\pi}{2},\quad |x|\leq 1$

(2) $\frac{3x}{2+\sqrt{1-x^2}}\leq \arcsin x\leq \frac{\pi x}{2+\sqrt{1-x^2}},\quad 0\leq x\leq 1\ \ (Shafer-Fink)$

(3) $2400<2401\Leftrightarrow 6\cdot 20^2<49^2\Leftrightarrow \sqrt{6}<\frac{^{5)}49}{20}=\frac{245}{100}=2,45$

(4) $288<289\Leftrightarrow 3^2\cdot 2\cdot 4^2<17^2\Leftrightarrow 3\sqrt{2}<\frac{^{25)}17}{4}=\frac{425}{100}=4,25$

(5) $3,14<\pi<3,15$

Din (1), (2) și (5) deducem că $\arccos\frac{1}{3}\leq \frac{7\pi-9+3\sqrt{2}}{14}<\frac{7\cdot 3,15-9+4,25}{14}=\frac{17,3}{14}$, iar folosind (3),(4) și (5) obținem inegalitatea cerută:

$$\frac{V_{piramidă}}{V_{\Omega}}=\frac{32}{7\pi-3\sqrt{6}\arccos\frac{1}{3}}<\frac{32}{7\cdot 3,14-3\cdot 2,45\cdot\frac{17,3}{14}}=\frac{448}{307,72-127,155}=\frac{448}{180,565}<\frac{448}{179,2}=2,5$$

3. THE NUMBERS of FIBONACCI and LUCAS - IDENTITIES
- PROOFS WITH FEW WORDS –
(VI)

By Dumitru M. Bătineţu-Giurgiu, Bucharest, Romania
and Neculai Stanciu, Buzău, Romania

Fibonacci

(1175 -1240)

François-Édouard-Anatole Lucas

(1842 – 1891)

$$F_0 = 0, F_1 = 1,$$
$$F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbf{N} \qquad \text{(F)}$$
$$L_0 = 2, L_1 = 1,$$
$$L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbf{N} \qquad \text{(L)}$$
$$r^2 - r - 1 = 0,$$
$$r_1 = \alpha = \frac{1+\sqrt{5}}{2}, r_2 = \beta = \frac{1-\sqrt{5}}{2}.$$

$(x_n)_{n\ge 0}$, **Fibonacci-Lucas**' s sequence
$$x_n = A\alpha^n + B\beta^n,\ \forall n \in \mathbf{N},$$

If $x_0 = 0 = F_0, x_1 = 1 = F_1$, then $A = \frac{1}{\sqrt{5}}, B = -\frac{1}{\sqrt{5}}$ so:
$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right),\ \forall n \in \mathbf{N} \text{ (**Binet**, 1843)},$$

If $x_0 = 2 = L_0, x_1 = 1 = L_1$, then $A = B = 1$, so
$$L_n = \alpha^n + \beta^n,\ \forall n \in \mathbf{N}.$$

Note that:
$$\alpha + \beta = 1 \text{ and } \alpha\beta = -1,$$

1.147. $F_{2n} = (F_{n+2} - F_{n-2})F_n,\ \forall n \in \mathbf{N}^* - \{1\}.$

Proof.
$$(F_{n+2} - F_{n-2})F_n = \frac{1}{(\alpha-\beta)^2}(\alpha^{n+2} - \beta^{n+2} - \alpha^{n-2} + \beta^{n-2})(\alpha^n - \beta^n) =$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha^{2n+2} - \alpha^n\beta^{n+2} - \alpha^{2n-2} + \alpha^n\beta^{n-2} - \alpha^{n+2}\beta^n + \beta^{2n+2} + \alpha^{n-2}\beta^n - \beta^{2n-2}) =$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha^{2n}(\alpha^2 - \alpha^{-2}) + \beta^{2n}(\beta^2 - \beta^{-2}) - \alpha^n\beta^n(\beta^2 - \beta^{-2} + \alpha^2 - \alpha^{-2})) =$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha^2 - \beta^2)(\alpha^{2n} - \beta^{2n}) - \frac{\alpha^n\beta^n}{(\alpha-\beta)^2}(\beta^2 - \alpha^2 + \alpha^2 - \beta^2) =$$
$$= \frac{(\alpha+\beta)(\alpha-\beta)}{(\alpha-\beta)^2}(\alpha^{2n} - \beta^{2n}) = \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) = F_{2n}.$$

1.148. $(L_{n+2} - L_{n-2})L_n = 5F_{2n},\ \forall n \in \mathbf{N}^*.$

Proof. $(L_{n+2} - L_{n-2})L_n = (\alpha^{n+2} + \beta^{n+2} - \alpha^{n-2} - \beta^{n-2})(\alpha^n + \beta^n) =$
$$= \alpha^{2n+2} + (\alpha\beta)^n\beta^2 - (\alpha\beta)^n\beta^{-2} - \alpha^{2n-2} + (\alpha\beta)^n\alpha^2 + \beta^{2n+2} - (\alpha\beta)^n\alpha^{-2} - \beta^{2n-2} =$$
$$= \alpha^{2n+2} - \alpha^{2n-2} + \beta^{2n+2} - \beta^{2n-2} + (\alpha\beta)^n(\beta^2 - \beta^{-2} + \alpha^2 - \alpha^{-2}) =$$
$$= \alpha^{2n}(\alpha^2 - \alpha^{-2}) + \beta^{2n}(\beta^2 - \beta^{-2}) = (\alpha^2 - \beta^2)(\alpha^{2n} - \beta^{2n}) =$$

$$=(\alpha-\beta)(\alpha^{2n}-\beta^{2n})=(\alpha-\beta)^2\cdot\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}=5F_{2n}.$$

1.149. $\sum_{k=0}^{n} C_{2k}(x)=B_n(x)C_n(x),\ \forall n\in\mathbf{N}^*$ (**Swamy**, 1966).

Proof. By **1.141** we have $C_{2k}(x)=B_k(x)C_k(x)-B_{k-1}(x)C_{k-1}(x),\forall k\in N^*$. So:

$$\sum_{k=0}^{n} C_{2k}(x)=\sum_{k=0}^{n} B_k(x)C_k(x)-\sum_{k=0}^{n} B_{k-1}(x)C_{k-1}(x)=$$

$$=\sum_{k=0}^{n} B_k(x)C_k(x)-\sum_{k=-1}^{n-1} B_k(x)C_k(x)=B_n(x)C_n(x)-B_{-1}(x)C_{-1}(x)$$, but

$B_{-1}(x)=0$ and $C_{-1}(x)=1$ and we are done.

1.150. $(x+2)B_{2n-1}(x)=B_n^2(x)-B_{n-1}^2(x),\ \forall n\in\mathbf{N}^*$.

Proof. $B_n^2(x)-B_{n-1}^2(x)=\frac{(r^n(x)-s^n(x))^2-(r^{n-1}(x)-s^{n-1}(x))^2}{(r(x)-s(x))^2}=$

$$=\frac{1}{(r(x)-s(x))^2}(r^{2n}(x)+s^{2n}(x)-2(r(x)s(x))^n-r^{2n-2}(x)-s^{2n-2}(x)+2(r(x)s(x))^{n-1})=$$

$$=\frac{1}{(r(x)-s(x))^2}(r^{2n-1}(x)(r(x)-(r(x))^{-1})+s^{2n-1}(x)(s(x)-(s(x))^{-1}))=$$

$$=\frac{r^{2n-1}(x)-s^{2n-1}(x)}{r(x)-s(x)}=B_{2n-1}(x).$$

1.151. $xB_n(x)=(x+1)C_n(x)-C_{n-1}(x),\ \forall n\in\mathbf{N}^*$.

Proof. $xB_n(x)=C_{n+1}(x)-C_n(x),\ \forall n\in\mathbf{N}$, but

$$C_{n+1}(x)=(x+2)C_n(x)-C_{n-1}(x),\ \forall n\in\mathbf{N}^*$$

$$xB_n(x)=(x+2)C_n(x)-C_{n-1}(x)-C_n(x)=(x+1)C_n(x)-C_{n-1}(x).$$

1.152. $C_{2n+1}(x)=B_n(x)C_{n+1}(x)-B_{n-1}(x)C_n(x),\ \forall n\in\mathbf{N}^*$.

Proof. In **1.140** we demonstrate that:

$C_{m+n}(x)=B_m(x)C_n(x)-B_{m-1}(x)C_{n-1}(x),\ \forall m,n\in\mathbf{N}^*$, and taking $m=n+1$ yields that $C_{2n+1}(x)=B_n(x)C_{n+1}(x)-B_{n-1}(x)C_n(x)$.

1.153. $\sum_{k=0}^{n} B_{2k}(x)=B_n^2(x),\ \forall n\in\mathbf{N}^*$ (**Swamy**, 1966).

Proof. $B_{m+n}(x)=B_m(x)B_n(x)-B_{m-1}(x)B_{n-1}(x),\ \forall m,n\in\mathbf{N}^*$, so:

$$B_{2k}(x)=B_k^2(x)-B_{k-1}^2(x),\forall k\in N^* \Leftrightarrow B_{2k+2}(x)=B_{k+1}^2(x)-B_k^2(x),\ \forall k\in\mathbf{N}, \text{ then}$$

$$\sum_{k=0}^{n}B_{2k}(x)=\sum_{k=0}^{n}B_k^2(x)-\sum_{k=0}^{n}B_{k-1}^2(x)=B_n^2(x)-B_{-1}^2(x),$$

and from $B_{-1}(x)=0$, yields $\sum_{k=0}^{n}B_{2k}(x)=B_n^2(x),\ \forall n\in\mathbf{N}^*$.

1.154. $\sum_{k=1}^{n}B_{2k-1}(x)=B_n(x)B_{n-1}(x),\ \forall n\in\mathbf{N}^*$ (**Swamy**, 1966).

Proof. $B_{2k-1}(x)=B_k(x)B_{k-1}(x)-B_{k-1}(x)B_{k-2}(x),\ \forall k\in\mathbf{N}^*$ so:

$$\sum_{k=1}^{n}B_{2k-1}(x)=\sum_{k=1}^{n}B_k(x)B_{k-1}(x)-\sum_{k=1}^{n}B_{k-1}(x)B_{k-2}(x)=$$

$$=\sum_{k=1}^{n}B_k(x)B_{k-1}(x)-\sum_{k=0}^{n-1}B_k(x)B_{k-1}(x)=B_n(x)B_{n-1}(x)-B_0(x)B_{-1}(x),$$

and since $B_{-1}(x)=0$, yields that $\sum_{k=1}^{n}B_{2k-1}(x)=B_n(x)B_{n-1}(x)$.

1.155. $\sum_{k=1}^{n}C_{2k-1}(x)=B_{n-1}(x)C_n(x),\ \forall n\in\mathbf{N}^*$ (**Swamy**, 1966).

Proof. By **1.152** we obtain: $C_{2k-1}(x)=B_{k-1}(x)C_k(x)-B_{k-2}(x)C_{k-1}(x),\ \forall k\in\mathbf{N}^*$**,** so

$$\sum_{k=1}^{n}C_{2k-1}(x)=\sum_{k=1}^{n}B_{k-1}(x)C_k(x)-\sum_{k=1}^{n}B_{k-2}(x)C_{k-1}(x)=$$

$$=\sum_{k=1}^{n}B_{k-1}(x)C_k(x)-\sum_{k=0}^{n-1}B_{k-1}(x)C_k(x)=B_{n-1}(x)C_n(x)-B_{-1}(x)C_0(x),$$

and by $B_{-1}(x)=0$ we get $\sum_{k=1}^{n}C_{2k-1}(x)=B_{n-1}(x)C_n(x)$.

1.156. $C_{2n}(x)-C_{2n-1}(x)=C_n^2(x)-C_{n-1}^2(x),\ \forall n\in\mathbf{N}^*$.

Proof. By **1.141** we have

$$C_{2n}(x)=B_n(x)C_n(x)-B_{n-1}(x)C_{n-1}(x),\ \forall n\in\mathbf{N}^*,$$

and by **1.152** we have

$$C_{2n-1}(x)=B_{n-1}(x)C_n(x)-B_{n-2}(x)C_{n-1}(x),\ \forall n\in\mathbf{N}^*, \text{ so:}$$

$$C_{2n}(x)-C_{2n-1}(x)=(B_n(x)-B_{n-1}(x))C_n(x)-(B_{n-1}(x)-B_{n-2}(x))C_{n-1}(x).$$

By **1.136** $C_n(x)=B_n(x)-B_{n-1}(x),\ \forall n\in\mathbf{N}^*$, yields:

$$C_{2n}(x)-C_{2n-1}(x)=C_n^2(x)-C_{n-1}^2(x).$$

1.157. $\sum_{k=0}^{2n}(-1)^k C_k(x)=C_n^2(x),\ \forall n\in\mathbf{N}^*$ (**Swamy**, 1966).

Proof. $\sum_{k=0}^{2n}(-1)^k C_k(x)=C_0-C_1+C_2-C_3+...-C_{2n-1}+C_{2n}=$

$$=\sum_{k=0}^{n}C_{2k}(x)-\sum_{k=1}^{n}C_{2k-1}(x).$$

By **1.149** $\sum_{k=0}^{n}C_{2k}(x)=B_n(x)C_n(x),\ \forall n\in\mathbf{N}^*$, and by **1.155**:

$$\sum_{k=1}^{n}C_{2k-1}(x)=B_{n-1}(x)C_n(x),\ \forall n\in\mathbf{N}^*,\text{ so:}$$

$$\sum_{k=0}^{2n}(-1)^k C_k(x)=\sum_{k=0}^{n}C_{2k}(x)-\sum_{k=1}^{n}C_{2k-1}(x)=B_n(x)C_n(x)-B_{n-1}(x)C_n(x)=$$

$=(B_n(x)-B_{n-1}(x))C_n(x),\ \forall n\in\mathbf{N}^*$ and $C_n(x)=B_n(x)-B_{n-1}(x),\ \forall n\in\mathbf{N}^*$,

yields $\sum_{k=0}^{2n}(-1)^k C_k(x)=C_n^2(x)$.

Other proof:

$$\sum_{k=0}^{2n}(-1)^k C_k(x)=C_{2n}(x)-C_{2n-1}(x)+C_{2n-2}(x)-C_{2n-3}(x)+...+C_2-C_1,$$

but by **1.156** $C_{2k}-C_{2k-1}=C_k^2-C_{k-1}^2,\ \forall k\in\mathbf{N}^*$, so:

$$\sum_{k=0}^{2n}(-1)^k C_k(x)=C_n^2(x)-C_{n-1}^2(x)+C_{n-1}^2(x)-C_{n-2}^2(x)+...+C_3^2(x)-C_2^2(x)+C_2^2(x)-$$

$$-C_1^2(x)+C_1^2(x)-C_0^2(x)=C_n^2(x)-C_0^2(x)=C_n^2(x).$$

1.158. $\sum_{k=0}^{2n}(-1)^k B_k(x)=B_n(x)C_n(x),\ \forall n\in\mathbf{N}^*$.

