REVISTA ELECTRONICĂ MATEINFO.RO

ISBN 2065-6432

FEBRUARIE 2014

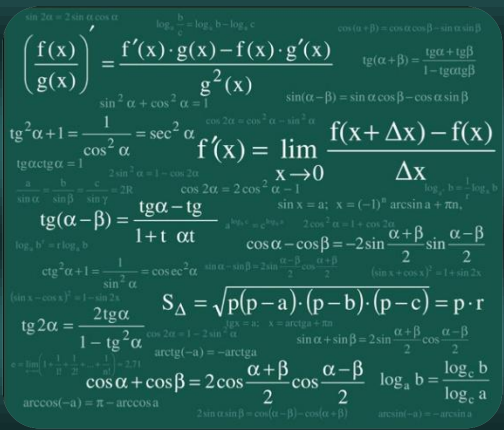
REVISTĂ LUNARĂ

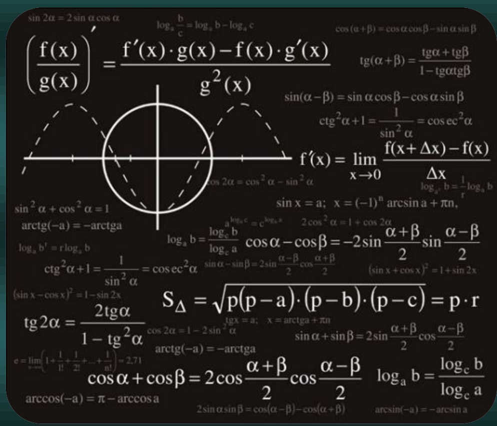
DIN FEBRUARIE 2009

DE 4 ANI ÎN FIECARE LUNĂ

WWW.MATEINFO.RO

revistaelectronica@mateinfo.ro





COORDONATOR: ANDREI OCTAVIAN DOBRE

REDACTORI PRINCIPALI ŞI SUSŢINĂTOR PERMANENŢI AI REVISTEI

NECULAI STANCIU, ROXANA MIHAELA STANCIU ŞI NELA CICEU

Articole:

1. Studiul monotoniei unor funcţii cu aplicaţii în demonstrarea unor inegalităţi.

Prof. Adrian Ioan Pop, Colegiul Naţional "Gheorghe Şincai", Baia Mare

Rezultatele teoretice ale analizei matematice permit obţinerea unor inegalităţi care prin metodele elementare ar fi fost greu de demonstrat.

Teoremă

Dacă f şi g sunt funcţii derivabile pe un interval $I=[x_0,\infty)$ astfel încât $f(x_0)=g(x_0)$ şi $f'(x)\geq g'(x), \forall x\in I$ atunci are loc inegalitatea $f(x)\geq g(x), \forall x\in I$.

Demonstraţie Fie h = f – g , $h(x_0)=f(x_0)-g(x_0)=0$, $h'(x)=f'(x)-g'(x)\geq 0, \forall x\in I$, (deoarece $f'(x)\geq g'(x), \forall x\in I$).

Din $h'(x)\geq 0, \forall x\in I$ $\Rightarrow$ h este monoton crescătoare pe I $\Rightarrow$ $h(x)\geq h(x_0), \forall x\geq x_0$ $\Rightarrow$ $h(x)\geq 0, \forall x\geq x_0$ $\Rightarrow$ $f(x)-g(x)\geq 0, \forall x\geq x_0$ $\Rightarrow$ $f(x)\geq g(x), \forall x\in I=[x_0,\infty)$.

Problema 1. Să se arate că $\frac{x}{1+x^2}<arctgx, \forall x>0$

$\frac{x}{1+x^2}<arctgx, \forall x>0 \Leftrightarrow \frac{x}{1+x^2}-arctgx<0, \forall x>0$.

Consider funcţia $f:\mathbf{R}\to\mathbf{R}, f(x)=\frac{x}{1+x^2}-arctgx$.

Din studiul monotoniei acestei funcţii vom deduce inegalitatea cerută.

$$f'(x)=\left(\frac{x}{1+x^2}-arctgx\right)'=\frac{-2x^2}{(1+x^2)^2}.$$

Alcătuim un tabel cu semnul primei derivate:

x	**- ∞**		**0**		**+ ∞**
$f'(x)$		**---------------**	**0**	**---------------------**	
$f(x)$		↘		↘	

Din acest tabel $\Rightarrow$ f este strict descrescătoare pe **R** $\Rightarrow$ $f(x) < f(0), \forall x > 0$ $\Rightarrow$ $f(x) < 0, \forall x > 0$ $\Rightarrow$

$\frac{x}{1+x^2} - arctgx < 0, \forall x > 0$ $\Rightarrow$ $\frac{x}{1+x^2} < arctgx, \forall x > 0$.

Problema 2. Să se demonstreze inegalitatea $\frac{\ln x + x}{\ln x - x} \geq \frac{1+e}{1-e}, \forall x \in (0,+\infty)$.

Rezolvare

Considerăm funcţia $g:(0,+\infty) \to \mathbf{R}, g(x) = \ln x - x$.

$g'(x) = \frac{1-x}{x}$. Ataşam ecuaţia $g'(x) = 0 \Leftrightarrow \frac{1-x}{x} = 0 \Leftrightarrow x = 1 \in (0,\infty)$

x	**0**		**1**		$+\infty$
$g'(x)$	/	+++++++++++	**0**	--------------------	
$g(x)$	/	↗	**g(1)**	↘	

Din acest tabel $\Rightarrow$ $g(x) \leq g(1), \forall x > 0$ $\Rightarrow$ $\ln x - x < -1, \forall x > 0$ şi prin urmare funcţia

$f:(0,+\infty) \to \mathbf{R}, f(x) = \frac{\ln x + x}{\ln x - x}$ este binedefinită.

Calculăm prima derivată a funcţiei f:

$$f'(x) = \left(\frac{\ln x + x}{\ln x - x}\right)' = \frac{(\ln x + x)'(\ln x - x) - (\ln x + x)(\ln x - x)'}{(\ln x - x)^2} = \frac{2\ln x - 2}{(\ln x - x)^2}.$$

Ataşăm ecuaţia $f'(x) = 0 \Leftrightarrow \frac{2\ln x - 2}{(\ln x - x)^2} = 0 \Leftrightarrow \ln x = 1 \Leftrightarrow x = e$

Tabelul cu monotonia şi punctele de extrem ale funcţiei f este:

x	**0**		**e**		$+\infty$
$f'(x)$		-------------------------	**0**	++++++++++++	
$f(x)$		↘	**f(e)**	↗	

Din tabelul de mai sus $\Rightarrow$ e este punct de minim global al funcţiei f $\Rightarrow f(x) \geq f(e), \forall x > 0$ $\Rightarrow$

$\frac{\ln x + x}{\ln x - x} \geq \frac{1+e}{1-e}, \forall x \in (0,+\infty)$.