Proof. $\sum_{k=0}^{2n}(-1)^k B_k(x)=\sum_{k=1}^{n}B_{2k}(x)-\sum_{k=1}^{n}B_{2k-1}(x)=B_n^2(x)-B_n(x)B_{n-1}(x)=$

$$=B_n(x)(B_n(x)-B_{n-1}(x)),$$

and taking account by

$$B_n(x)-B_{n-1}(x)=C_n(x),\ \forall n\in\mathbf{N}^*,$$

we get the result.

1.159. If $(x_n)_{n\ge 0}$ is the positive real sequence with $x_0=L_0=2$,

$$x_1=L_1=1\text{ and }\sqrt{F_n^2+F_{2n}^2}+\sqrt{1+x_n^2}=\sqrt{(x_n+F_{2n})^2+(1+F_n)^2},\ \forall n\in\mathbf{N}^*,$$

then $(x_n)_{n\ge 0}$ is the sequence of Lucas.

(**D.M. Bătineţu-Giurgiu and N. Stanciu**, 2013)

Proof. $(x_n+F_{2n})^2+(1+F_n)^2=1+x_n^2+F_n^2+F_{2n}^2+2\sqrt{1+x_n^2}\cdot\sqrt{F_n^2+F_{2n}^2}\Leftrightarrow$

$\Leftrightarrow 1+F_n^2+x_n^2+F_{2n}^2+2F_n+2x_nF_{2n}=1+x_n^2+F_n^2+F_{2n}^2+2\sqrt{(1+x_n^2)(F_n^2+F_{2n}^2)}$

$\Leftrightarrow F_n+x_nF_{2n}=\sqrt{(1+x_n^2)(F_n^2+F_{2n}^2)}\Leftrightarrow F_n^2+2F_nF_{2n}x_n+x_n^2F_{2n}^2=F_n^2+F_{2n}^2+x_n^2F_n^2+x_n^2F_{2n}^2$

$\Leftrightarrow x_n^2F_n^2+F_{2n}^2-2F_nF_{2n}x_n=0\Leftrightarrow(x_nF_n-F_{2n})^2=0\Leftrightarrow$ (1) $x_n=\frac{F_{2n}}{F_n},\ \forall n\in\mathbf{N}^*$.

$$F_n=\frac{1}{\sqrt{5}}(\alpha^n-\beta^n) \text{ and } L_n=\alpha^n+\beta^n,\ \forall n\in\mathbf{N}^*,$$

so:

$$x_n=\frac{\alpha^{2n}-\beta^{2n}}{\alpha^n-\beta^n}=\alpha^n+\beta^n=L_n,\forall n\in\mathbf{N}^*,$$

and because $x_0=L_0$ yields that $x_n=L_n,\ \forall n\in\mathbf{N}^*$.

1.160. If $(x_n)_{n\geq 0}$ is the positive real sequence with $x_0=0$,

$$x_1=1 \text{ şi } \sqrt{L_n^2+F_{2n}^2}+\sqrt{1+x_n^2}=\sqrt{(x_n+F_{2n})^2+(1+L_n)^2},\ \forall n\in\mathbf{N}^*,$$

then the sequence $(x_n)_{n\geq 0}$ is *Fibonacci*' s sequence.

(**D.M. Bătineţu-Giurgiu and N. Stanciu**, 2013)

Proof. $(x_n+F_{2n})^2+(1+L_n)^2=1+x_n^2+L_n^2+F_{2n}^2+2\sqrt{1+x_n^2}\cdot\sqrt{L_n^2+F_{2n}^2}\Leftrightarrow$

$\Leftrightarrow x_n^2+2x_nF_{2n}+F_{2n}^2+1+L_n^2+2L_n=1+x_n^2+L_n^2+F_{2n}^2+2\sqrt{(1+x_n^2)(L_n^2+F_{2n}^2)}$

$\Leftrightarrow L_n+x_nF_{2n}=\sqrt{(1+x_n^2)(L_n^2+F_{2n}^2)}\Leftrightarrow L_n^2+2L_nF_{2n}x_n+x_n^2F_{2n}^2=L_n^2+F_{2n}^2+x_n^2L_n^2+x_n^2F_{2n}^2$

$\Leftrightarrow x_n^2L_n^2+F_{2n}^2-2L_nF_{2n}x_n=0\Leftrightarrow(x_nL_n-F_{2n})^2=0\Leftrightarrow x_n=\frac{F_{2n}}{L_n}\Leftrightarrow$

$\Leftrightarrow x_n=\frac{1}{\sqrt{5}}\cdot\frac{\alpha^{2n}-\beta^{2n}}{\alpha^n+\beta^n}=\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)=F_n,\forall n\in\mathbf{N}^*$ and because $x_0=F_0$ yields that $x_n=F_n,\forall n\in\mathbf{N}^*$.

1.161. $5^mL_{2m+1}\cdot\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k=5^nL_{2n+1}\cdot\sum_{p=0}^{2m+1}\binom{2m+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k,\ \forall m,n\in\mathbf{N}^*$.

(**D.M. Bătineţu-Giurgiu and N. Stanciu**, 2013)

Proof. $L_{2m+1}\cdot\frac{1}{5^n}\cdot\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k=L_{2n+1}\cdot\frac{1}{5^m}\cdot\sum_{p=0}^{2m+1}\binom{2m+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k,\ \forall m,n\in\mathbf{N}^*$.

We will prove that:

$$\frac{1}{5^n}\cdot\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k = L_{2n+1}, \forall n\in\mathbf{N}^*.$$

Indeed,

$$F_k=\frac{1}{\sqrt{5}}(\alpha^k-\beta^k) \text{ iar } L_k=\alpha^k+\beta^k, \forall n\in\mathbf{N}, \text{ and then:}$$

$$\frac{1}{5^n}\cdot\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k=\frac{1}{5^n\sqrt{5}}\cdot\sum_{p=0}^{2n+1}\sum_{k=0}^{p}\binom{2n+1}{p}\binom{p}{k}(\alpha^k-\beta^k)=$$

$$=\frac{1}{(\sqrt{5})^{2n+1}}\sum_{p=0}^{2n+1}\left((\alpha+1)^p-(\beta+1)^p\right)\binom{2n+1}{p}=\frac{1}{(\sqrt{5})^{2n+1}}\left((\alpha+2)^{2n+1}-(\beta+2)^{2n+1}\right), \forall n\in\mathbf{N}^*$$

Since:

$$\alpha+2=\frac{1+\sqrt{5}}{2}+2=\frac{5+\sqrt{5}}{2}=\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)=\alpha\sqrt{5},$$

$$\beta+2=\frac{1-\sqrt{5}}{2}+2=\frac{5-\sqrt{5}}{2}=-\sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)=-\beta\sqrt{5}, \text{ so:}$$

$$(\alpha+2)^{2n+1}-(\beta+2)^{2n+1}=(\sqrt{5})^{2n+1}(\alpha^{2n+1}-(-\beta)^{2n+1})=(\sqrt{5})^{2n+1}(\alpha^{2n+1}+\beta^{2n+1})=$$
$$=(\sqrt{5})^{2n+1}L_{2n+1}, \forall n\in\mathbf{N}^* \text{ and we are done.}$$

1.162. $5^m F_{2m+1}\cdot\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k=5^n F_{2n+1}\cdot\sum_{p=0}^{2m+1}\binom{2m+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k, \forall m,n\in\mathbf{N}^*.$

(**D.M. Bătineţu-Giurgiu and N. Stanciu**, 2013)

Proof. $F_k=\frac{1}{\sqrt{5}}(\alpha^k-\beta^k)$ iar $L_k=\alpha^k+\beta^k$, $\forall n\in\mathbf{N}$, so:

$$\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k=\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}(\alpha^k+\beta^k)=$$

$$=\sum_{p=0}^{2n+1}\left((\alpha+1)^p+(\beta+1)^p\right)\binom{2n+1}{p}=\sum_{p=0}^{2n+1}\binom{2n+1}{p}(\alpha+1)^p+\sum_{p=0}^{2n+1}\binom{2n+1}{p}(\beta+1)^p=$$
$$=(\alpha+2)^{2n+1}+(\beta+2)^{2n+1}, \forall n\in\mathbf{N}^*.$$

Since, $\alpha+2=\sqrt{5}$, $\beta+2=-\beta\sqrt{5}$, we deduce:

$$\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k=(\alpha^{2n+1}-\beta^{2n+1})(\sqrt{5})^{2n+1}=$$

$$=\frac{1}{\sqrt{5}}(\alpha^{2n+1}-\beta^{2n+1})(\sqrt{5})^{2(n+1)}=5^{n+1}F_{2n+1}.$$

Hence,

$$F_{2m+1}\cdot\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k=5^{n+1}F_{2m+1}F_{2n+1},\ \forall m,n\in\mathbf{N}^*,\text{ respectively}$$

$$F_{2n+1}\cdot\sum_{p=0}^{2m+1}\binom{2m+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k=5^{m+1}F_{2m+1}F_{2n+1},\ \forall m,n\in\mathbf{N}^*,$$

and we are done.

1.163. If $(a_n)_{n\geq1}$ is a positive real sequence and there exists $r\in\mathbf{R}_+^*$ such that: $\sum_{k=1}^{n}a_kL_k=a_nL_{n+2}-r(L_{n+3}-7)-3a_1$, $\forall n\in\mathbf{N}^*$, then $(a_n)_{n\geq1}$ is a arithmetic progression with the ratio r.

(**D.M. Bătineţu-Giurgiu and N. Stanciu**, 2013)

Proof. We prove by mathematical induction:

$$n=1\Rightarrow a_1L_1=a_1L_3-r(L_4-7)-3a_1,$$

so by : $L_1=1, L_3=4, L_4=7$, we obtain:

$$a_1=4a_1-r(7-7)-3a_1\Leftrightarrow a_1=a_1,\text{ true.}$$

We suppose, $a_k=a_1+(k-1)r,\forall k=\overline{1,n}$ and we will prove that: $a_{n+1}=a_1+nr$. Indeed,

$$\sum_{k=1}^{n+1}a_kL_k=a_{n+1}L_{n+3}-r(L_{n+4}-7)-3a_1\Leftrightarrow$$

$$\Leftrightarrow\sum_{k=1}^{n}a_kL_k+a_{n+1}L_{n+1}=a_{n+1}L_{n+3}-r(L_{n+4}-7)-3a_1\Leftrightarrow$$
$$\Leftrightarrow a_nL_{n+2}-r(L_{n+3}-7)-3a_1+a_{n+1}L_{n+1}=a_{n+1}L_{n+3}-r(L_{n+4}-7)-3a_1\Leftrightarrow$$
$$\Leftrightarrow a_{n+1}(L_{n+3}-L_{n+1})=a_nL_{n+2}+r(L_{n+4}-L_{n+3})\Leftrightarrow a_{n+1}L_{n+2}=a_nL_{n+2}+rL_{n+2}\Leftrightarrow$$
$$\Leftrightarrow a_{n+1}L_{n+2}=(a_n+r)L_{n+2}\Leftrightarrow a_{n+1}=a_n+r=a_1+nr,\ \forall n\in\mathbf{N}^*,$$

and the proof is complete.

1.164. If $(u_n)_{n\geq1}$ is a arithmetic progression with the ratio $r>0$ with $u_1\in\mathbf{R}_+^*$ and $(x_n)_{n\geq0}$ is a sequence with $x_0=0, x_1=x_2=1$, and

$$\sum_{k=1}^{n}u_kx_k=u_nx_{n+2}+r(x_4-x_{n+3})-x_2u_1,\ \forall n\in\mathbf{N}^*,$$

then $(x_n)_{n\geq0}$ is the sequence of *Fibonacci*.

(**D.M. Bătineţu-Giurgiu and N. Stanciu**, 2013)

Proof. We note that $x_2 = x_1 + x_0$ and we will prove by mathematical induction that: $x_{n+2} = x_{n+1} + x_n$, $\forall n \in \mathbf{N}$.

For $n = 0$, $x_2 = x_1 + x_0$, true.

For $n = 1$:

$$u_1 x_1 = u_1 x_3 + r(x_4 - x_4) - x_2 u_1 \Leftrightarrow u_1(x_3 - x_2 - x_1) = 0 \Leftrightarrow x_3 = x_2 + x_1, \text{ true.}$$

We suppose that $x_{n+2} = x_{n+1} + x_n$, is true for any $k = \overline{0,n}$, i.e we will prove that $x_{n+3} = x_{n+2} + x_{n+1}$.

$$\sum_{k=1}^{n} u_k x_k = u_n x_{n+2} + r(x_4 - x_{n+3}) - x_2 u_1, \text{ and } \sum_{k=1}^{n-1} u_k x_k = u_{n-1} x_{n+1} + r(x_4 - x_{n+2}) - x_2 u_1,$$

so:

$$u_n x_n = u_n x_{n+2} - u_{n-1} x_{n+1} + r(x_{n+2} - x_{n+3}) \Leftrightarrow$$
$$\Leftrightarrow r(x_{n+3} - x_{n+2}) = u_n(x_{n+2} - x_n) - (u_n - r)x_{n+1} \Leftrightarrow$$
$$\Leftrightarrow r(x_{n+3} - x_{n+2} - x_{n+1}) = u_n(x_{n+2} - x_{n+1} - x_n) \Leftrightarrow$$
$$\Leftrightarrow r(x_{n+3} - x_{n+2} - x_{n+1}) = u_n(x_{n+2} - x_{n+1} - x_n),$$

and by $x_{n+2} = x_{n+1} + x_n$, hence:

$r(x_{n+3} - x_{n+2} - x_{n+1}) = 0$, $\forall n \in \mathbf{N}$, because $r > 0$, yields:

$$x_{n+3} = x_{n+2} + x_{n+1}, \text{ and we are done.}$$

1.165. $l_{n+1}(x) = f_{n+2}(x) + f_n(x)$, $\forall n \in \mathbf{N}$, $\forall x \in \mathbf{R}$.

Proof. $f_{n+2}(x) + f_n(x) = \dfrac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)} + \dfrac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} =$

$$= \frac{1}{\alpha(x) - \beta(x)}\left(\alpha^{n+1}(x)(\alpha(x) + \alpha^{-1}(x)) - \beta^{n+1}(x)(\beta(x) + \beta^{-1}(x))\right) =$$
$$= \frac{1}{\alpha(x) - \beta(x)}\left(\alpha^{n+1}(x)(\alpha(x) - \beta(x)) - \beta^{n+1}(x)(\beta(x) - \alpha(x))\right) =$$
$$= \alpha^{n+1}(x) + \beta^{n+1}(x) = l_{n+1}(x).$$

Observation. For $x = 1$ we obtain **1.15.**

1.166. $2 f_{n+k}(x) = f_n(x) \cdot l_k(x) + f_k(x) \cdot l_n(x)$, $\forall k, n \in \mathbf{N}$, $\forall x \in \mathbf{R}$.

Proof. $f_n(x) \cdot l_k(x) + f_k(x) \cdot l_n(x) =$

$$= \frac{1}{\alpha(x) - \beta(x)}\left((\alpha^n(x) - \beta^n(x))(\alpha^k(x) + \beta^k(x)) + (\alpha^k(x) - \beta^k(x))(\alpha^n(x) + \beta^n(x))\right) =$$
$$= \frac{2}{\alpha(x) - \beta(x)}\left(\alpha^{n+k}(x) - \beta^{n+k}(x)\right) = 2 f_{n+k}(x).$$

1.167. $xf_n(x)+l_n(x)=2f_{n+1}(x), \forall n\in\mathbf{N},\ \forall x\in\mathbf{R}.$

Proof. If we take $k=1$, then:

$$2f_{n+1}(x)=f_n(x)\cdot l_1(x)+f_1(x)\cdot l_n(x),$$

and because $l_1(x)=x$, $f_1(x)=1$ we obtain $xf_n(x)+l_n(x)=2f_{n+1}(x)$.

1.168. $(x^2+4)\cdot f_n(x)+x\cdot l_n(x)=2l_{n+1}(x), \forall n\in\mathbf{N},\ \forall x\in\mathbf{R}.$

Proof. $2l_{n+1}(x)-xl_n(x)=2(\alpha^{n+1}(x)+\beta^{n+1}(x))-x(\alpha^n(x)+\beta^n(x))=\alpha^n(x)(2\alpha(x)-x)+$

$$+2\beta^n(x)(2\beta(x)-x)=\alpha^n(x)(x+\sqrt{x^2+4}-x)+2\beta^n(x)(x-\sqrt{x^2+4}-x)=$$

$$=\sqrt{x^2+4}(\alpha^n(x)-\beta^n(x))=\frac{x^2+4}{\sqrt{x^2+4}}(\alpha^n(x)-\beta^n(x))=$$

$$=\frac{x^2+4}{\alpha(x)-\beta(x)}(\alpha^n(x)-\beta^n(x))=(x^2+4)f_n(x).$$

1.169.

$$2\cdot\begin{pmatrix} f_{n+1}(x)\\ f_n(x)\end{pmatrix}=\begin{pmatrix} x & 1\\ x^2+4 & x\end{pmatrix}\cdot\begin{pmatrix} f_n(x)\\ l_n(x)\end{pmatrix},\ \forall n\in\mathbf{N},\ \forall x\in\mathbf{R}.$$

Proof.

$$\begin{pmatrix} x & 1\\ x^2+4 & x\end{pmatrix}\cdot\begin{pmatrix} f_n(x)\\ l_n(x)\end{pmatrix}=\begin{pmatrix} xf_{n+1}(x)+l_n(x)\\ (x^2+4)f_n(x)+xl_n(x)\end{pmatrix},$$

So by above we get the conclusion.

1.170. $2\cdot l_{n+k}(x)=(x^2+4)f_n(x)f_k(x)+l_n(x)l_k(x),\ \forall k,n\in\mathbf{N},\ \forall x\in\mathbf{R}.$

Proof. $(x^2+4)f_n(x)f_k(x)+l_n(x)l_k(x)=$

$$=\frac{x^2+4}{(\alpha(x)-\beta(x))^2}(\alpha^n(x)-\beta^n(x))(\alpha^k(x)-\beta^k(x))+(\alpha^n(x)+\beta^n(x)(\alpha^k(x)+\beta^k(x))=$$

$$=\alpha^{n+k}(x)+\beta^{n+k}(x)-\alpha^n(x)\beta^k(x)-\alpha^k(x)\beta^n(x)+\alpha^{n+k}(x)+\beta^{n+k}(x)+$$

$$+\alpha^n(x)\beta^k(x)+\alpha^k(x)\beta^n(x)=2(\alpha^{n+k}(x)+\beta^{n+k}(x))=2l_{n+k}(x).$$

1.171.

$$2\cdot\begin{pmatrix} f_{n+k}(x)\\ l_{n+k}(x)\end{pmatrix}=\begin{pmatrix} l_k(x) & f_k(x)\\ (x^2+4)f_k(x) & l_k(x)\end{pmatrix}\cdot\begin{pmatrix} f_n(x)\\ l_n(x)\end{pmatrix},\forall k,n\in\mathbf{N},\ \forall x\in\mathbf{R}.$$

Proof.

$$\begin{pmatrix} l_k(x) & f_k(x) \\ (x^2+4)f_k(x) & l_k(x) \end{pmatrix} \cdot \begin{pmatrix} f_n(x) \\ l_n(x) \end{pmatrix} = \begin{pmatrix} f_n(x)l_k(x) + f_k(x)l_n(x) \\ (x^2+4)f_n(x)f_k(x) + l_k(x)l_n(x) \end{pmatrix},$$

so by above yields the conclusion.

1.172. $f_n^2(x) = \dfrac{1}{x^2+4}(l_n^2(x) - 4(-1)^n)$, $\forall n \in \mathbf{N},\ \forall x \in \mathbf{R}$.

Proof.

$$f_n^2(x) = \frac{1}{(\alpha(x)-\beta(x))^2}(\alpha^n(x)-\beta^n(x))^2 = \frac{1}{x^2+4}(\alpha^{2n}(x)+\beta^{2n}(x)-2(\alpha(x)\beta(x))^n) =$$

$$= \frac{1}{x^2+4}(\alpha^{2n}(x)+\beta^{2n}(x)+2(\alpha(x)\beta(x))^n - 4(\alpha(x)\beta(x))^n) =$$

$$= \frac{1}{x^2+4}(((\alpha^n(x)+\beta^n(x))^2 - 4(-1)^n) = \frac{1}{x^2+4}(l_n^2(x)-4(-1)^n).$$

1.173. $f_n^2(x) = \dfrac{1}{x^2+4}(l_{2n}(x) - 2(-1)^n)$, $\forall n \in \mathbf{N},\ \forall x \in \mathbf{R}$.

Proof. $f_n^2(x) = \dfrac{1}{x^2+4}(\alpha^{2n}(x)+\beta^{2n}(x)-2(\alpha(x)\beta(x)^n) =$

$$= \frac{1}{x^2+4}(l_{2n}(x)-2(-1)^n).$$

1.174. $\sum_{k=0}^{m} F_{n+k} = F_n F_{m+1} + F_{n+1}F_{m+2} - F_{n+1}$.

(**Cezar Popovici,** 1911)

Proof.

(1) $\sum_{k=0}^{m} F_{n+k} = \sum_{i=0}^{n+m} F_i - \sum_{i=0}^{n+m} F_i = (F_{n+m+2}-1)-(F_{n+1}-1) = F_{n+m+2} - F_{n+1}$.

Also we have:

(2) $F_n F_{m+1} + F_{n+1}F_{m+2} = \dfrac{1}{5}(\alpha^n-\beta^n)(\alpha^{m+1}-\beta^{m+1}) + \dfrac{1}{5}(\alpha^{n+1}-\beta^{n+1})(\alpha^{m+2}-\beta^{m+2}) =$

$$= \frac{1}{5}(\alpha^{m+n+1} - \alpha^n\beta^{m+1} - \alpha^{m+1}\beta^n + \beta^{m+n+1} + \alpha^{m+n+3} - \alpha^{n+1}\beta^{m+2} - \alpha^{m+2}\beta^{n+1} + \beta^{m+n+3}) =$$

$$= \frac{1}{5}(\alpha^{m+n+2}(\alpha^{-1}+\alpha) + \beta^{m+n+2}(\beta^{-1}+\beta) - \alpha^n\beta^{m+1}(1+\alpha\beta) - \alpha^{m+1}\beta^n(1+\alpha\beta)) =$$

$$= \frac{1}{5}(\alpha-\beta)(\alpha^{m+n+2}-\beta^{m+n-2}) = \frac{\alpha-\beta}{5}(\alpha^{m+n+2}-\beta^{m+n+2}) =$$

$$= \frac{1}{\sqrt{5}}(\alpha^{m+n+2}-\beta^{m+n+2}) = F_{m+n+2}.$$

From (1) and (2) we deduce that:

$$\sum_{k=0}^{m} F_{n+k} = F_n F_{m+1} + F_{n+1} F_{m+2} - F_{n+1},$$

and we are done.

1.175. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{F_m}{F_{m+2}}\right)^k = \left(\frac{F_{m+1}}{F_{m+2}}\right)^{2n}, \forall m, n \in \mathbf{N}^*.$

(**D.M. Bătinețu-Giurgiu and N. Stanciu**, 2013)

Proof. We have that:

$$(*)\ \left(\frac{x}{x+y}\right)^{2n} = \left(1 - \frac{y}{x+y}\right)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{y}{x+y}\right)^k, \forall n \in \mathbf{N}^*, \forall x, y \in \mathbf{R}_+^*.$$

If we take:

$$x = F_{m+1}, y = F_m,$$

yields:

$$x + y = F_m + F_{m+1} = F_{m+2},$$

so by (*) :

$$\left(\frac{F_{m+1}}{F_{m+2}}\right)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{F_m}{F_{m+2}}\right)^k, \forall m, n \in \mathbf{N}^*,$$

and the proof is complete.

1.176. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{F_{m+1}}{F_{m+2}}\right)^k = \left(\frac{F_m}{F_{m+2}}\right)^{2n}, \forall m, n \in \mathbf{N}^*.$

(**N. Stanciu**, and **G. Tica** 2013)

Proof.

$$(*)\ \left(\frac{x}{x+y}\right)^{2n} = \left(1 - \frac{y}{x+y}\right)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{y}{x+y}\right)^k, \forall n \in \mathbf{N}^*, \forall x, y \in \mathbf{R}_+^*.$$

Setting:

$$x = F_m, y = F_{m+1},$$

hence:

$$x + y = F_m + F_{m+1} = F_{m+2},$$

then by (*):

$$\left(\frac{F_m}{F_{m+2}}\right)^{2n} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{F_{m+1}}{F_{m+2}}\right)^k, \forall m, n \in \mathbf{N}^*,$$

and we are done.

1.177. $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\left(\frac{L_m}{L_{m+2}}\right)^k=\left(\frac{L_{m+1}}{L_{m+2}}\right)^{2n}, \forall m,n\in\mathbf{N}^*.$

(**D.M. Bătineţu-Giurgiu and G. Tica**, 2013)

Proof.

$$(*)\quad \left(\frac{x}{x+y}\right)^{2n}=\left(1-\frac{y}{x+y}\right)^{2n}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\left(\frac{y}{x+y}\right)^k, \forall n\in\mathbf{N}^*, \forall x,y\in\mathbf{R}_+^*.$$

Putting:

$$x=L_{m+1}, y=L_m,$$

we get:

$$x+y=L_m+L_{m+1}=L_{m+2},$$

then by (*) yields that:

$$\left(\frac{L_{m+1}}{L_{m+2}}\right)^{2n}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\left(\frac{L_m}{L_{m+2}}\right)^k, \forall m,n\in\mathbf{N}^*,$$

q.e.d.

1.178. $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\left(\frac{L_{m+1}}{L_{m+2}}\right)^k=\left(\frac{L_m}{L_{m+2}}\right)^{2n}, \forall m,n\in\mathbf{N}^*.$

(**D.M. Bătineţu-Giurgiu**, 2013)

Proof.

$$(*)\quad \left(\frac{x}{x+y}\right)^{2n}=\left(1-\frac{y}{x+y}\right)^{2n}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\left(\frac{y}{x+y}\right)^k, \forall n\in\mathbf{N}^*, \forall x,y\in\mathbf{R}_+^*.$$

Putting:

$$x=L_m, y=L_{m+1},$$

hence:

$$x+y=L_m+L_{m+1}=L_{m+2},$$

then (*) yields that:

$$\left(\frac{L_m}{L_{m+2}}\right)^{2n}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\left(\frac{L_{m+1}}{L_{m+2}}\right)^k, \forall m,n\in\mathbf{N}^*,$$

And we are done.

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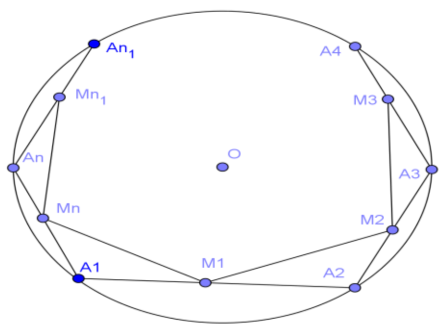
4. Generalizarea problemei CO. 5204 GM 5/2011

Gheorghe Alexe si George-Florin Serban, profesor, Liceul Pedagogic"D.P.Perpessicius", Braila

Fie poligonul regulat cu n laturi $A_1A_2...A_n,\ n\geq 3, n\in N$ si punctele $M_1, M_2, ...M_n$ astfel ca $M_1\in[A_1A_2], \mathrm{A}_1M_1\geq M_1A_2, M_2\in[A_2A_3], \mathrm{A}_2M_2\geq M_2A_3, ..., M_n\in[A_nA_1], \mathrm{A}_nM_n\geq M_nA_1$. Sa se arate ca: $S_{[M_1M_2...M_n]}\geq(\cos^2\frac{\pi}{n})S_{[\mathrm{A}_1A_2...\mathrm{A}_n]}$.



Solutie:

$l_n = a = A_1A_2 = A_2A_3 = ... = A_nA_1,\ S_{[\mathrm{A}_1A_2...\mathrm{A}_n]} = \frac{n}{4}a^2ctg\frac{\pi}{n}, (\forall)n\in N\setminus\{0,1,2\}$.