Problema 3. Să se demonstreze inegalitatea $xarctgx > \ln(x^2+1), \forall x \in (0,+\infty)$.

Rezolvare: Consider funcţia $f:\mathbf{R} \to \mathbf{R}, f(x) = xarctgx - \ln(x^2+1)$.

Din studiul monotoniei acestei funcţii o să deducem inegalitatea cerută.

Calculăm prima şi a doua derivată a funcţiei f: $f'(x) = arctgx - \frac{x}{x^2+1}$, $f''(x) = \frac{x^2}{(x^2+1)^2} \Rightarrow$

$f''(x) > 0, \forall x \in \mathbf{R}^* \Rightarrow f'$ este strict crescătoare pe **R**. $\Rightarrow f'(x) > f'(0), \forall x \in (0,\infty) \overset{f'(0)=0}{\Rightarrow}$

$f'(x) > 0, \forall x \in (0,\infty) \Rightarrow f$ este strict crescătoare pe $[0,\infty) \Rightarrow f(x) > f(0), \forall x \in (0,\infty) \Rightarrow$

$f(x) > 0, \forall x \in (0,\infty) \Rightarrow xarctgx - \ln(x^2+1) > 0, \forall x > 0 \Rightarrow xarctgx > \ln(x^2+1), \forall x \in (0,\infty)$

Problema 4. Să se arate că $\forall x \in (0,\infty)$ are loc inegalitatea $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

Consider funcţia $f:\mathbf{R} \to \mathbf{R}, f(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!}$. Calculăm:

$$f'(x) = \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!}\right)' = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!},\ f''(x) = \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!}\right)' = e^x - 1 - \frac{x}{1!},$$

$f'''(x) = \left(e^x - 1 - \frac{x}{1!}\right)' = e^x - 1$. Ataşăm ecuaţia $f'''(x) = 0 \Leftrightarrow e^x - 1 = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0 \Rightarrow$

$f'''(x) = e^x - 1 > 0, \forall x > 0 \Rightarrow f''$ este strict crescătoare pe $[0,+\infty) \Rightarrow f''(x) > f''(0), \forall x > 0 \Rightarrow$

$f''(x) > 0, \forall x > 0 \Rightarrow f'$ este strict crescătoare pe $[0,+\infty) \Rightarrow f'(x) > f'(0), \forall x > 0 \overset{f'(0)=0}{\Rightarrow}$

$f'(x) > 0, \forall x > 0 \Rightarrow f$ este strict crescătoare pe $[0,+\infty) \Rightarrow f(x) > f(0), \forall x > 0 \Rightarrow$

$$f(x) > 0, \forall x > 0 \Rightarrow e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} > 0, \forall x > 0 \Rightarrow e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}, \forall x \in (0,+\infty)$$

Problema 5. Să se arate:

1) $\ln(x-1) \cdot \ln(x+1) \le \ln^2 x$ pentru orice x > 2;

2) $\int_x^{x+1} \sin t^3 dt \le \frac{2}{3x^2}$ pentru orice x > 0.

Rezolvare:

a) Întrucât $\ln x > 0$ şi $\ln(x+1) > 0, \forall x > 1$ inegalitatea $\ln(x-1) \cdot \ln(x+1) \le \ln^2 x, \forall x > 2$ este echivalentă cu $\frac{\ln(x-1)}{\ln x} \le \frac{\ln x}{\ln(x+1)}, \forall x > 2$.

Consider funcţia $f:(1,+\infty)\to\mathbf{R}, f(x)=\dfrac{\ln(x-1)}{\ln x}$. Studiind monotonia acestei funcţii o să demonstrăm inegalitatea cerută.

$$f'(x)=\left(\frac{\ln(x-1)}{\ln x}\right)'=\frac{\frac{1}{x-1}\cdot\ln x-\frac{1}{x}\cdot\ln(x-1)}{\ln^2 x}=\frac{x\ln x-(x-1)\ln(x-1)}{x(x-1)\ln^2 x}.\ (1)$$

Considerând funcţia $g:(0,+\infty)\to\mathbf{R}, g(x)=x\ln x \Rightarrow g'(x)=\ln x+1>0, \forall x>e^{-1} \Rightarrow$ g este strict crescă toare pe $[e^{-1},+\infty) \Rightarrow g(x)>g(x-1), \forall x>2 \Rightarrow x\ln x-(x-1)\ln(x-1)>0, \forall x>2 \overset{1)}{\Rightarrow}$

$f'(x)>0, \forall x>2 \Rightarrow$ f strict crescătoare pe $(2,+\infty) \Rightarrow$

$$f(x+1)>f(x), \forall x>2 \Rightarrow \frac{\ln x}{\ln(x+1)}\ge\frac{\ln(x-1)}{\ln x}, \forall x>2 \Rightarrow \ln(x-1)\cdot\ln(x+1)\le\ln^2 x, \forall x>2$$

2) $$I=\int_x^{x+1}\sin t^3 dt=\int_x^{x+1}\frac{\left(\cos t^3\right)'}{-3t^2}dt=\frac{-1}{3}\left(\frac{\cos t^3}{t^2}\Bigg|_x^{x+1}-\int_x^{x+1}\cos t^3\cdot\frac{-2}{t^3}dt\right)=\frac{-\cos t^3}{3t^2}\Bigg|_x^{x+1}-\frac{2}{3}\int_x^{x+1}\frac{\cos t^3}{t^3}dt$$

Dar $\cos t^3\ge-1\Rightarrow-\cos t^3\le-1\Rightarrow\dfrac{-\cos t^3}{t^3}\le\dfrac{1}{t^3}, \forall t>0 \Rightarrow$

$$-\frac{2}{3}\int_x^{x+1}\frac{\cos t^3}{t^3}dt\le\frac{2}{3}\int_x^{x+1}\frac{1}{t^3}dt=\frac{-1}{3t^2}\Bigg|_x^{x+1}=\frac{1}{3}\left(\frac{1}{x^2}-\frac{1}{(x+1)^2}\right).$$

Prin urmare

$$I=\int_x^{x+1}\sin t^3 dt=\frac{-\cos t^3}{3t^2}\Bigg|_x^{x+1}-\frac{2}{3}\int_x^{x+1}\frac{\cos t^3}{t^3}dt=-\frac{\cos(x+1)^3}{3(x+1)^2}+\frac{\cos t^3}{3x^2}+\frac{1}{3}\left(\frac{1}{x^2}-\frac{1}{(x+1)^2}\right)$$

$$\le\frac{1}{3}\left[\frac{1}{(x+1)^2}+\frac{1}{x^2}+\frac{1}{x^2}-\frac{1}{(x+1)^2}\right]=\frac{2}{3\cdot x^2}, \forall x>0\Rightarrow$$

$$I=\int_x^{x+1}\sin t^3 dt\le\frac{2}{3x^2}, \forall x>0.$$

Problema 6. (Inegalitatea lui Hölder)

Dacă $a_1,a_2,\ldots,a_n\ge0, b_1,b_2,\ldots,b_n\ge0; p>1, q>1$ şi $\dfrac{1}{p}+\dfrac{1}{q}=1$ atunci

$$\sum_{i=1}^n a_i b_i\le\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\cdot\left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}$$

Demonstraţie Considerăm funcţia $\varphi:[0,+\infty)\to\mathbf{R}, \varphi(x)=x^\alpha-\alpha x,$ unde $\alpha\in(0,1)$ este un parametru. $\varphi'(x)=\alpha\cdot x^{\alpha-1}-\alpha=\alpha(x^{\alpha-1}-1)$. Din tabelul de variaţie al funcţiei φ

x	0		1		$+\infty$
$\varphi'(x)$		+++++++++++	0	---------------------	
$\varphi(x)$	0	↗	$1-\alpha$	↘	

rezultă că $\varphi(x) \le 1-\alpha \Rightarrow x^{\alpha} - \alpha x \le 1-\alpha, \forall x > 0$

Pentru orice A > 0, B > 0 substituind $x = \frac{A}{B}$ şi $\alpha = \frac{1}{p} \Rightarrow 1-\alpha = 1-\frac{1}{p} = \frac{1}{q}$. Prin urmare:

$$\left(\frac{A}{B}\right)^{\frac{1}{p}} - \frac{1}{p}\cdot\frac{A}{B} \le \frac{1}{q}\Big|\cdot B \Rightarrow A^{\frac{1}{p}}\cdot B^{\frac{1}{q}} \le \frac{A}{p}+\frac{B}{q}.$$

Punând în inegalitatea de mai sus $A = \frac{a_i^p}{\sum_{i=1}^n a_i^p}, B = \frac{b_i^q}{\sum_{i=1}^n b_i^q}$ şi adunând inegalităţile obţinute $\Rightarrow$

$$\frac{\sum_{i=1}^n a_{ib_i}}{\left(\sum_{i=1}^n a_i^p\right)^p \cdot \left(\sum_{i=1}^n b_i^q\right)^q} \le \frac{1}{p}\cdot\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^p} + \frac{1}{q}\cdot\frac{\sum_{i=1}^n b_i^q}{\sum_{i=1}^n b_i^q} = \frac{1}{p}+\frac{1}{q} = 1 \Rightarrow$$

$$\Rightarrow \sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}.$$

Pentru p = 2, q = 2 înlocuite în inegaliutatea de mai sus se obţine inegalitatea lui

Cauchy-Buniakowski-Schwartz : $\left(\sum_{i=1}^n a_i b_i\right)^2 \le \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2$ inegalitate folosită des în rezolvarea

altor probleme.

Bibliografie

[1] Roşculeţ M. , Analiză matematică , Editura Didactică şi Pedagogică, Bucureşti 1966.

[2] Andrica D. , Duca D. , Purdea I. , Agratini O. , Manual pentru clasa a XI-a , Editura Gil, Zalău 2001.

[3] Andrica D. , Duca D. , Purdea I. , Matematică de bază , Editura Studium, Cluj Napoca 2000.

[4] Gussi Gh. , Stănăşilă O. , Stoica T. , Elemente de analiză matematică, Manual pentru clasa a XI-a, Editura Didactică şi Pedagogică, Bucureşti 1999.

[5] Ganga M. , Elemente de analiză matematică, Manual pentru clasa a XI-a, Editura Mathpress, Ploieşti 2001.

2. ŞIRURI DE ARII DE SUPRAFEŢE TRIUNGHIULARE

Prof. Văcărean Sorina, Colegiul Naţional „George Bariţiu”, Cluj-Napoca

1. Se consideră $\triangle ABC$. Fie $A_1 \in (BC)$, $B_1 \in (CA)$, $C_1 \in (AB)$ astfel încât $\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{k}$, $A_2 \in (B_1C_1)$, $B_2 \in (C_1A_1)$, $C_2 \in (A_1B_1)$ astfel încât $\frac{B_1A_2}{B_1C_1} = \frac{C_1B_2}{C_1A_1} = \frac{A_1C_2}{A_1B_1} = \frac{1}{k}$, $A_3 \in (B_2C_2)$, $B_3 \in (C_2A_2)$, $C_3 \in (A_2B_2)$ astfel încât $\frac{B_2A_3}{B_2C_2} = \frac{C_2B_3}{C_2A_2} = \frac{A_2C_3}{A_2B_2} = \frac{1}{k}$, …, $A_n \in (B_{n-1}C_{n-1})$, $B_n \in (C_{n-1}A_{n-1})$, $C_n \in (A_{n-1}B_{n-1})$ astfel încât $\frac{B_{n-1}A_n}{B_{n-1}C_{n-1}} = \frac{C_{n-1}B_n}{C_{n-1}A_{n-1}} = \frac{A_{n-1}C_n}{A_{n-1}B_{n-1}} = \frac{1}{k}$, $n \in \mathbb{N}^*$, $k \in \mathbb{N}^* \setminus \{1\}$, unde $A_0 = A$, $B_0 = B$ şi $C_0 = C$. Se notează cu s_i aria suprafeţei $\triangle A_iB_iC_i$, $i \in \mathbb{N}$, exprimată în u^2.

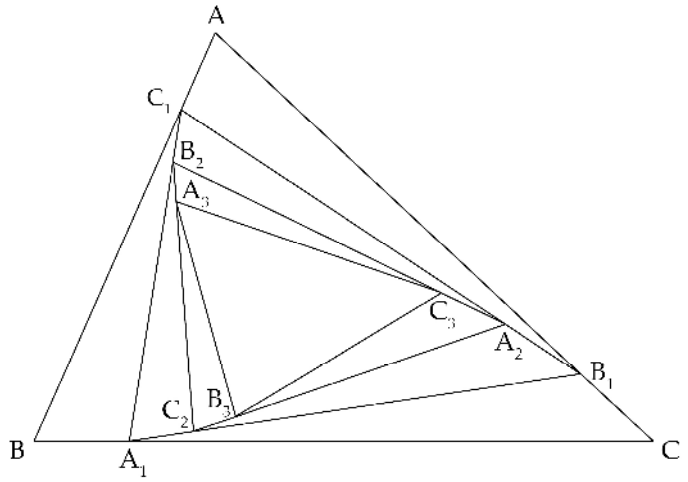
a) Calculaţi s_n în funcţie de s_0, $n \in \mathbb{N}$.

b) Calculaţi $\lim_{n\to\infty} s_n$.

c) Calculaţi $\sum_{i=0}^{n} s_i$, $n \in \mathbb{N}$.

d) Arătaţi că $s_n^{\,2} = s_{n-1} \cdot s_{n+1}$, $\forall n \in \mathbb{N}^*$.

Rezolvare:



a) i) Calculăm s_1 în funcţie de s_0.