$\mu(A_1A_2A_3) = \frac{1}{2}\frac{(n-2)2\pi}{n} = \frac{(n-2)\pi}{n} = ... = \mu(A_nA_1A_2)$. Notam $A_1M_1 = x_1, M_1A_2 = a - x_1$,

$x_1\geq a - x_1, \Rightarrow \frac{a}{2}\leq x_1\leq a,\ A_2M_2 = x_2, M_2A_3 = a - x_2, \Rightarrow \frac{a}{2}\leq x_2\leq a,\ A_nM_n = x_n, M_nA_1 = a - x_n,$

$\Rightarrow \frac{a}{2}\leq x_n\leq a.\ S_{[M_1A_2M_2]} = \frac{1}{2}(a - x_1)x_2\sin\frac{(n-2)\pi}{n} = \frac{1}{2}(a - x_1)x_2\sin\frac{2\pi}{n},$

$S_{[M_1A_3M_3]} = \frac{1}{2}(a - x_2)x_3\sin\frac{2\pi}{n}, ..., S_{[M_nA_1M_1]} = \frac{1}{2}(a - x_n)x_1\sin\frac{2\pi}{n}.$

$S_{[M_1M_2...M_n]} = S_{[\mathrm{A}_1A_2...\mathrm{A}_n]} - (S_{[M_1A_2M_2]} + S_{[M_1A_3M_3]} + ... + S_{[M_nA_1M_1]}),$

$S_{[M_1M_2...M_n]} = \frac{n}{4}a^2ctg\frac{\pi}{n} - \frac{1}{2}[(a - x_1)x_2 + (a - x_2)x_3 + ... + (a - x_n)x_1]\sin\frac{2\pi}{n}.$

Dar $x_1 \in [\frac{a}{2}, a], \Leftrightarrow (\exists)\, t_1 \in [0,1],\ x_1 = (1-t_1)\frac{a}{2} + t_1 a = (1+t_1)\frac{a}{2},\ ,\ a - x_1 = (1-t_1)\frac{a}{2}.$

Dar $x_2 \in [\frac{a}{2}, a], \Leftrightarrow (\exists)\, t_2 \in [0,1],\ x_2 = (1+t_2)\frac{a}{2},\ \ a - x_2 = (1-t_2)\frac{a}{2}.$

Deci $(a-x_1)x_2 = (1+t_2)(1-t_1)\frac{a^2}{4}, (a-x_2)x_3 = (1+t_3)(1-t_2)\frac{a^2}{4},$

$(a-x_n)x_1 = (1+t_1)(1-t_n)\frac{a^2}{4}$, unde $t_1, t_2, ... t_n \in [0,1].$

$$S = (a-x_1)x_2 + (a-x_2)x_3 + ... + (a-x_n)x_1 = (1+t_2)(1-t_1)\frac{a^2}{4} + (1+t_3)(1-t_2)\frac{a^2}{4} + ... + (1+t_1)(1-t_n)\frac{a^2}{4}.$$

Vom arata ca $S \le \frac{na^2}{4}$. Deci $(1+t_2)(1-t_1)\frac{a^2}{4} + (1+t_3)(1-t_2)\frac{a^2}{4} + ... + (1+t_1)(1-t_n)\frac{a^2}{4} \le \frac{na^2}{4},$

$(1+t_2)(1-t_1) + (1+t_3)(1-t_2) + ... + (1+t_1)(1-t_n) \le n,$

$(1-t_1+t_2-t_1 t_2) + (1-t_2+t_3-t_2 t_3) + ... + (1-t_n+t_1-t_1 t_n) \le n,\quad n - (t_1t_2 + t_2t_3 + ... + t_1 t_n) \le n,$

$t_1t_2 + t_2t_3 + ... + t_1 t_n \ge 0, (A).$ Deci $(a-x_1)x_2 + (a-x_2)x_3 + ... + (a-x_n)x_1 \le \frac{na^2}{4}.$

$$-\frac{1}{2}[(a-x_1)x_2 + (a-x_2)x_3 + ... + (a-x_n)x_1]\sin\frac{2\pi}{n} \le -\frac{na^2}{8}\sin\frac{2\pi}{n},\quad \frac{2\pi}{n} \in (0,\pi), n \ge 3, \sin\frac{2\pi}{n} > 0.$$

$$S_{[M_1M_2...M_n]} \ge \frac{n}{4}a^2 ctg\frac{\pi}{n} - \frac{na^2}{8}\sin\frac{2\pi}{n} = \frac{na^2}{4}\cos\frac{\pi}{n}\left(\frac{1}{\sin\frac{\pi}{n}} - \sin\frac{\pi}{n}\right) = \frac{na^2}{4}\cos\frac{\pi}{n}\frac{\cos^2\frac{\pi}{n}}{\sin\frac{\pi}{n}},$$

$$S_{[M_1M_2...M_n]} \ge \frac{na^2}{4}\cos\frac{\pi}{n}\frac{\cos^2\frac{\pi}{n}}{\sin\frac{\pi}{n}} = \frac{na^2}{4}\text{ctg}\frac{\pi}{n}\cos^2\frac{\pi}{n},\ S_{[A_1A_2...A_n]} = \frac{n}{4}a^2 ctg\frac{\pi}{n}, (\forall) n \in N \setminus \{0,1,2\}.$$

Rezulta $S_{[M_1M_2...M_n]} \ge \frac{na^2}{4}\text{ctg}\frac{\pi}{n}\cos^2\frac{\pi}{n} = (\cos^2\frac{\pi}{n})S_{[A_1A_2...A_n]},\quad S_{[M_1M_2...M_n]} \ge (\cos^2\frac{\pi}{n})S_{[A_1A_2...A_n]}.$

Cazuri particulare: 1)Pentru n=3, $A_1A_2A_3$ triunghi echilateral,

$S_{[M_1M_2M_3]} \ge (\cos^2\frac{\pi}{3})S_{[A_1A_2A_3]} \Rightarrow S_{[M_1M_2M_3]} \ge \frac{1}{4}S_{[A_1A_2A_3]}$, sau $S_{[A_1A_2A_3]} \le 4 \cdot S_{[M_1M_2M_3]}$, este problema CO 5204 din GM 5\ 2011, autor Vasile Pop, profesor Cluj-Napoca.

2)Pentru n=4, $A_1A_2A_3A_4$ este patrat,

$S_{[M_1M_2M_3M_4]} \ge (\cos^2\frac{\pi}{4})S_{[A_1A_2A_3A_4]} \Rightarrow S_{[M_1M_2M_3M_4]} \ge \frac{1}{2}S_{[A_1A_2A_3A_4]},$

Este problema 3 de la Concursul de Matematica Argument, Editia a II-a,2010, autor Vasile Pop, profesor Cluj-Napoca.

5. APLICAȚII ALE UNEI INEGALITĂȚI ALGEBRICE ÎN TRIUNGHI

Marin Chirciu[1]

Articolul își propune ca pornind de la o inegalitate algebrică să obținem o clasă de inegalități geometrice într-un triunghi oarecare.

Lemă.

Dacă $x, y, z \in (0, \infty)$ atunci este adevărată inegalitatea

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}.$$

Soluție.

Cu inegalitatea mediilor avem $\frac{x}{y}+\frac{x}{y}+\frac{y}{z} \geq 3\sqrt[3]{\frac{x^2}{yz}}=\frac{3x}{\sqrt[3]{xyz}}$. Scriind și celelalte două inegalități analoage și adunând se obține concluzia. Egalitatea are loc dacă și numai dacă $x=y=z$.

Remarcă.

Să observăm că inegalitatea demonstrată este mai tare decât $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3$, deoarece $\frac{x+y+z}{\sqrt[3]{xyz}} \geq 3$ (AM-GM), ceea ce va genera prin particularizarea variabilelor x, y, z cu elemente ale unui triunghi inegalități mai tari decât cele obținute din inegalitatea mediilor.

În continuare vom prezenta aplicații la inegalitatea de mai sus.

Aplicația 1.

In ΔABC

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt[3]{\frac{2p^2}{Rr}} \geq 3.$$

Soluție.

În **Lemă** punem $(x, y, z)=(a, b, c)$.

Obținem $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}=\frac{2p}{\sqrt[3]{abc}}=\frac{2p}{\sqrt[3]{4Rrp}}=\sqrt[3]{\frac{8p^3}{4Rrp}}=\sqrt[3]{\frac{2p^2}{Rr}} \geq 3$.

Ultima inegalitate este adevărată din AM-GM, $\frac{a+b+c}{\sqrt[3]{abc}} \geq 3$, care la rândul ei generează o nouă inegalitate în triunghi și anume $2p^2 \geq 27Rr$ (C.Coșniță și F. Turtoiu, 1965).

Egalitatea are loc dacă și numai dacă $a=b=c$, adică pentru triunghiul echilateral.

Aplicația 2.

In ΔABC

[1] *Profesor, Colegiul Național „Zinca Golescu", Pitești*

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{p^2-r^2-4Rr}{\sqrt[3]{2p^2r^2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(a^2,b^2,c^2)$. Folosim $\sum a^2=2(p^2-r^2-4Rr)$ și $abc=4prR$.

Aplicația 3.

In ΔABC

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\geq 5-\frac{4r}{R}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(a^3,b^3,c^3)$. Folosim $\sum a^3=2p(p^2-3r^2-6Rr)$, $abc=4prR$ și inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.

Aplicația 4.

In ΔABC

$$\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\geq\frac{p^2+r^2+4Rr}{2\sqrt[3]{2p^2r^2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(h_a,h_b,h_c)$. Folosim $\sum h_a=\frac{p^2+r^2+4Rr}{2R}$ și $\prod h_a=\frac{2p^2r^2}{R}$.

Aplicația 5.

In ΔABC

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(r_a,r_b,r_c)$. Folosim $\sum r_a=4R+r$, $\prod r_a=rp^2$ și inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$. Ultima inegalitate este inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicația 6.

In ΔABC

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}}\geq 4-\frac{2r}{R}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)$. Folosim $\sum\sin^2\frac{A}{2}=1-\frac{2r}{R}$, $\prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 7.

In ΔABC

$$\frac{\operatorname{tg}\frac{A}{2}}{\operatorname{tg}\frac{B}{2}}+\frac{\operatorname{tg}\frac{B}{2}}{\operatorname{tg}\frac{C}{2}}+\frac{\operatorname{tg}\frac{C}{2}}{\operatorname{tg}\frac{A}{2}}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\operatorname{tg}\frac{A}{2},\operatorname{tg}\frac{B}{2},\operatorname{tg}\frac{C}{2}\right)$. Folosim $\sum\operatorname{tg}\frac{A}{2}=\frac{4R+r}{p}$, $\prod\operatorname{tg}\frac{A}{2}=\frac{r}{p}$ și inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$. Ultima inegalitate este inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicația 8.

In ΔABC

$$\frac{\operatorname{ctg}\frac{A}{2}}{\operatorname{ctg}\frac{B}{2}}+\frac{\operatorname{ctg}\frac{B}{2}}{\operatorname{ctg}\frac{C}{2}}+\frac{\operatorname{ctg}\frac{C}{2}}{\operatorname{ctg}\frac{A}{2}}\geq\sqrt[3]{\frac{p^2}{r^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\operatorname{ctg}\frac{A}{2},\operatorname{ctg}\frac{B}{2},\operatorname{ctg}\frac{C}{2}\right)$. Folosim $\sum\operatorname{ctg}\frac{A}{2}=\frac{p}{r}$, $\prod\operatorname{ctg}\frac{A}{2}=\frac{p}{r}$.

Ultima inegalitate este inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$.

Aplicația 9.

In ΔABC

$$\frac{\operatorname{tg}^2\frac{A}{2}}{\operatorname{tg}^2\frac{B}{2}}+\frac{\operatorname{tg}^2\frac{B}{2}}{\operatorname{tg}^2\frac{C}{2}}+\frac{\operatorname{tg}^2\frac{C}{2}}{\operatorname{tg}^2\frac{A}{2}}\geq 6\left(\frac{4R+r}{p}\right)^2-3\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\operatorname{tg}^2\frac{A}{2},\operatorname{tg}^2\frac{B}{2},\operatorname{tg}^2\frac{C}{2}\right)$.

Folosim $\sum\operatorname{tg}^2\frac{A}{2}=\left(\frac{4R+r}{p}\right)^2-2$, $\prod\operatorname{tg}\frac{A}{2}=\frac{r}{p}$ și inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$. Ultima inegalitate este inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicația 10.

In ΔABC

$$\frac{\operatorname{ctg}^2\frac{A}{2}}{\operatorname{ctg}^2\frac{B}{2}}+\frac{\operatorname{ctg}^2\frac{B}{2}}{\operatorname{ctg}^2\frac{C}{2}}+\frac{\operatorname{ctg}^2\frac{C}{2}}{\operatorname{ctg}^2\frac{A}{2}}\geq\frac{\sqrt{3}}{p}(8R-7r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\text{ctg}^2\frac{A}{2},\text{ctg}^2\frac{B}{2},\text{ctg}^2\frac{C}{2}\right)$. Folosim $\sum \text{ctg}^2\frac{A}{2}=\frac{p^2-r^2-8Rr}{r^2}$,

$\prod \text{ctg}\frac{A}{2}=\frac{p}{r}$,

inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$ și inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicația 11.

In ΔABC

$$\frac{a\cdot\sin A}{b\cdot\sin B}+\frac{b\cdot\sin B}{c\cdot\sin C}+\frac{c\cdot\sin C}{a\cdot\sin A}\geq\frac{p^2-r^2-4Rr}{\sqrt[3]{2p^2r^2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(a\cdot\sin A,b\cdot\sin B,c\cdot\sin C)$.

Folosim $\sum a\cdot\sin A=\frac{p^2-r^2-4Rr}{R}$, $\prod\sin\frac{A}{2}=\frac{r}{4R}$, $abc=4prR$.

Este echivalentă cu inegalitatea **2).**

Aplicația 12.

In ΔABC

$$\frac{a\cdot\sin^2\frac{A}{2}}{b\cdot\sin^2\frac{B}{2}}+\frac{b\cdot\sin^2\frac{B}{2}}{c\cdot\sin^2\frac{C}{2}}+\frac{c\cdot\sin^2\frac{C}{2}}{a\cdot\sin^2\frac{A}{2}}\geq\frac{2p}{\sqrt{3}}\left(\frac{1}{r}-\frac{1}{R}\right)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot\sin^2\frac{A}{2},b\cdot\sin^2\frac{B}{2},c\cdot\sin^2\frac{C}{2}\right)$. Folosim $\sum a\cdot\sin^2\frac{A}{2}=p\left(1-\frac{r}{R}\right)$,

$\prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}$, $abc=4prR$ și inegalitatea lui Mitrinović $p\leq\frac{R\sqrt{3}}{2}$.

Ultima inegalitate rezultă din inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

Problema se reduce la inegalitatea $2p^2(R-r)\geq 3Rr(4R+r)$, adevărată din

$2(16Rr-5r^2)(R-r)\geq 3Rr(4R+r)\Leftrightarrow 4R^2-9Rr+2r^2\geq 0\Leftrightarrow (R-2r)(4R-r)\geq 0$.

Aplicația 13.