Se arată că $A_{\triangle C_1BA_1} = A_{\triangle B_1A_1C} = A_{\triangle AC_1B_1} = \frac{k-1}{k^2} s_0$.

$$s_1 = A_{\triangle A_1B_1C_1} = A_{\triangle ABC} - 3A_{\triangle C_1BA_1} = s_0 - 3 \cdot \frac{k-1}{k^2} s_0 = \frac{k^2 - 3k + 3}{k^2} s_0.$$

ii) Calculăm s_2 în funcţie de s_0.

Urmând raţionamentul de la i), $s_2 = \frac{k^2-3k+3}{k^2} s_1 = \left(\frac{k^2-3k+3}{k^2}\right)^2 s_0$.

iii) Calculăm s_3 în funcţie de s_0.

Urmând raţionamentul de la i), $s_3 = \frac{k^2-3k+3}{k^2} s_2 = \left(\frac{k^2-3k+3}{k^2}\right)^3 s_0$.

iv) Calculăm s_n în funcţie de s_{n-1}, $n \in \mathbb{N}^*$.

Urmând raţionamentul de la i), obţinem formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$:

$s_n = \frac{k^2-3k+3}{k^2} s_{n-1}$, $n \in \mathbb{N}^*$, $s_0 > 0$.

v) Calculăm s_n în funcţie de s_0, $n \in \mathbb{N}$.

Pornind de la formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$, obţinem:

$s_n = \left(\frac{k^2-3k+3}{k^2}\right)^n s_0$, $n \in \mathbb{N}$.

b) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(\frac{k^2-3k+3}{k^2}\right)^n s_0 = 0$, întrucât $\frac{k^2-3k+3}{k^2} < 1$, $\forall k \in \mathbb{N}^* \setminus \{1\}$.

c) $\sum\limits_{i=0}^{n} s_i = \sum\limits_{i=0}^{n}\left(\frac{k^2-3k+3}{k^2}\right)^i s_0 = \frac{k^2}{3(k-1)}\left[1-\left(\frac{k^2-3k+3}{k^2}\right)^{n+1}\right] s_0$.

d) Aplicând formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$, avem:

$s_{n+1} \cdot s_{n-1} = \frac{k^2-3k+3}{k^2} s_n \cdot s_{n-1} = s_n{}^2$, $\forall n \in \mathbb{N}^*$.

2. Acelaşi text ca la problema 1, în ipoteza că valoarea comună a tuturor şirurilor de rapoarte egale este $\frac{a}{b}$, $a,b \in \mathbb{N}^*$, $a < b$.

Rezolvare:

a) i) Calculăm s_1 în funcţie de s_0.

Se arată că $A_{\triangle C_1BA_1} = A_{\triangle B_1A_1C} = A_{\triangle AC_1B_1} = \frac{a(b-a)}{b^2} s_0$.

$s_1 = A_{\triangle A_1B_1C_1} = A_{\triangle ABC} - 3A_{\triangle C_1BA_1} = s_0 - 3\cdot\frac{a(b-a)}{b^2} s_0 = \frac{3a^2-3ab+b^2}{b^2} s_0$.

ii) Calculăm s_2 în funcţie de s_0.

Urmând raţionamentul de la i), $s_2 = \frac{3a^2-3ab+b^2}{b^2} s_1 = \left(\frac{3a^2-3ab+b^2}{b^2}\right)^2 s_0$.

iii) Calculăm s_3 în funcţie de s_0.

Urmând raţionamentul de la i), $s_3 = \frac{3a^2-3ab+b^2}{b^2} s_2 = \left(\frac{3a^2-3ab+b^2}{b^2}\right)^3 s_0$.

iv) Calculăm s_n în funcţie de s_{n-1}, $n\in\mathbb{N}^*$.

Urmând raţionamentul de la i), obţinem formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$:

$$s_n=\frac{3a^2-3ab+b^2}{b^2}s_{n-1},\ n\in\mathbb{N}^*,\ s_0>0.$$

v) Calculăm s_n în funcţie de s_0, $n\in\mathbb{N}$.

Pornind de la formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$, obţinem:

$$s_n=\left(\frac{3a^2-3ab+b^2}{b^2}\right)^n s_0,\ n\in\mathbb{N}.$$

b) $\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}\left(\frac{3a^2-3ab+b^2}{b^2}\right)^n s_0=0$, întrucât $\frac{3a^2-3ab+b^2}{b^2}<1$, $\forall a,b\in\mathbb{N}^*$, $a<b$.

c) $\sum\limits_{i=0}^{n}s_i=\sum\limits_{i=0}^{n}\left(\frac{3a^2-3ab+b^2}{b^2}\right)^i s_0=\frac{b^2}{3a(b-1)}\left[1-\left(\frac{3a^2-3ab+b^2}{b^2}\right)^{n+1}\right]s_0$.

d) Aplicând formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$, avem:

$$s_{n+1}\cdot s_{n-1}=\frac{3a^2-3ab+b^2}{b^2}s_n\cdot s_{n-1}=s_n^{\ 2},\ \forall n\in\mathbb{N}^*.$$

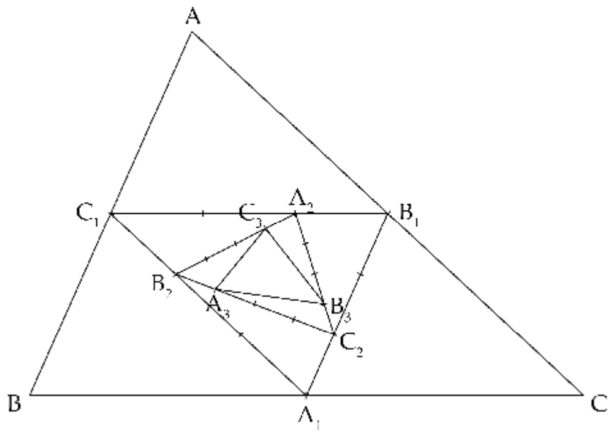
3. Se consideră $\triangle ABC$. Fie $A_1\in(BC)$, $B_1\in(CA)$, $C_1\in(AB)$ astfel încât $\frac{BA_1}{BC}=\frac{CB_1}{CA}=\frac{AC_1}{AB}=\frac{1}{2}$, $A_2\in(B_1C_1)$, $B_2\in(C_1A_1)$, $C_2\in(A_1B_1)$ astfel încât $\frac{B_1A_2}{B_1C_1}=\frac{C_1B_2}{C_1A_1}=\frac{A_1C_2}{A_1B_1}=\frac{1}{3}$, $A_3\in(B_2C_2)$, $B_3\in(C_2A_2)$, $C_3\in(A_2B_2)$ astfel încât $\frac{B_2A_3}{B_2C_2}=\frac{C_2B_3}{C_2A_2}=\frac{A_2C_3}{A_2B_2}=\frac{1}{4}$, …, $A_n\in(B_{n-1}C_{n-1})$, $B_n\in(C_{n-1}A_{n-1})$, $C_n\in(A_{n-1}B_{n-1})$ astfel încât $\frac{B_{n-1}A_n}{B_{n-1}C_{n-1}}=\frac{C_{n-1}B_n}{C_{n-1}A_{n-1}}=\frac{A_{n-1}C_n}{A_{n-1}B_{n-1}}=\frac{1}{n+1}$, $n\in\mathbb{N}^*$, unde $A_0=A$, $B_0=B$ şi $C_0=C$. Se notează cu s_j aria suprafeţei $\triangle A_jB_jC_j$, $j\in\mathbb{N}$, exprimată în u^2.

a) Calculaţi s_n în funcţie de s_0, $n\in\mathbb{N}$.

b) Calculaţi $\lim\limits_{n\to\infty}s_n$.

c) Arătaţi că $s_n^{\ 2}<s_{n-1}\cdot s_{n+1}$, $\forall n\in\mathbb{N}^*$.