In ΔABC

$$\frac{a\cdot\cos^2\frac{A}{2}}{b\cdot\cos^2\frac{B}{2}}+\frac{b\cdot\cos^2\frac{B}{2}}{c\cdot\cos^2\frac{C}{2}}+\frac{c\cdot\cos^2\frac{C}{2}}{a\cdot\cos^2\frac{A}{2}}\geq\left(1+\frac{r}{R}\right)\sqrt[3]{\frac{4R}{r}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot\cos^2\frac{A}{2},b\cdot\cos^2\frac{B}{2},c\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum a\cdot\cos^2\frac{A}{2}=p\left(1+\frac{r}{R}\right)$, $\prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}$ și $abc=4prR$.

Aplicația 14.

In ΔABC

$$\frac{a\cdot\operatorname{tg}\frac{A}{2}}{b\cdot\operatorname{tg}\frac{B}{2}}+\frac{b\cdot\operatorname{tg}\frac{B}{2}}{c\cdot\operatorname{tg}\frac{C}{2}}+\frac{c\cdot\operatorname{tg}\frac{C}{2}}{a\cdot\operatorname{tg}\frac{A}{2}}\geq 4-\frac{2r}{R}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot\operatorname{tg}\frac{A}{2},b\cdot\operatorname{tg}\frac{B}{2},c\cdot\operatorname{tg}\frac{C}{2}\right)$.

Folosim $\sum a\cdot\operatorname{tg}\frac{A}{2}=2(2R-r)$, $\prod\operatorname{tg}\frac{A}{2}=\frac{r}{p}$, $abc=4prR$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 15.

In ΔABC

$$\frac{a\cdot\operatorname{ctg}\frac{A}{2}}{b\cdot\operatorname{ctg}\frac{B}{2}}+\frac{b\cdot\operatorname{ctg}\frac{B}{2}}{c\cdot\operatorname{ctg}\frac{C}{2}}+\frac{c\cdot\operatorname{ctg}\frac{C}{2}}{a\cdot\operatorname{ctg}\frac{A}{2}}\geq\frac{2(4R+r)}{\sqrt[3]{4p^2R}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot\operatorname{ctg}\frac{A}{2},b\cdot\operatorname{ctg}\frac{B}{2},c\cdot\operatorname{ctg}\frac{C}{2}\right)$.

Folosim $\sum a\cdot\operatorname{ctg}\frac{A}{2}=2(4R+r)$, $\prod\operatorname{ctg}\frac{A}{2}=\frac{p}{r}$ și $abc=4prR$.

Aplicația 16.

In ΔABC

$$\frac{\frac{1}{a}\cdot\sin^2\frac{A}{2}}{\frac{1}{b}\cdot\sin^2\frac{B}{2}}+\frac{\frac{1}{b}\cdot\sin^2\frac{B}{2}}{\frac{1}{c}\cdot\sin^2\frac{C}{2}}+\frac{\frac{1}{c}\cdot\sin^2\frac{C}{2}}{\frac{1}{a}\cdot\sin^2\frac{A}{2}}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{a}\cdot\sin^2\frac{A}{2},\frac{1}{b}\cdot\sin^2\frac{B}{2},\frac{1}{c}\cdot\sin^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{a}\cdot\sin^2\frac{A}{2}=\frac{4R+r}{4pR}$, $\prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}$, $abc=4prR$ și inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$. Ultima inegalitate este inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicația 17.

In ΔABC

$$\frac{\frac{1}{a}\cdot\cos^2\frac{A}{2}}{\frac{1}{b}\cdot\cos^2\frac{B}{2}}+\frac{\frac{1}{b}\cdot\cos^2\frac{B}{2}}{\frac{1}{c}\cdot\cos^2\frac{C}{2}}+\frac{\frac{1}{c}\cdot\cos^2\frac{C}{2}}{\frac{1}{a}\cdot\cos^2\frac{A}{2}}\ge\sqrt[3]{\frac{p^2}{r^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{a}\cdot\cos^2\frac{A}{2},\frac{1}{b}\cdot\cos^2\frac{B}{2},\frac{1}{c}\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{a}\cdot\cos^2\frac{A}{2}=\frac{p}{4Rr}$, $\prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}$ și $abc=4prR$.

Ultima inegalitate este inegalitatea lui Mitrinović $p\ge 3r\sqrt{3}$.

Aplicația 18

In ΔABC

$$\frac{r_a\cdot\operatorname{ctg}^2\frac{A}{2}}{r_b\cdot\operatorname{ctg}^2\frac{B}{2}}+\frac{r_b\cdot\operatorname{ctg}^2\frac{B}{2}}{r_c\cdot\operatorname{ctg}^2\frac{C}{2}}+\frac{r_c\cdot\operatorname{ctg}^2\frac{C}{2}}{r_a\cdot\operatorname{ctg}^2\frac{A}{2}}\ge\sqrt[3]{\frac{p^2}{r^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(r_a\cdot\operatorname{ctg}^2\frac{A}{2},r_b\cdot\operatorname{ctg}^2\frac{B}{2},r_c\cdot\operatorname{ctg}^2\frac{C}{2}\right)$.

Folosim $\sum r_a\cdot\operatorname{ctg}^2\frac{A}{2}=\frac{p^2}{r}$, $\prod\operatorname{ctg}\frac{A}{2}=\frac{p}{r}$.

Aplicația 19.

In ΔABC

$$\frac{\frac{1}{r_a}\operatorname{tg}^2\frac{A}{2}}{\frac{1}{r_b}\operatorname{tg}^2\frac{B}{2}}+\frac{\frac{1}{r_b}\operatorname{tg}^2\frac{B}{2}}{\frac{1}{r_c}\operatorname{tg}^2\frac{C}{2}}+\frac{\frac{1}{r_c}\operatorname{tg}^2\frac{C}{2}}{\frac{1}{r_a}\operatorname{tg}^2\frac{A}{2}}\ge\frac{\sqrt{3}}{p}(4R+r)\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{r_a}\operatorname{tg}^2\frac{A}{2},\frac{1}{r_b}\operatorname{tg}^2\frac{B}{2},\frac{1}{r_c}\operatorname{tg}^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{r_a}\operatorname{tg}^2\frac{A}{2}=\frac{4R+r}{p^2}$, $\prod\operatorname{tg}\frac{A}{2}=\frac{r}{p}$ și inegalitatea lui Mitrinović $p\ge 3r\sqrt{3}$.

Ultima inegalitate este inegalitatea lui Doucet $4R+r\ge p\sqrt{3}$.

Aplicația 20.

In ΔABC

$$\frac{r_a\cdot\sin^2\frac{A}{2}}{r_b\cdot\sin^2\frac{B}{2}}+\frac{r_b\cdot\sin^2\frac{B}{2}}{r_c\cdot\sin^2\frac{C}{2}}+\frac{r_c\cdot\sin^2\frac{C}{2}}{r_a\cdot\sin^2\frac{A}{2}}\ge\frac{3R}{2r}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(r_a\cdot\sin^2\frac{A}{2},r_b\cdot\sin^2\frac{B}{2},r_c\cdot\sin^2\frac{C}{2}\right)$.

Folosim $\sum r_a\cdot\sin^2\frac{A}{2}=\frac{8R^2+2Rr-p^2}{2R}$, $\prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}$, $r_ar_br_c=rp^2$, inegalitatea lui Mitrinović $p\le\frac{3R\sqrt{3}}{2}$, inegalitatea lui Gerretsen $p^2\le 4R^2+4Rr+3r^2$ și inegalitatea lui Euler $R\ge 2r$.

Aplicația 21.

In ΔABC

$$\frac{r_a\cdot\cos^2\frac{A}{2}}{r_b\cdot\cos^2\frac{B}{2}}+\frac{r_b\cdot\cos^2\frac{B}{2}}{r_c\cdot\cos^2\frac{C}{2}}+\frac{r_c\cdot\cos^2\frac{C}{2}}{r_a\cdot\cos^2\frac{A}{2}}\ge\sqrt[3]{\frac{2p^2}{Rr}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(r_a\cdot\cos^2\frac{A}{2},r_b\cdot\cos^2\frac{B}{2},r_c\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum r_a\cdot\cos^2\frac{A}{2}=\frac{p^2}{2R}$, $\prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}$, $r_ar_br_c=rp^2$.

Ultima inegalitate este $2p^2\ge 27Rr$ (C.Coșniță și F. Turtoiu, 1965).

Aplicația 22.

In ΔABC

$$\frac{\frac{1}{p-a}\cdot\cos^2\frac{A}{2}}{\frac{1}{p-b}\cdot\cos^2\frac{B}{2}}+\frac{\frac{1}{p-b}\cdot\cos^2\frac{B}{2}}{\frac{1}{p-c}\cdot\cos^2\frac{C}{2}}+\frac{\frac{1}{p-c}\cdot\cos^2\frac{C}{2}}{\frac{1}{p-a}\cdot\cos^2\frac{A}{2}}\ge\sqrt[3]{\frac{2p^2}{Rr}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{p-a}\cdot\cos^2\frac{A}{2},\frac{1}{p-b}\cdot\cos^2\frac{B}{2},\frac{1}{p-c}\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{p-a}\cdot\cos^2\frac{A}{2}=\frac{p}{2Rr}$, $\prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}$, $\prod(p-a)=r^2p$.

Aplicația 23.

In ΔABC

$$\frac{\frac{1}{r_a}\cdot\sin^2\frac{A}{2}}{\frac{1}{r_b}\cdot\sin^2\frac{B}{2}}+\frac{\frac{1}{r_b}\cdot\sin^2\frac{B}{2}}{\frac{1}{r_c}\cdot\sin^2\frac{C}{2}}+\frac{\frac{1}{r_c}\cdot\sin^2\frac{C}{2}}{\frac{1}{r_a}\cdot\sin^2\frac{A}{2}}\ge\sqrt[3]{\frac{2p^2}{Rr}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{r_a}\cdot\sin^2\frac{A}{2},\frac{1}{r_b}\cdot\sin^2\frac{B}{2},\frac{1}{r_c}\cdot\sin^2\frac{C}{2}\right)$.

Folosim $\sum \frac{1}{r_a}\cdot \sin^2\frac{A}{2}=\frac{1}{2R}, \prod \sin^2\frac{A}{2}=\frac{r^2}{16R^2}, r_a r_b r_c = rp^2$.

Aplicația 24.

In ΔABC

$$\frac{\frac{1}{r_a}\cdot\cos^2\frac{A}{2}}{\frac{1}{r_b}\cdot\cos^2\frac{B}{2}}+\frac{\frac{1}{r_b}\cdot\cos^2\frac{B}{2}}{\frac{1}{r_c}\cdot\cos^2\frac{C}{2}}+\frac{\frac{1}{r_c}\cdot\cos^2\frac{C}{2}}{\frac{1}{r_a}\cdot\cos^2\frac{A}{2}}\geq\left(\frac{2R}{r}-1\right)\sqrt[3]{\frac{2r}{R}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{r_a}\cdot\cos^2\frac{A}{2},\frac{1}{r_b}\cdot\cos^2\frac{B}{2},\frac{1}{r_c}\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{r_a}\cdot\cos^2\frac{A}{2}=\frac{2R-r}{2Rr}, \prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}, r_a r_b r_c = rp^2$.

Aplicația 25.

In ΔABC

$$\frac{h_a\cdot\sin^2\frac{A}{2}}{h_b\cdot\sin^2\frac{B}{2}}+\frac{h_b\cdot\sin^2\frac{B}{2}}{h_c\cdot\sin^2\frac{C}{2}}+\frac{h_c\cdot\sin^2\frac{C}{2}}{h_a\cdot\sin^2\frac{A}{2}}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(h_a\cdot\sin^2\frac{A}{2},h_b\cdot\sin^2\frac{B}{2},h_c\cdot\sin^2\frac{C}{2}\right)$.

Folosim $\sum h_a\cdot\sin^2\frac{A}{2}=\frac{r(4R+r)}{2R}, \prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}, h_a h_b h_c=\frac{2r^2p^2}{R}$ și inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$. Ultima inegalitate este inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicația 26.

In ΔABC

$$\frac{h_a\cdot\cos^2\frac{A}{2}}{h_b\cdot\cos^2\frac{B}{2}}+\frac{h_b\cdot\cos^2\frac{B}{2}}{h_c\cdot\cos^2\frac{C}{2}}+\frac{h_c\cdot\cos^2\frac{C}{2}}{h_a\cdot\cos^2\frac{A}{2}}\geq\sqrt[3]{\frac{p^2}{r^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(h_a\cdot\cos^2\frac{A}{2},h_b\cdot\cos^2\frac{B}{2},h_c\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum h_a\cdot\cos^2\frac{A}{2}=\frac{p^2}{2R}, \prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}, h_a h_b h_c=\frac{2r^2p^2}{R}$.

Ultima inegalitate este inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$.

Aplicația 27.

In ΔABC

$$\frac{\frac{1}{h_a}\cdot\sin^2\frac{A}{2}}{\frac{1}{h_b}\cdot\sin^2\frac{B}{2}}+\frac{\frac{1}{h_b}\cdot\sin^2\frac{B}{2}}{\frac{1}{h_c}\cdot\sin^2\frac{C}{2}}+\frac{\frac{1}{h_c}\cdot\sin^2\frac{C}{2}}{\frac{1}{h_a}\cdot\sin^2\frac{A}{2}}\geq\frac{2p}{\sqrt{3}}\left(\frac{1}{r}-\frac{1}{R}\right)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{h_a}\cdot\sin^2\frac{A}{2},\frac{1}{h_b}\cdot\sin^2\frac{B}{2},\frac{1}{h_c}\cdot\sin^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{h_a}\cdot\sin^2\frac{A}{2}=\frac{R-2r}{2Rr}$, $\prod\sin^2\frac{A}{2}=\frac{r^2}{16R^2}$, $h_ah_bh_c=\frac{2r^2p^2}{R}$ și inegalitatea lui Mitrinović $p\leq\frac{3R\sqrt{3}}{2}$.

Pentru a doua inegalitate vezi **12)**.

Aplicația 28.

In ΔABC

$$\frac{\frac{1}{h_a}\cdot\cos^2\frac{A}{2}}{\frac{1}{h_b}\cdot\cos^2\frac{B}{2}}+\frac{\frac{1}{h_b}\cdot\cos^2\frac{B}{2}}{\frac{1}{h_c}\cdot\cos^2\frac{C}{2}}+\frac{\frac{1}{h_c}\cdot\cos^2\frac{C}{2}}{\frac{1}{h_a}\cdot\cos^2\frac{A}{2}}\geq\frac{2(R+r)}{\sqrt[3]{2R^2r}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{h_a}\cdot\cos^2\frac{A}{2},\frac{1}{h_b}\cdot\cos^2\frac{B}{2},\frac{1}{h_c}\cdot\cos^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{h_a}\cdot\cos^2\frac{A}{2}=\frac{R+r}{2Rr}$, $\prod\cos^2\frac{A}{2}=\frac{p^2}{16R^2}$, $h_ah_bh_c=\frac{2r^2p^2}{R}$.

Aplicația 29.

In ΔABC

$$\frac{\frac{1}{h_a}\cdot\sec^2\frac{A}{2}}{\frac{1}{h_b}\cdot\sec^2\frac{B}{2}}+\frac{\frac{1}{h_b}\cdot\sec^2\frac{B}{2}}{\frac{1}{h_c}\cdot\sec^2\frac{C}{2}}+\frac{\frac{1}{h_c}\cdot\sec^2\frac{C}{2}}{\frac{1}{h_a}\cdot\sec^2\frac{A}{2}}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{h_a}\cdot\sec^2\frac{A}{2},\frac{1}{h_b}\cdot\sec^2\frac{B}{2},\frac{1}{h_c}\cdot\sec^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{h_a}\cdot\sec^2\frac{A}{2}=\frac{2R(R+r)}{rp^2}$, $\prod\sec^2\frac{A}{2}=\frac{16R^2}{p^2}$, $h_ah_bh_c=\frac{2r^2p^2}{R}$ și inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$.

Aplicația 30.

In ΔABC

$$\frac{\frac{1}{h_a}\cdot\csc^2\frac{A}{2}}{\frac{1}{h_b}\cdot\csc^2\frac{B}{2}}+\frac{\frac{1}{h_b}\cdot\csc^2\frac{B}{2}}{\frac{1}{h_c}\cdot\csc^2\frac{C}{2}}+\frac{\frac{1}{h_c}\cdot\csc^2\frac{C}{2}}{\frac{1}{h_a}\cdot\csc^2\frac{A}{2}}\geq\sqrt[3]{\frac{p^2}{r^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{h_a}\cdot\csc^2\frac{A}{2},\frac{1}{h_b}\cdot\csc^2\frac{B}{2},\frac{1}{h_c}\cdot\csc^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{h_a}\cdot\csc^2\frac{A}{2}=\frac{2R}{r^2}$, $\prod\csc^2\frac{A}{2}=\frac{16R^2}{r^2}$, $h_ah_bh_c=\frac{2r^2p^2}{R}$.

Aplicația 31.

In ΔABC

$$\frac{a^2\cdot\text{tg}\frac{A}{2}}{b^2\cdot\text{tg}\frac{B}{2}}+\frac{b^2\cdot\text{tg}\frac{B}{2}}{c^2\cdot\text{tg}\frac{C}{2}}+\frac{c^2\cdot\text{tg}\frac{C}{2}}{a^2\cdot\text{tg}\frac{A}{2}}\geq\left(\frac{R}{r}-1\right)\sqrt[3]{\frac{4p^2}{R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a^2\cdot\text{tg}\frac{A}{2},b^2\cdot\text{tg}\frac{B}{2},c^2\cdot\text{tg}\frac{C}{2}\right)$.

Folosim $\sum a^2\cdot\text{tg}\frac{A}{2}=4p(R-r)$, $\prod\text{tg}\frac{A}{2}=\frac{r}{p}$ și $abc=4prR$.

Aplicația 32.

In ΔABC

$$\frac{a^2\cdot\text{ctg}\frac{A}{2}}{b^2\cdot\text{ctg}\frac{B}{2}}+\frac{b^2\cdot\text{ctg}\frac{B}{2}}{c^2\cdot\text{ctg}\frac{C}{2}}+\frac{c^2\cdot\text{ctg}\frac{C}{2}}{a^2\cdot\text{ctg}\frac{A}{2}}\geq\frac{2(R+r)}{\sqrt[3]{2R^2r}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a^2\cdot\text{ctg}\frac{A}{2},b^2\cdot\text{ctg}\frac{B}{2},c^2\cdot\text{ctg}\frac{C}{2}\right)$.

Folosim $\sum a^2\cdot\text{ctg}\frac{A}{2}=4p(R+r)$, $\prod\text{ctg}\frac{A}{2}=\frac{p}{r}$ și $abc=4prR$.

Aplicația 33.

In ΔABC

$$\frac{\frac{1}{a}\cdot\text{tg}\frac{A}{2}}{\frac{1}{b}\cdot\text{tg}\frac{B}{2}}+\frac{\frac{1}{b}\cdot\text{tg}\frac{B}{2}}{\frac{1}{c}\cdot\text{tg}\frac{C}{2}}+\frac{\frac{1}{c}\cdot\text{tg}\frac{C}{2}}{\frac{1}{a}\cdot\text{tg}\frac{A}{2}}\geq\frac{1}{2}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\sqrt[3]{\frac{p^2}{2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{a}\cdot\text{tg}\frac{A}{2},\frac{1}{b}\cdot\text{tg}\frac{B}{2},\frac{1}{c}\cdot\text{tg}\frac{C}{2}\right)$.

Folosim $\sum \frac{1}{a}\cdot \operatorname{tg}\frac{A}{2}=\frac{1}{4R}\left[1+\left(\frac{4R+r}{p}\right)^2\right]$, $\prod \operatorname{tg}\frac{A}{2}=\frac{r}{p}$ și $abc=4prR$.

Aplicația 34.

In ΔABC

$$\frac{\frac{1}{a}\cdot \operatorname{ctg}\frac{A}{2}}{\frac{1}{b}\cdot \operatorname{ctg}\frac{B}{2}}+\frac{\frac{1}{b}\cdot \operatorname{ctg}\frac{B}{2}}{\frac{1}{c}\cdot \operatorname{ctg}\frac{C}{2}}+\frac{\frac{1}{c}\cdot \operatorname{ctg}\frac{C}{2}}{\frac{1}{a}\cdot \operatorname{ctg}\frac{A}{2}}\geq 4-\frac{2r}{R}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{a}\cdot \operatorname{ctg}\frac{A}{2},\frac{1}{b}\cdot \operatorname{ctg}\frac{B}{2},\frac{1}{c}\cdot \operatorname{ctg}\frac{C}{2}\right)$.

Folosim $\sum \frac{1}{a}\cdot \operatorname{ctg}\frac{A}{2}=\frac{p^2+r^2-8Rr}{4Rr^2}$, $\prod \operatorname{ctg}\frac{A}{2}=\frac{p}{r}$, $abc=4prR$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

In ΔABC

$$\frac{\csc A}{\csc B}+\frac{\csc B}{\csc C}+\frac{\csc C}{\csc A}\geq \frac{p^2+r^2+4Rr}{2\sqrt[3]{2p^2r^2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(\csc A,\csc B,\csc C)$.

Folosim $\sum \csc A=\frac{p^2+r^2+4Rr}{2rp}$, $\prod \csc A=\frac{2R^2}{rp}$ și $abc=4prR$.

Aplicația 36.

In ΔABC

$$\frac{a^2\cdot \sin A}{b^2\cdot \sin B}+\frac{b^2\cdot \sin B}{c^2\cdot \sin C}+\frac{c^2\cdot \sin C}{a^2\cdot \sin A}\geq 5-\frac{4r}{R}\geq 3.$$

Soluție.

Vezi teorema sinusurilor și **Aplicația 3.**

Aplicația 37.

In ΔABC

$$\frac{a\cdot \operatorname{tg}^2\frac{A}{2}}{b\cdot \operatorname{tg}^2\frac{B}{2}}+\frac{b\cdot \operatorname{tg}^2\frac{B}{2}}{c\cdot \operatorname{tg}^2\frac{C}{2}}+\frac{c\cdot \operatorname{tg}^2\frac{C}{2}}{a\cdot \operatorname{tg}^2\frac{A}{2}}\geq \frac{9R}{4r}\sqrt[3]{\frac{2R^2}{p^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot \operatorname{tg}^2\frac{A}{2},b\cdot \operatorname{tg}^2\frac{B}{2},c\cdot \operatorname{tg}^2\frac{C}{2}\right)$.

Folosim $\sum a\cdot \text{tg}^2\frac{A}{2}=\frac{4R(4R+r)-2p^2}{p}$, $\prod \text{tg}\frac{A}{2}=\frac{r}{p}$, $abc=4prR$, inegalitatea lui Gerretsen $p^2\le 4R^2+4Rr+3r^2$, inegalitatea lui Mitrinović $p\le\frac{3R\sqrt{3}}{2}$ şi inegalitatea lui Euler $R\ge 2r$.

Aplicația 38

In ΔABC

$$\frac{a\cdot \text{ctg}^2\frac{A}{2}}{b\cdot \text{ctg}^2\frac{B}{2}}+\frac{b\cdot \text{ctg}^2\frac{B}{2}}{c\cdot \text{ctg}^2\frac{C}{2}}+\frac{c\cdot \text{ctg}^2\frac{C}{2}}{a\cdot \text{ctg}^2\frac{A}{2}}\ge\frac{2(2R-r)}{\sqrt[3]{4Rr^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot \text{ctg}^2\frac{A}{2},b\cdot \text{ctg}^2\frac{B}{2},c\cdot \text{ctg}^2\frac{C}{2}\right)$.

Folosim $\sum a\cdot \text{ctg}^2\frac{A}{2}=\frac{2p(2R-r)}{r}$, $\prod \text{ctg}\frac{A}{2}=\frac{p}{r}$ şi $abc=4prR$.

Aplicația 39

In ΔABC

$$\frac{\frac{1}{p-a}\cdot \text{ctg}^2\frac{A}{2}}{\frac{1}{p-b}\cdot \text{ctg}^2\frac{B}{2}}+\frac{\frac{1}{p-b}\cdot \text{ctg}^2\frac{B}{2}}{\frac{1}{p-c}\cdot \text{ctg}^2\frac{C}{2}}+\frac{\frac{1}{p-c}\cdot \text{ctg}^2\frac{C}{2}}{\frac{1}{p-a}\cdot \text{ctg}^2\frac{A}{2}}\ge\sqrt[3]{\frac{p^2}{r^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{p-a}\cdot \text{ctg}^2\frac{A}{2},\frac{1}{p-b}\cdot \text{ctg}^2\frac{B}{2},\frac{1}{p-c}\cdot \text{ctg}^2\frac{C}{2}\right)$.

Folosim $\sum\frac{1}{p-a}\cdot \text{ctg}^2\frac{A}{2}=\frac{p}{r^2}$, $\prod \text{ctg}\frac{A}{2}=\frac{p}{r}$ şi $abc=4prR$.

Aplicația 40

In ΔABC

$$\frac{a\cdot \csc^2\frac{A}{2}}{b\cdot \csc^2\frac{B}{2}}+\frac{b\cdot \csc^2\frac{B}{2}}{c\cdot \csc^2\frac{C}{2}}+\frac{c\cdot \csc^2\frac{C}{2}}{a\cdot \csc^2\frac{A}{2}}\ge\sqrt[3]{\frac{p^2}{r^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot \csc^2\frac{A}{2},b\cdot \csc^2\frac{B}{2},c\cdot \csc^2\frac{C}{2}\right)$.

Folosim $\sum a\cdot \csc^2\frac{A}{2}=\frac{4Rp}{r}$, $\prod \csc\frac{A}{2}=\frac{4R}{r}$ şi $abc=4prR$.

Aplicația 41

In ΔABC

$$\frac{a\cdot\sec^2\frac{A}{2}}{b\cdot\sec^2\frac{B}{2}}+\frac{b\cdot\sec^2\frac{B}{2}}{c\cdot\sec^2\frac{C}{2}}+\frac{c\cdot\sec^2\frac{C}{2}}{a\cdot\sec^2\frac{A}{2}}\ge\frac{\sqrt{3}}{p}(4R+r)\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a\cdot\sec^2\frac{A}{2},b\cdot\sec^2\frac{B}{2},c\cdot\sec^2\frac{C}{2}\right)$.

Folosim $\sum a\cdot\sec^2\frac{A}{2}=\frac{4R(4R+r)}{p}$, $\prod\sec\frac{A}{2}=\frac{4R}{p}$, $abc=4prR$, inegalitatea lui Mitrinović $p\ge 3r\sqrt{3}$ și inegalitatea lui Doucet $4R+r\ge p\sqrt{3}$.

Aplicația 42.

In ΔABC

$$\frac{a\cdot r_a}{b\cdot r_b}+\frac{b\cdot r_b}{c\cdot r_c}+\frac{c\cdot r_c}{a\cdot r_a}\ge 4-\frac{2r}{R}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(a\cdot r_a,b\cdot r_b,c\cdot r_c)$.

Folosim $\sum a\cdot r_a=2p(2R-r)$, $abc=4prR$, $r_ar_br_c=rp^2$ și inegalitatea lui Euler $R\ge 2r$.

Aplicația 43.

In ΔABC

$$\frac{\frac{1}{a}\cdot r_a}{\frac{1}{b}\cdot r_b}+\frac{\frac{1}{b}\cdot r_b}{\frac{1}{c}\cdot r_c}+\frac{\frac{1}{c}\cdot r_c}{\frac{1}{a}\cdot r_a}\ge\frac{1}{2}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\sqrt[3]{\frac{p^2}{2R^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{r_a}{a},\frac{r_b}{b},\frac{r_c}{c}\right)$.

Folosim $\sum\frac{r_a}{a}=\frac{1}{4R}\left[1+\left(\frac{4R+r}{p}\right)^2\right]$, $abc=4prR$, $r_ar_br_c=rp^2$.

Aplicația 44.

In ΔABC

$$\frac{a^2\cdot\frac{1}{r_a}}{b^2\cdot\frac{1}{r_b}}+\frac{b^2\cdot\frac{1}{r_b}}{c^2\cdot\frac{1}{r_c}}+\frac{c^2\cdot\frac{1}{r_c}}{a^2\cdot\frac{1}{r_a}}\ge\frac{2(R+r)}{\sqrt[3]{2R^2r}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a^2}{r_a},\frac{b^2}{r_b},\frac{c^2}{r_c}\right)$.

Folosim $\sum\frac{a^2}{r_a}=4(R+r)$, $abc=4prR$, $r_ar_br_c=rp^2$.

Aplicația 45.

In ΔABC

$$\frac{a\cdot\frac{1}{r_a^2}}{b\cdot\frac{1}{r_b^2}}+\frac{b\cdot\frac{1}{r_b^2}}{c\cdot\frac{1}{r_c^2}}+\frac{c\cdot\frac{1}{r_c^2}}{a\cdot\frac{1}{r_a^2}}\ge\left(\frac{2R}{r}-1\right)\sqrt[3]{\frac{2r}{R}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{r_a^2},\frac{b}{r_b^2},\frac{c}{r_c^2}\right)$.

Folosim $\sum\frac{a}{r_a^2}=\frac{2(2R-r)}{rp}$, $abc=4prR$, $r_ar_br_c=rp^2$.

Aplicația 46.