Rezolvare:



a) Utilizăm formula găsită la problema 1 a) iv):

(1) $s_i = \dfrac{k^2-3k+3}{k^2} s_{i-1}$, $i \in \mathbb{N}^*$, $k \in \mathbb{N}^* \setminus \{1\}$, $s_0 > 0$.

i) Calculăm s_1 în funcţie de s_0.

Din (1), pentru $i=1$ şi $k=2$ avem: $s_1 = \dfrac{1}{4} s_0$.

ii) Calculăm s_2 în funcţie de s_0.

Din (1), pentru $i=2$ şi $k=3$ şi din i), avem: $s_2 = \dfrac{1}{3} s_1 = \dfrac{1}{4} \cdot \dfrac{1}{3} s_0$.

iii) Calculăm s_3 în funcţie de s_0.

Din (1), pentru $i=3$ şi $k=4$ şi din ii), avem: $s_3 = \dfrac{7}{16} s_2 = \dfrac{1}{4} \cdot \dfrac{1}{3} \cdot \dfrac{7}{16} s_0$.

iv) Calculăm s_4 în funcţie de s_0.

Din (1), pentru $i=4$ şi $k=5$ şi di iii), avem: $s_4 = \dfrac{13}{25} s_3 = \dfrac{1}{4} \cdot \dfrac{1}{3} \cdot \dfrac{7}{16} \cdot \dfrac{13}{25} s_0$.

v) Calculăm s_n în funcţie de s_{n-1}, $n \in \mathbb{N}^*$.

Din (1), pentru $i=n$ şi $k=n+1$ obţinem formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$:

$$s_n = \frac{n^2-n+1}{(n+1)^2} s_{n-1},\ n \in \mathbb{N}^*,\ s_0 > 0.$$

vi) Calculăm s_n în funcţie de s_0, $n \in \mathbb{N}^*$.

Pornind de la formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$, obţinem:

$$s_n = \left(\prod_{i=1}^{n} \frac{i^2-i+1}{(i+1)^2} \right) s_0,\ n \in \mathbb{N}^*.$$

b) Din $\dfrac{i^2-i+1}{(i+1)^2} < \dfrac{i}{i+1}$, $\forall i \in \mathbb{N}^*$ şi din faptul că $\dfrac{i^2-i+1}{(i+1)^2} > 0$, $\dfrac{i}{i+1} > 0$, $\forall i \in \mathbb{N}^*$, rezultă că

$\prod_{i=1}^{n}\frac{i^2-i+1}{(i+1)^2}<\prod_{i=1}^{n}\frac{i}{i+1}$. Cum $\prod_{i=1}^{n}\frac{i}{i+1}=\frac{1}{n+1}$, urmează că $\prod_{i=1}^{n}\frac{i^2-i+1}{(i+1)^2}<\frac{1}{n+1}$, de unde $s_n<\frac{1}{n+1}s_0$, $\forall n\in\mathbb{N}^*$. Cum şirul $(s_n)_{n\in\mathbb{N}}$ are termenii strict pozitivi şi $\lim_{n\to\infty}\frac{1}{n+1}s_0=0$, rezultă, pe baza criteriului majorării, că $\lim_{n\to\infty}s_n=0$.

c) Aplicând formula de recurenţă ce caracterizează şirul $(s_n)_{n\in\mathbb{N}}$, avem:

$$s_n^{\ 2}<s_{n-1}\cdot s_{n+1}\Leftrightarrow\left[\frac{n^2-n+1}{(n+1)^2}\right]^2 s_{n-1}^{\ 2}<s_{n-1}\cdot\frac{n^2+n+1}{(n+2)^2}\cdot\frac{n^2-n+1}{(n+1)^2}s_{n-1}\Leftrightarrow n^2+n-1>0\Leftrightarrow$$

$\Leftrightarrow n\in\left(\frac{\sqrt{5}-1}{2},\infty\right)\cap\mathbb{N}^*\Leftrightarrow n\in\mathbb{N}^*$, ceea ce este adevărat.

3. Solutions to problems 5271 and 5273 from School Science and Mathematics Journal other than those published in School Science and Mathematics Journal - February 2014 -

by Nela Ciceu, Roșiori, Bacău
and
Roxana Mihaela Stanciu, Buzău

- **5271:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with $\overline{AB}=x, \overline{BC}=y$, and $\overline{BD}=2\overline{AD}=2\overline{CD}$.

  Express the radius of the circum-circle in terms of x and y.

Solution. By Ptolemy's theorem easily we obtain that

$$AC=\frac{x+y}{2}.$$

The triangle ABC exists if only if

$$\frac{1}{3} < \frac{x}{y} < 3.$$

Easily yields

$$R = \frac{x \cdot y \cdot \frac{x+y}{2}}{4\sqrt{\frac{3(x+y)}{4} \cdot \frac{3y-x}{4} \cdot \frac{3x-y}{4} \cdot \frac{x+y}{4}}} = \frac{2xy}{\sqrt{3(3x-y)(3y-x)}},$$

and we are done.

- **5273:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

  Solve in the positive integers the equation $abcd + abc = (a+1)(b+1)(c+1)$.

Solution. We have to solve in positive integers the equation:

$$(d+1)abc = (a+1)(b+1)(c+1) \Leftrightarrow dabc - ab - bc - ac - a - b = c+1 \Leftrightarrow$$

$$\Leftrightarrow (dc-1)ab - (c+1)a - (c+1)b = c+1 \qquad (1)$$

If $dc = 1$, then $d = c = 1$ and is easy to see that equation (1) has no solutions.

We can multiply (1) with $dc-1$ and we obtain successively:

$$(dc-1)^2 ab - (dc-1)(c+1)a - (dc-1)(c+1)b = (dc-1)(c+1) \Leftrightarrow$$

$$\Leftrightarrow [(dc-1)a - (c+1)][(dc-1)b - (c+1)] = (c+1)^2 + (dc-1)(c+1) \Leftrightarrow$$

$$\Leftrightarrow [(dc-1)a - (c+1)][(dc-1)b - (c+1)] = (d+1)c(c+1) \qquad (*)$$

From symmetry we can assume that $a \ge b \ge c$.