In ΔABC

$$\frac{a^2\cdot\frac{1}{r_a^2}}{b^2\cdot\frac{1}{r_b^2}}+\frac{b^2\cdot\frac{1}{r_b^2}}{c^2\cdot\frac{1}{r_c^2}}+\frac{c^2\cdot\frac{1}{r_c^2}}{a^2\cdot\frac{1}{r_a^2}}\ge\left[\left(\frac{4R+r}{p}\right)^2-1\right]\sqrt[3]{\frac{p^2}{2R^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a^2}{r_a^2},\frac{b^2}{r_b^2},\frac{c^2}{r_c^2}\right)$.

Folosim $\sum\frac{a^2}{r_a^2}=2\left[\left(\frac{4R+r}{p}\right)^2-1\right]$, $abc=4prR$, $r_ar_br_c=rp^2$.

Aplicația 47.

In ΔABC

$$\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}\ge 3\left[\left(\frac{4R+r}{p}\right)^2-2\right]\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(r_a^2,r_b^2,r_c^2\right)$.

Folosim $\sum r_a^2=(4R+r)^2-2p^2$, $r_ar_br_c=rp^2$.

Aplicația 48.

In ΔABC

$$\frac{r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}+\frac{r_c^3}{r_a^3}\ge\frac{3R}{2r}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(r_a^3,r_b^3,r_c^3\right)$.

Folosim $\sum r_a^3 = (4R+r)^3 - 12Rp^2$, $r_a r_b r_c = rp^2$, inegalitatea lui Blundon $p^2 \le \dfrac{R(4R+r)^2}{2(2R-r)}$ și inegalitatea lui Euler $R \ge 2r$.

Aplicația 49.

In ΔABC

$$\frac{a^2\cdot r_a}{b^2\cdot r_b}+\frac{b^2\cdot r_b}{c^2\cdot r_c}+\frac{c^2\cdot r_c}{a^2\cdot r_a}\ge\left(\frac{R}{r}-1\right)\sqrt[3]{\frac{4p^2}{R^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a^2\cdot r_a, b^2\cdot r_b, c^2\cdot r_c\right)$.

Folosim $\sum a^2\cdot r_a = 4p^2(R-r)$, $abc = 4prR$, $r_a r_b r_c = rp^2$.

Aplicația 50.

In ΔABC

$$\frac{\frac{1}{a}\cdot(r_b+r_c)}{\frac{1}{b}\cdot(r_c+r_a)}+\frac{\frac{1}{b}\cdot(r_c+r_a)}{\frac{1}{c}\cdot(r_a+r_b)}+\frac{\frac{1}{c}\cdot(r_a+r_b)}{\frac{1}{a}\cdot(r_b+r_c)}\ge\sqrt[3]{\frac{p^2}{r^2}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{r_b+r_c}{a},\frac{r_c+r_a}{b},\frac{r_a+r_b}{c}\right)$.

Folosim $\sum\frac{r_b+r_c}{a}=\frac{p}{r}$, $\prod(r_b+r_c)=4Rp^2$ și $abc=4prR$.

Aplicația 51.

In ΔABC

$$\frac{\frac{1}{r_a}\cdot(r_b+r_c)}{\frac{1}{r_b}\cdot(r_c+r_a)}+\frac{\frac{1}{r_b}\cdot(r_c+r_a)}{\frac{1}{r_c}\cdot(r_a+r_b)}+\frac{\frac{1}{r_c}\cdot(r_a+r_b)}{\frac{1}{r_a}\cdot(r_b+r_c)}\ge\left(\frac{2R}{r}-1\right)\sqrt[3]{\frac{2r}{R}}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{r_b+r_c}{r_a},\frac{r_c+r_a}{r_b},\frac{r_a+r_b}{r_c}\right)$.

Folosim $\sum\frac{r_b+r_c}{r_a}=\frac{2(2R-r)}{r}$, $\prod(r_b+r_c)=4Rp^2$ și $r_a r_b r_c = rp^2$.

Aplicația 52.

In ΔABC

$$\frac{a^2\cdot\frac{1}{h_a}}{b^2\cdot\frac{1}{h_b}}+\frac{b^2\cdot\frac{1}{h_b}}{c^2\cdot\frac{1}{h_c}}+\frac{c^2\cdot\frac{1}{h_c}}{a^2\cdot\frac{1}{h_a}}\ge 5-\frac{4r}{R}\ge 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a^2}{h_a},\frac{b^2}{h_b},\frac{c^2}{h_c}\right)$.

Folosim $\sum\frac{a^2}{h_a}=\frac{p^2-3r^2-6Rr}{r}$, $abc=4prR$, $h_ah_bh_c=\frac{2r^2p^2}{R}$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 53.

In ΔABC

$$\frac{a\cdot\frac{1}{h_a^2}}{b\cdot\frac{1}{h_b^2}}+\frac{b\cdot\frac{1}{h_b^2}}{c\cdot\frac{1}{h_c^2}}+\frac{c\cdot\frac{1}{h_c^2}}{a\cdot\frac{1}{h_a^2}}\geq 5-\frac{4r}{R}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{h_a^2},\frac{b}{h_b^2},\frac{c}{h_c^2}\right)$.

Folosim $\sum\frac{a}{h_a^2}=\frac{p^2-3r^2-6Rr}{2r^2p}$, $abc=4prR$, $h_ah_bh_c=\frac{2r^2p^2}{R}$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 54.

In ΔABC

$$\frac{\frac{1}{h_a}\cdot(r_b+r_c)}{\frac{1}{h_b}\cdot(r_c+r_a)}+\frac{\frac{1}{h_b}\cdot(r_c+r_a)}{\frac{1}{h_c}\cdot(r_a+r_b)}+\frac{\frac{1}{h_c}\cdot(r_a+r_b)}{\frac{1}{h_a}\cdot(r_b+r_c)}\geq\left(\frac{R}{r}+1\right)\sqrt[3]{\frac{4r^2}{R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{r_b+r_c}{h_a},\frac{r_c+r_a}{h_b},\frac{r_a+r_b}{h_c}\right)$.

Folosim $\sum\frac{r_b+r_c}{h_a}=\frac{2(R+r)}{r}$, $\prod(r_b+r_c)=4Rp^2$ și $h_ah_bh_c=\frac{2r^2p^2}{R}$.

Aplicația 55.

In ΔABC

$$\frac{h_a\cdot r_a}{h_b\cdot r_b}+\frac{h_b\cdot r_b}{h_c\cdot r_c}+\frac{h_c\cdot r_c}{h_a\cdot r_a}\geq\frac{1}{2}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\sqrt[3]{\frac{p^2}{2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(h_a\cdot r_a,h_b\cdot r_b,h_c\cdot r_c)$.

Folosim $\sum h_a\cdot r_a=\frac{r}{2R}\left[p^2+(4R+r)^2\right]$, $h_ah_bh_c=\frac{2r^2p^2}{R}$, $r_ar_br_c=rp^2$.

Aplicația 56.

In ΔABC

$$\frac{a\cdot\frac{1}{r_a}}{b\cdot\frac{1}{r_b}}+\frac{b\cdot\frac{1}{r_b}}{c\cdot\frac{1}{r_c}}+\frac{c\cdot\frac{1}{r_c}}{a\cdot\frac{1}{r_a}}\geq\frac{2(4R+r)}{\sqrt[3]{4Rp^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{r_a},\frac{b}{r_b},\frac{c}{r_c}\right)$.

Folosim $\sum\frac{a}{r_a}=\frac{2(4R+r)}{p}$, $abc=4prR$, $r_ar_br_c=rp^2$.

Aplicația 57.

In ΔABC

$$\frac{a\cdot\frac{1}{h_a}}{b\cdot\frac{1}{h_b}}+\frac{b\cdot\frac{1}{h_b}}{c\cdot\frac{1}{h_c}}+\frac{c\cdot\frac{1}{h_c}}{a\cdot\frac{1}{h_a}}\geq\frac{p^2-r^2-4Rr}{\sqrt[3]{2p^2r^2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{h_a},\frac{b}{h_b},\frac{c}{h_c}\right)$.

Folosim $\sum\frac{a}{h_a}=\frac{p^2-r^2-4Rr}{rp}$, $abc=4prR$, $h_ah_bh_c=\frac{2r^2p^2}{R}$.

Aplicația 58.

In ΔABC

$$\frac{(p-a)\cdot\frac{1}{h_ar_a}}{(p-b)\cdot\frac{1}{h_br_b}}+\frac{(p-b)\cdot\frac{1}{h_br_b}}{(p-c)\cdot\frac{1}{h_cr_c}}+\frac{(p-c)\cdot\frac{1}{h_cr_c}}{(p-a)\cdot\frac{1}{h_ar_a}}\geq\left(\frac{2R}{r}-1\right)\sqrt[3]{\frac{2r}{R}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{p-a}{h_ar_a},\frac{p-b}{h_br_b},\frac{p-c}{h_cr_c}\right)$.

Folosim $\sum\frac{p-a}{h_ar_a}=\frac{2R-r}{rp}$, $\prod(p-a)=r^2p$, $h_ah_bh_c=\frac{2r^2p^2}{R}$ și $r_ar_br_c=rp^2$.

Aplicația 59.

In ΔABC

$$\frac{a\cdot\frac{1}{h_a+2r_a}}{b\cdot\frac{1}{h_b+2r_b}}+\frac{b\cdot\frac{1}{h_b+2r_b}}{c\cdot\frac{1}{h_c+2r_c}}+\frac{c\cdot\frac{1}{h_c+2r_c}}{a\cdot\frac{1}{h_a+2r_a}}\geq\frac{2(R+r)}{\sqrt[3]{2R^2r}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{h_a+2r_a},\frac{b}{h_b+2r_b},\frac{c}{h_c+2r_c}\right)$.

Folosim $\sum\frac{a}{h_a+2r_a}=\frac{2(R+r)}{p}$, $\prod(h_a+2r_a)=\frac{2p^4}{R}$ şi $abc=4prR$.

Aplicaţia 60.

In ΔABC

$$\frac{r_a^2\cdot\frac{1}{h_a}}{r_b^2\cdot\frac{1}{h_b}}+\frac{r_b^2\cdot\frac{1}{h_b}}{r_c^2\cdot\frac{1}{h_c}}+\frac{r_c^2\cdot\frac{1}{h_c}}{r_a^2\cdot\frac{1}{h_a}}\geq\frac{3R}{2r}\geq 3.$$

Soluţie.

În **Lemă** punem $(x,y,z)=\left(\frac{r_a^2}{h_a},\frac{r_b^2}{h_b},\frac{r_c^2}{h_c}\right)$.

Folosim $\sum\frac{r_a^2}{h_a}=\frac{4R(4R+r)-p^2}{r}$, $r_ar_br_c=rp^2$, $h_ah_bh_c=\frac{2r^2p^2}{R}$, inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$, inegalitatea lui Mitrinović $p\leq\frac{3R\sqrt{3}}{2}$ şi inegalitatea lui Euler $R\geq 2r$.

Aplicaţia 61.

In ΔABC

$$\frac{r_a\cdot\frac{1}{p-a}}{r_b\cdot\frac{1}{p-b}}+\frac{r_b\cdot\frac{1}{p-b}}{r_c\cdot\frac{1}{p-c}}+\frac{r_c\cdot\frac{1}{p-c}}{r_a\cdot\frac{1}{p-a}}\geq 3\left[\left(\frac{4R+r}{p}\right)^2-2\right]\geq 3.$$

Soluţie.

În **Lemă** punem $(x,y,z)=\left(\frac{r_a}{p-a},\frac{r_b}{p-b},\frac{r_c}{p-c}\right)$.

Folosim $\sum\frac{r_a}{p-a}=\frac{p}{r}\left[\left(\frac{4R+r}{p}\right)^2-2\right]$, $\prod(p-a)=r^2p$, $r_ar_br_c=rp^2$, inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$ şi inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$.

Aplicaţia 62.

In ΔABC

$$\frac{\frac{1}{p-a}}{\frac{1}{p-b}}+\frac{\frac{1}{p-b}}{\frac{1}{p-c}}+\frac{\frac{1}{p-c}}{\frac{1}{p-a}}\geq\left(\frac{4R}{r}+1\right)\sqrt[3]{\frac{r^2}{p^2}}\geq 3.$$

Soluţie.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{p-a},\frac{1}{p-b},\frac{1}{p-c}\right)$.

Folosim $\sum\frac{1}{p-a}=\frac{4R+r}{rp}$, $\prod(p-a)=r^2p$.

Aplicația 63.

In ΔABC

$$\frac{\frac{1}{a}\cdot(p-a)}{\frac{1}{b}\cdot(p-b)}+\frac{\frac{1}{b}\cdot(p-b)}{\frac{1}{c}\cdot(p-c)}+\frac{\frac{1}{c}\cdot(p-c)}{\frac{1}{a}\cdot(p-a)}\geq 4-\frac{2r}{R}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{p-a}{a},\frac{p-b}{b},\frac{p-c}{c}\right)$.

Folosim $\sum\frac{p-a}{a}=\frac{p^2+r^2-8Rr}{4Rr}$, $\prod(p-a)=r^2p$, $abc=4prR$.

Aplicația 64.

In ΔABC

$$\frac{a\cdot\frac{1}{p-a}}{b\cdot\frac{1}{p-b}}+\frac{b\cdot\frac{1}{p-b}}{c\cdot\frac{1}{p-c}}+\frac{c\cdot\frac{1}{p-c}}{a\cdot\frac{1}{p-a}}\geq\left(\frac{2R}{r}-1\right)\sqrt[3]{\frac{2r}{R}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{p-a},\frac{b}{p-b},\frac{c}{p-c}\right)$.

Folosim $\sum\frac{a}{p-a}=\frac{2(2R-r)}{r}$, $\prod(p-a)=r^2p$, $abc=4prR$.

Aplicația 65.

In ΔABC

$$\frac{a^2\cdot\frac{1}{p-a}}{b^2\cdot\frac{1}{p-b}}+\frac{b^2\cdot\frac{1}{p-b}}{c^2\cdot\frac{1}{p-c}}+\frac{c^2\cdot\frac{1}{p-c}}{a^2\cdot\frac{1}{p-a}}\geq\left(\frac{R}{r}-1\right)\sqrt[3]{\frac{4p^2}{R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a^2}{p-a},\frac{b^2}{p-b},\frac{c^2}{p-c}\right)$.

Folosim $\sum\frac{a^2}{p-a}=\frac{4p(R-r)}{r}$, $\prod(p-a)=r^2p$, $abc=4prR$.

Aplicația 66.

In ΔABC

$$\frac{\frac{1}{a^2}\cdot(p-a)}{\frac{1}{b^2}\cdot(p-b)}+\frac{\frac{1}{b^2}\cdot(p-b)}{\frac{1}{c^2}\cdot(p-c)}+\frac{\frac{1}{c^2}\cdot(p-c)}{\frac{1}{a^2}\cdot(p-a)}\geq\frac{1}{3}\left(4-\frac{2r}{R}\right)^2\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{p-a}{a^2},\frac{p-b}{b^2},\frac{p-c}{c^2}\right)$.

Folosim $\sum\frac{p-a}{a^2}=\frac{p^2\left(p^2+r^2-12Rr\right)+r^3(4R+r)}{16R^2r^2p^2}$, $\prod(p-a)=r^2p$, $abc=4prR$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$, inegalitatea lui Mitrinović $p\leq\frac{3R\sqrt{3}}{2}$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 67.

In ΔABC

$$\frac{\frac{1}{a(p-a)}}{\frac{1}{b(p-b)}}+\frac{\frac{1}{b(p-b)}}{\frac{1}{c(p-c)}}+\frac{\frac{1}{c(p-c)}}{\frac{1}{a(p-a)}}\geq\frac{1}{2}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\sqrt[3]{\frac{p^2}{2R^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{1}{a(p-a)},\frac{1}{b(p-b)},\frac{1}{c(p-c)}\right)$.

Folosim $\sum\frac{1}{a(p-a)}=\frac{1}{4Rr}\left[1+\left(\frac{4R+r}{p}\right)^2\right]$, $\prod(p-a)=r^2p$ și $abc=4prR$.

Aplicația 68.

In ΔABC

$$\frac{a(p-a)}{b(p-b)}+\frac{b(p-b)}{c(p-c)}+\frac{c(p-c)}{a(p-a)}\geq\frac{2(4R+r)}{\sqrt[3]{4Rp^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a(p-a),b(p-b),c(p-c)\right)$.

Folosim $\sum a(p-a)=2r(4R+r)$, $\prod(p-a)=r^2p$ și $abc=4prR$.

Aplicația 69.

In ΔABC

$$\frac{a^2(p-a)}{b^2(p-b)}+\frac{b^2(p-b)}{c^2(p-c)}+\frac{c^2(p-c)}{a^2(p-a)}\geq\frac{2(R+r)}{\sqrt[3]{2R^2r}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a^2(p-a),b^2(p-b),c^2(p-c)\right)$.

Folosim $\sum a^2(p-a)=4rp(R+r)$, $\prod(p-a)=r^2p$ și $abc=4prR$.

Aplicația 70.

In ΔABC

$$\frac{a\cdot\frac{1}{(p-a)^2}}{b\cdot\frac{1}{(p-b)^2}}+\frac{b\cdot\frac{1}{(p-b)^2}}{c\cdot\frac{1}{(p-c)^2}}+\frac{c\cdot\frac{1}{(p-c)^2}}{a\cdot\frac{1}{(p-a)^2}}\geq\frac{3R}{2r}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a}{(p-a)^2},\frac{b}{(p-b)^2},\frac{c}{(p-c)^2}\right)$.

Folosim $\sum\frac{a}{(p-a)^2}=\frac{4R(4R+r)-2p^2}{r^2p}$, $\prod(p-a)=r^2p$, $abc=4prR$, inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$, inegalitatea lui Mitrinović $p\leq\frac{3R\sqrt{3}}{2}$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 71.

In ΔABC

$$\frac{\frac{1}{a}\cdot(p-a)^2}{\frac{1}{b}\cdot(p-b)^2}+\frac{\frac{1}{b}\cdot(p-b)^2}{\frac{1}{c}\cdot(p-c)^2}+\frac{\frac{1}{c}\cdot(p-c)^2}{\frac{1}{a}\cdot(p-a)^2}\geq\frac{2p}{\sqrt{3}}\left(\frac{1}{r}-\frac{1}{R}\right)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{(p-a)^2}{a},\frac{(p-b)^2}{b},\frac{(p-c)^2}{c}\right)$.

Folosim $\sum\frac{(p-a)^2}{a}=\frac{p(p^2+r^2-12Rr)}{4Rr}$, $\prod(p-a)=r^2p$, $abc=4prR$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$, inegalitatea lui Mitrinović $p\leq\frac{3R\sqrt{3}}{2}$ și inegalitatea lui Euler $R\geq 2r$.

Pentru a doua inegalitate vezi **12)**.

Aplicația 72.

In ΔABC

$$\frac{p-a}{p-b}+\frac{p-b}{p-c}+\frac{p-c}{p-a}\geq\sqrt[3]{\frac{p^2}{r^2}}\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=(p-a,p-b,p-c)$.

Folosim $\sum(p-a)=p$, $\prod(p-a)=r^2p$.

Aplicația 73.

In ΔABC

$$\left(\frac{p-a}{p-b}\right)^2+\left(\frac{p-b}{p-c}\right)^2+\left(\frac{p-c}{p-a}\right)^2\geq\frac{\sqrt{3}}{p}(8R-7r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left((p-a)^2,(p-b)^2,(p-c)^2\right)$.

Folosim $\sum(p-a)^2=p^2-2r^2-8Rr$, $\prod(p-a)=r^2p$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$, inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 74.

In ΔABC

$$\left(\frac{p-a}{p-b}\right)^3+\left(\frac{p-b}{p-c}\right)^3+\left(\frac{p-c}{p-a}\right)^3\geq\frac{4R}{r}-5\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left((p-a)^3,(p-b)^3,(p-c)^3\right)$.

Folosim $\sum(p-a)^3=p\left(p^2-12Rr\right)$, $\prod(p-a)=r^2p$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 75.

In ΔABC

$$\frac{a^2(p-a)^2}{b^2(p-b)^2}+\frac{b^2(p-b)^2}{c^2(p-c)^2}+\frac{c^2(p-c)^2}{a^2(p-a)^2}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(a^2(p-a)^2,b^2(p-b)^2,c^2(p-c)^2\right)$.

Folosim $\sum a^2(p-a)^2=2r^2\left[(4R+r)^2-p^2\right]$, $\prod(p-a)=r^2p$, inegalitatea lui Gerretsen $p^2\leq 4R^2+4Rr+3r^2$, inegalitatea lui Mitrinović $p\leq\frac{3R\sqrt{3}}{2}$, inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$ și inegalitatea lui Euler $R\geq 2r$.

Aplicația 76.

In ΔABC

$$\frac{a\cdot\sin^2 A}{b\cdot\sin^2 B}+\frac{b\cdot\sin^2 B}{c\cdot\sin^2 C}+\frac{c\cdot\sin^2 C}{a\cdot\sin^2 A}\geq 5-\frac{4r}{R}\geq 3.$$

Soluție.

Vezi teorema sinusurilor și **Aplicația 3.**

Aplicația 76.

In ΔABC

$$\frac{a^3\cdot\frac{1}{p-a}}{b^3\cdot\frac{1}{p-b}}+\frac{b^3\cdot\frac{1}{p-b}}{c^3\cdot\frac{1}{p-c}}+\frac{c^3\cdot\frac{1}{p-c}}{a^3\cdot\frac{1}{p-a}}\geq\frac{\sqrt{3}}{p}(4R+r)\geq 3.$$

Soluție.

În **Lemă** punem $(x,y,z)=\left(\frac{a^3}{p-a},\frac{b^3}{p-b},\frac{c^3}{p-c}\right)$.

Folosim $\sum\frac{a^3}{p-a}=\frac{2p^2(2R-3r)+2r^2(4R+r)}{r}$, $\prod(p-a)=r^2p$, $abc=4prR$, inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$, inegalitatea lui Mitrinović $p\geq 3r\sqrt{3}$, inegalitatea lui Doucet $4R+r\geq p\sqrt{3}$ și inegalitatea lui Euler $R\geq 2r$.

La toate inegalitățile de mai sus egalitatea are loc dacă și numai dacă triunghiul este echilateral.

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6. POLINOMUL DE INTERPOLARE AL LUI LAGRANGE

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Se numeşte polinom de interpolare Lagrange un polinom f cu grad $f \le n-1$ care ia valorile date $f_1, f_2, ..., f_n$ in punctele diferite $x_1, x_2, ..., x_n$ adică

$$f(x_1) = f_1 \quad , \quad f(x_2) = f_2 \quad ,..., \quad f(x_n) = f_n \qquad (1)$$

Se observă că acest polinom este:

$$f(x) = f_1 \frac{(x-x_2)(x-x_3)\cdot ...\cdot(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdot ...\cdot(x_1-x_n)} + f_2 \frac{(x-x_1)(x-x_3)\cdot ...\cdot(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdot ...\cdot(x_2-x_n)} + ... \qquad (2)$$

$$... + f_n \frac{(x-x_1)(x-x_2)\cdot ...\cdot(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdot ...\cdot(x_n-x_{n-1})}$$

deoarece inlocuind pe x cu $x_1, x_2, ..., x_n$ vedem că sunt indeplinite condiţiile (1) şi in plus grad $f(x) \le n-1$.

Polinoamele

$$l_1(x) = \frac{(x-x_2)(x-x_3)\cdot ...\cdot(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdot ...\cdot(x_1-x_n)}$$

$$l_2(x) = \frac{(x-x_1)(x-x_3)\cdot ...\cdot(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdot ...\cdot(x_2-x_n)}$$

....

$$l_n(x) = \frac{(x-x_1)(x-x_2)\cdot ...\cdot(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdot ...\cdot(x_n-x_{n-1})}$$

se numesc polinoamele lui Lagrange.

Cu ajutorul polinoamelor lui Lagrange formula (2) se scrie:

$$f(x) = \sum_{k=1}^{n} l_k(x)\cdot f(x_k) \qquad \text{(formula de interpolare a lui Lagrange)}$$

Alt mod de a introduce polinoamele lui Lagrange

Să considerăm polinomul $\varphi(x) = (x-x_1)(x-x_2)\cdot ...\cdot(x-x_n)$ care are rădăcinile $x_1, x_2, ..., x_n$.

Avem

$$\varphi'(x) = (x-x_2)(x-x_3)...(x-x_n) + (x-x_1)(x-x_3)...(x-x_n) + ... + (x-x_1)(x-x_2)...(x-x_{n-1})$$

Mai departe obţinem:

$$\varphi'(x_1) = (x_1-x_2)(x_1-x_3)...(x_1-x_n)$$

$$\varphi'(x_2) = (x_2-x_1)(x_2-x_3)...(x_2-x_n)$$

....

$$\varphi'(x_n) = (x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})$$

Polinoamele lui Lagrange se pot scrie astfel:

$$l_1(x)=\frac{1}{x-x_1}\cdot\frac{\varphi(x)}{\varphi'(x_1)}$$

$$l_2(x)=\frac{1}{x-x_2}\cdot\frac{\varphi(x)}{\varphi'(x_2)}$$

$$l_n(x)=\frac{1}{x-x_n}\cdot\frac{\varphi(x)}{\varphi'(x_n)}$$

iar formula de interpolare se poate scrie sub forma:

$$f(x)=\frac{\varphi(x)}{x-x_1}\cdot\frac{f(x_1)}{\varphi'(x_1)}+\frac{\varphi(x)}{x-x_2}\cdot\frac{f(x_2)}{\varphi'(x_2)}+\ldots+\frac{\varphi(x)}{x-x_n}\cdot\frac{f(x_n)}{\varphi'(x_n)}$$

Aplicaţia 1:Identităţile lui Euler

In formula de interpolare a lui Lagrange să considerăm $f(x)=x^p$ unde $p=1,2,\ldots,n-1$

$$x^p=\frac{\varphi(x)}{x-x_1}\cdot\frac{x_1^p}{\varphi'(x_1)}+\frac{\varphi(x)}{x-x_2}\cdot\frac{x_2^p}{\varphi'(x_2)}+\ldots+\frac{\varphi(x)}{x-x_n}\cdot\frac{x_n^p}{\varphi'(x_n)} \qquad (3)$$

Pentru $x=0$ şi după o impărţire cu $\varphi(0)$ obţinem identităţile lui Euler:

$$\frac{x_1^{p-1}}{\varphi'(x_1)}+\frac{x_2^{p-1}}{\varphi'(x_2)}+\ldots+\frac{x_n^{p-1}}{\varphi'(x_n)}=0 \quad \text{pentru } p=1,2,\ldots,n-1$$

Aplicaţia 2: Descompunerea in funcţii raţionale simple.

Dacă in formula (3) luăm $p=0$ obţinem

$$1=\frac{\varphi(x)}{x-x_1}\cdot\frac{1}{\varphi'(x_1)}+\frac{\varphi(x)}{x-x_2}\cdot\frac{1}{\varphi'(x_2)}+\ldots+\frac{\varphi(x)}{x-x_n}\cdot\frac{1}{\varphi'(x_n)}$$

de unde rezultă

$$\frac{1}{\varphi(x)}=\frac{1}{(x-x_1)(x-x_2)\ldots(x-x_n)}=\frac{1}{\varphi'(x_1)}\cdot\frac{1}{x-x_1}+\frac{1}{\varphi'(x_2)}\cdot\frac{1}{x-x_2}+\ldots+\frac{1}{\varphi'(x_n)}\cdot\frac{1}{x-x_n}$$

care reprezintă descompunerea funcţiei raţionale $\frac{1}{\varphi(x)}$ in funcţii raţionale simple.

Aplicaţia 3: Restul impărţirii unui polinom f prin $(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)$

Din teorema impărţirii cu rest obţinem că există in mod unic polinoamele $q(x)$ şi $r(x)$ astfel incât:

$f=(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)q(x)+r(x)$ şi $\operatorname{grad} r(x)\le n-1$

Avem:

$$r(x_1)=f(x_1) \quad , \quad r(x_2)=f(x_2) \quad ,\ldots, \quad r(x_n)=f(x_n). \qquad (4)$$

Rezultă că restul impărţirii lui f la $(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)$ este polinomul de interpolare al lui Lagrange care indeplineşte condiţiile (4) deci:

$$r(x)=f(x_1)\frac{(x-x_2)(x-x_3)\cdot\ldots\cdot(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdot\ldots\cdot(x_1-x_n)}+f(x_2)\frac{(x-x_1)(x-x_3)\cdot\ldots\cdot(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdot\ldots\cdot(x_2-x_n)}+\ldots$$

$$\ldots + f(x_n)\frac{(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdot\ldots\cdot(x_n-x_{n-1})}$$

Ex.Restul impărţirii polinomului $f(x)=x^{10}-2x+7$ la $x(x-1)(x+1)$ este:

$$r(x)=f(0)\frac{(x-1)(x+1)}{(-1)\cdot 1}+f(1)\frac{x(x+1)}{1\cdot 2}+f(-1)\frac{x(x-1)}{(-1)\cdot(-2)}=-7(x^2-1)+3(x^2+x)+5(x^2-x)$$

$$\Rightarrow r(x)=x^2-2x+7$$