Then $(dc-1)a - (c+1) \ge dc^2 - 2c - 1, (dc-1)b - (c+1) \ge dc^2 - 2c - 1$.

We analyze the cases when the expression $dc^2 - 2c - 1$ is positive or negative:

Case 1. If $d \ge 3$, then $dc^2 - 2c - 1 \ge 3c^2 - 2c - 1 = (c-1)(3c+1) \ge 0$ and by the relation (*) results that

$$(d+1)c(c+1) = [(dc-1)a - (c+1)][(dc-1)a - (c+1)] \ge (dc^2 - 2c - 1)^2 \Leftrightarrow$$

$$\Leftrightarrow d^2c^4 + 4c^2 + 1 - 4dc^3 - 2dc^2 + 4c - (d+1)c^2 - (d+1)c \le 0 \Leftrightarrow$$

$$\Leftrightarrow d^2c^4 - 4dc^3 - 3(d-1)c^2 - (d-3)c + 1 \le 0 \qquad (2)$$

If $d \ge 8$, then we have:

$$d^2c^4 - 4dc^3 - 3(d-1)c^2 - (d-3)c + 1 \ge 8dc^4 - 4dc^3 - 3dc^2 - dc + 3c^2 + 3c + 1 =$$

$$= 4dc^3(c-1) + 3dc^2(c^2-1) + dc(c^3-1) + 3c^2 + 3c + 1 > 0,$$

and the inequality (2) is not true.

(1.1) $d = 7$.

The inequality (2) becomes

$$49c^4 - 28c^3 - 18c^2 - 4c + 1 \le 0 \Leftrightarrow (c-1)(49c^3 + 21c^2 + 3c - 1) \le 0$$

and since $49c^3 + 21c^2 + 3c - 1 > 0$, we have $c = 1$.

The relation (*) becomes

$$(6a-2)(6b-2) = 16 \Leftrightarrow (3a-1)(3b-1) = 4,$$

and we obtain the solution $a = b = c = 1$.

(1.2) $d = 6$**.**

The inequality (2) becomes

$$36c^4 - 24c^3 - 15c^2 - 3c + 1 \le 0.$$

For $c \ge 2$ we have

$$36c^4 - 24c^3 - 15c^2 - 3c + 1 \ge 72c^3 - 24c^3 - 15c^2 - 3c + 1 =$$
$$= 48c^3 - 15c^2 - 3c + 1 = 30c^3 + 15c^2(c-1) + 3c(c^2-1) + 1 > 0,$$

and so $c = 1$.

The relation (*) becomes

$$(5a-2)(5b-2) = 12,$$

and we no obtain solutions.

(1.3) $d = 5$.

The inequality (2) becomes

$$25c^4 - 20c^3 - 12c^2 - 2c + 1 \le 0.$$

For $c \ge 2$ we have

$$25c^4 - 20c^3 - 12c^2 - 2c + 1 \ge 50c^3 - 20c^3 - 12c^2 - 2c + 1 =$$
$$= 30c^3 - 12c^2 - 2c + 1 = 16c^3 + 12c^2(c-1) + 2c(c^2-1) + 1 > 0.$$

so $c = 1$.

The relation (*) becomes

$$(4a-2)(4b-2) = 12 \Leftrightarrow (2a-1)(2b-1) = 3,$$

and we obtain the solution $a = 2, b = 1, c = 1$.

(1.4) $d = 4$.

The inequality (2) becomes

$$16c^4 - 16c^3 - 9c^2 - c + 1 \le 0.$$

For $c \ge 2$ we have

$$16c^4 - 16c^3 - 9c^2 - c + 1 \le 32c^3 - 16c^3 - 9c^2 - c + 1 =$$
$$= 16c^3 - 9c^2 - c + 1 = 6c^3 + 9c^2(c-1) + c(c^2-1) + 1 > 0,$$

hence $c = 1$.

The relation (*) becomes

$$(3a-2)(3b-2) = 10,$$

and we obtain the solution $a = 4, b = 1, c = 1$.

(1.5) $d = 3$.

The inequality (2) becomes

$$9c^4 - 12c^3 - 6c^2 + 1 \le 0.$$

For $c \ge 2$ we have

$$9c^4 - 12c^3 - 6c^2 + 1 \ge 18c^3 - 12c^3 - 6c^2 + 1 = 6c^3 - 6c^2 + 1 = 6c^2(c-1) + 1 > 0$$

hence $c = 1$.

The relation (*) becomes

$$(2a-2)(2b-2) = 8 \Leftrightarrow (a-1)(b-1) = 2,$$

and we obtain the solution $a = 3, b = 2, c = 1$.

Cases 2. $d = 2$.

(2.1) For $c = 1$ (*) becomes

$$(a-2)(b-2) = 6,$$

and we obtain the solutions

$$a = 8, b = 3, c = 1 \text{ and } a = 5, b = 4, c = 1.$$

(2.2) For $c \ge 2$,

$$2c^2 - 2c - 1 \ge 4c - 2c - 1 = 2c - 1 > 0$$

and then (2) is true, i.e

$$4c^4 - 8c^3 - 6c^2 + c + 1 \le 0.$$

For $c \geq 3$ we have

$$4c^4 - 8c^3 - 6c^2 + c + 1 \geq 12c^3 - 8c^3 - 6c^2 + c + 1 =$$

$$= 4c^3 - 6c^2 + c + 1 = 2c^3 + 2c^2(c-3) + c + 1 > 0,$$

hence $c = 2$.

(*) becomes

$$(3a-3)(3b-3) = 18 \Leftrightarrow (a-1)(b-1) = 2$$

and we obtain the solution

$$a = 3, b = 2, c = 2.$$

Cases 3. $d = 1$.

Since $c \neq 1$, we have:

(3.1) For $c = 2$ (*) becomes

$$(a-3)(b-3) = 12,$$

and we obtain the solutions

$$(a = 15, b = 4, c = 2), (a = 9, b = 5, c = 2) \text{ and } (a = 7, b = 6, c = 2).$$

(3.2) For $c = 3$ (*) becomes

$$(2a-4)(2b-4) = 24 \Leftrightarrow (a-2)(b-2) = 6$$

and we obtain the solutions

$$(a = 8, b = 3, c = 3), \text{ şi } (a = 5, b = 4, c = 3).$$

(3.3) For $c \geq 4$ we have

$$c^2 - 2c - 1 \geq 4c - 2c - 1 = 2c - 1 > 0$$

and must that (2) $c^4 - 4c^3 + 2c + 1 \leq 0$, to be true.

Since $c^4 - 4c^3 + 2c + 1 = c^3(c-4) + 2c + 1 > 0$, has no solutions.

From the above we obtain the solutions:

a	b	c	d
1	1	1	7
2	1	1	5
4	1	1	4
3	2	1	3
5	4	1	2
8	3	1	2
3	2	2	2
15	4	2	1
9	5	2	1
7	6	2	1
8	3	3	1
5	4	3	1

The solution is complete.

4. The solutions of some problems of Mathematical Reflections

Prof. Codreanu Ioan Viorel, Satulung Secondary School, Maramureș

J 273. Let a,b,c be real numbers greater than or equal to 1. Prove that

$$\frac{a^3+2}{b^2-b+1}+\frac{b^3+2}{c^2-c+1}+\frac{c^3+2}{a^2-a+1}\ge 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Using the **AM-GM Inequality** we get

$$\sum\frac{a^3+2}{b^2-b+1}\ge 3\sqrt[3]{\prod\frac{a^3+2}{a^2-a+1}}$$

We have $\frac{a^3+2}{a^2-a+1}\ge 3 \Leftrightarrow (a-1)^3\ge 0$, which is clearly true. Similarly $\frac{b^3+2}{b^2-b+1}\ge 3$ and $\frac{c^3+2}{c^2-c+1}\ge 3$. So it follows that $\sum\frac{a^3+2}{b^2-b+1}\ge 9$.

S 274. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{a}{ca+1}+\frac{b}{ab+1}+\frac{c}{bc+1}\le\frac{1}{2}\left(a^2+b^2+c^2\right)$$

Proposed by Sayan Das, Kolkata, India

Solution by Ioan Viorel Codreanu, Maramures, Romania

Using the **AM-GM Inequality** and the condition $abc=1$, we get

$$\sum\frac{a}{ca+1}\le\sum\frac{a}{2\sqrt{ca}}=\frac{1}{2}\sum\sqrt{\frac{a}{c}}=\frac{1}{2}\sum a\sqrt{b}$$

so it suffices to prove that

$$\sum a\sqrt{b}\le\sum a^2.$$

From the **Cauchy-Schwarz Inequality,** we have

$$\left(\sum a\sqrt{b}\right)^2\le\left(\sum a^2\right)\left(\sum a\right)$$

so it suffices to prove that

$$\sum a\le\sum a^2.$$

From the **Cauchy-Schwarz Inequality**, we have

$$\left(\sum a\right)^2\le 3\sum a^2$$

and so it suffices to prove that

$$\sum a^2 \ge 3$$

which follows from the **AM-GM Inequality**

$$\sum a^2 \ge 3\sqrt[3]{\prod a^2} = 3.$$

J 280. Let a,b,c,d be positive real numbers. Prove that

$$2(ab+cd)(ac+bd)(ad+bc) \ge (abc+bcd+cda+dab)^2$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Rewrite the inequality in the form

$$\left(a^2bc+d^2bc+c^2ad+b^2ad\right)(bc+bc+ad+ad) \ge (abc+bcd+cda+dab)^2$$

and this follows immediately from the **Cauchy-Schwarz Inequality**.

S 282. Let ABC be a triangle, G its centroid, and O its circumcenter. Lines AG, BG, CG intersect the circumcircle of triangle ABC at A', B', C'. Denote by G' the centroid of triangle $A'B'C'$. Prove that $OG \ge OG'$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramureş, Romania

From $\angle BAG \equiv \angle A'B'G$ and $\angle AGB \equiv \angle B'GA'$ it follows that ΔABC is similarly of $\Delta B'GA'$.

So $\frac{AG}{B'G} = \frac{BG}{A'G} = \frac{AB}{A'B'}$ from where we obtain $A'B' = \frac{AB \cdot A'G}{BG}$. But $A'G \cdot GA = R^2 - OG^2$ (the power of the point G to the circumcircle of triangle ΔABC) and we get

$A'B' = \frac{AB}{AG \cdot BG} \cdot \left(R^2 - OG^2\right)$, namely $A'B' = \frac{9}{4} \cdot \frac{c}{m_a m_b}\left(R^2 - OG^2\right)$. Using the known equalities

$OG^2 = R^2 - \frac{\sum a^2}{9}$ and $OG'^2 = R^2 - \frac{\sum A'B'^2}{9}$ we have $A'B' = \frac{c\sum a^2}{4m_a m_b}$ and

$$OG \ge OG' \Leftrightarrow OG^2 \ge OG'^2 \Leftrightarrow \sum A'B'^2 \ge \sum a^2 \Leftrightarrow \left(\sum a^2\right)\left(\sum \frac{a^2}{m_b{}^2 m_c{}^2}\right) \ge 16.$$

Using the **Cauchy-Schwarz Inequality** we get

$$\left(\sum a^2\right)\left(\sum \frac{a^2}{m_b{}^2 m_c{}^2}\right) \ge \left(\sum \frac{a^2}{m_b m_c}\right)^2.$$

It is enough to prove the inequality

$$\sum \frac{a^2}{m_b m_c} \geq 4 .$$

It is known that the medians of a triangle form a new triangle, which we will note by $\Delta A_0B_0C_0$. In this new triangle we will note: $m_a{}', m_b{}', m_c{}'$ the medians lengths $\Delta A_0B_0C_0$ and a', b', c' the side lengths $\Delta A_0B_0C_0$. We have $m_a{}' = \frac{3}{4}a, m_b{}' = \frac{3}{4}b, m_c{}' = \frac{3}{4}c$ şi $a' = m_a, b' = m_b, c' = m_c$. The dual inequality is

$$\sum \frac{m_a'^2}{b'c'} \geq \frac{9}{4} .$$

To prove the inequality

$$\sum \frac{m_a{}^2}{bc} \geq \frac{9}{4} \Leftrightarrow \sum \frac{2(b^2+c^2)-a^2}{bc} \geq 9 \Leftrightarrow 2\sum \frac{(b+c)^2}{bc} - \sum \frac{a^2}{bc} \geq 21 .$$

Using the equalities $\sum \frac{(a+b)^2}{ab} = \frac{s^2+r^2+10Rr}{2Rr}$ and $\sum \frac{a^2}{bc} = \frac{s^2-3r^2-6Rr}{2Rr}$ the last inequality becomes

$$s^2 \geq 16Rr - 5r^2$$

the last inequality is known the inequality **Gerretsen** and solution ends.

J 283. Let a, b, c be positive real numbers. Prove that

$$\frac{2a+1}{b+c} + \frac{2b+1}{c+a} + \frac{2c+1}{a+b} \geq 3 + \frac{9}{2(a+b+c)}$$

Proposed by Zarif Ibragimov, Sam SU, Samarkand, Uzbekistan

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Using the **AM-HM Inequality** we get

$$\sum \frac{1}{b+c} \geq \frac{9}{\sum (b+c)} = \frac{9}{2\sum a}$$

But, by applying the **Nesbitt Inequality**

$$\sum \frac{a}{b+c} \geq \frac{3}{2}$$

we obtain

$$\sum \frac{2a+1}{b+c} = 2\sum \frac{a}{b+c} + \sum \frac{1}{b+c} \geq 3 + \frac{9}{2\sum a} .$$

J 285. Let a,b,c be the sidelengths of a triangle. Prove that

$$8<\frac{(a+b+c)\left(2ab+2bc+2ca-a^2-b^2-c^2\right)}{abc}\leq 9$$

Proposed by Adithya Ganesh, Plano, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

We know that $\sum a=2s, \sum ab=s^2+r^2+4Rr, \sum a^2=2\left(s^2-r^2-4Rr\right)$ and $\prod a=4sRr$.

The inequality of the problem is equivalent to

$$8<\frac{2s\left(2s^2+2r^2+8Rr-2s^2+2r^2+8Rr\right)}{4sRr}\leq 9$$

or

$$8Rr<2r^2+8Rr\leq 9Rr.$$

But this is easy, because we clearly have $8Rr<2r^2+8Rr$ and from the **Euler Inequality** $R\geq 2r$, so $2r^2+8Rr\leq Rr+8Rr=9Rr$.

J 286. Let $ABCD$ be a square inscribed in a circle. If P is a point on the arc AB, find the maximum of the expressions

$$\frac{PC\cdot PD}{PA\cdot PB}$$

Proposed by Panagiote Ligouras, Noci, Italy

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Let the measure of arch $AP=2x, x\in\left(0,\frac{\pi}{4}\right)$. Then the measures of the arches BP, CP, DP

$\frac{\pi}{2}-2x, \pi-2x$, respectevly $\frac{\pi}{2}+2x$. Let R the radius of the circumcenter.

We have $PA=2R\sin x, PB=2R\sin\left(\frac{\pi}{4}-x\right), PC=2R\sin\left(\frac{\pi}{2}-x\right)$ and $PD=2R\sin\left(\frac{\pi}{4}+x\right)$.

Then $\frac{PA\cdot PB}{PC\cdot PD}=\frac{\sin x\cdot\sin\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)\cdot\sin\left(\frac{\pi}{4}+x\right)}=\frac{\sin x}{\cos x}\cdot\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos\left(\frac{\pi}{4}-x\right)}=tgx\cdot tg\left(\frac{\pi}{4}-x\right)$.

We consider the function $f:\left(0,\frac{\pi}{4}\right)\to R, f(x)=tgx\cdot tg\left(\frac{\pi}{4}-x\right)$. We have

$f'(x)=\frac{\sin\left(\frac{\pi}{2}-2x\right)-\sin 2x}{2\cos^2 x\cdot\cos^2\left(\frac{\pi}{2}-x\right)}$ and $f'(x)=0 \Leftrightarrow \sin\left(\frac{\pi}{2}-2x\right)=\sin 2x \Leftrightarrow x=\frac{\pi}{8}$. But, $f'(x)\geq 0$ for $x\in\left(0,\frac{\pi}{8}\right]$ and $f'(x)\leq 0$ for $x\in\left[\frac{\pi}{8},\frac{\pi}{4}\right)$, namely f is increasing on to $\left(0,\frac{\pi}{8}\right]$ and f is decreasing on to $\left[\frac{\pi}{8},\frac{\pi}{4}\right)$. A short analysis with derivatives show that the maximum is attained when $x=\frac{\pi}{8}$ and so the maximum value is:

$$f_{\max}(x)=f\left(\frac{\pi}{8}\right)=tg^2\frac{\pi}{8}=\left(\sqrt{2}-1\right)^2=3-2\sqrt{2}$$

because $tg\frac{\pi}{8}=\frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}=\frac{\sqrt{2}}{2+\sqrt{2}}=\sqrt{2}-1$.

J 287. Let n be a positive integer and let $a_1,a_2,...,a_n$ be real numbers in the interval $\left(0,\frac{1}{n}\right)$.

Prove that

$$\log_{1-a_1}(1-na_2)+\log_{1-a_2}(1-na_3)+...+\log_{1-a_n}(1-na_1)\geq n^2$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Because $a_k\in\left(0,\frac{1}{n}\right),\forall k=\overline{1,n}$, we have $-na_k\in(-1,0),\forall k=\overline{1,n}$. Using the **Bernoulli Inequality** $(1+x)^\alpha\leq 1+\alpha x, x>-1, 0<\alpha\leq 1$, we get

$$(1-na_k)^{\frac{1}{n}}\leq 1-\frac{1}{n}\cdot na_k=1-a_k,\forall k=\overline{1,n},$$

and because $1-a_k<1,\forall k=\overline{1,n}$, we get

$$\log_{1-a_k}(1-na_{k+1})\geq n\log_{1-a_k}(1-a_{k+1}),\forall k=\overline{1,n}\text{, where } a_{n+1}=a_1.$$

We have

$$\sum_{k=1}^{n}\log_{1-a_k}(1-na_{k+1})\ge n\sum_{k=1}^{n}\log_{1-a_k}(1-a_{k+1}) \quad (1).$$

Using the **AM-GM Inequality** we obtain

$$\sum_{k=1}^{n}\log_{1-a_k}(1-a_{k+1})=\sum_{k=1}^{n}\frac{\ln(1-a_{k+1})}{\ln(1-a_k)}\ge n\sqrt[n]{\prod_{k=1}^{n}\frac{\ln(1-a_{k+1})}{\ln(1-a_k)}}=n \quad (2).$$

From (1) and (2) we get

$$\sum_{k=1}^{n}\log_{1-a_k}(1-na_{k+1})\ge n^2.$$

S 285. Let be a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$\frac{a}{b^2+c^2+2}+\frac{b}{c^2+b^2+2}+\frac{c}{a^2+b^2+2}\le\frac{1}{8}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Because $ab+bc+ca=1$, we have

$b^2+c^2+2=b^2+c^2+2(ab+bc+ca)=(b+c)^2+2a(b+c)=(b+c)(b+c+2a)$ and

$\frac{a}{b^2+c^2+2}=\frac{a}{(b+c)(b+c+2a)}=\frac{1}{2}\left(\frac{1}{b+c}-\frac{1}{b+c+2a}\right)$. Similarly,

$\frac{b}{c^2+a^2+2}=\frac{1}{2}\left(\frac{1}{c+a}-\frac{1}{c+a+2b}\right)$ and $\frac{c}{a^2+b^2+2}=\frac{1}{2}\left(\frac{1}{a+b}-\frac{1}{a+b+2c}\right)$.

The inequality becomes

$$\sum\frac{1}{a+b}\le\frac{1}{4}\sum\frac{1}{a}+\sum\frac{1}{a+b+2c}.$$

We consider the convex function $f:(0,\infty)\to R, f(x)=\frac{1}{x}, f''(x)=\frac{2}{x^3}>0, \forall x\in(0,\infty)$.

Using the **Popoviciu Inequality**:

$$\frac{f(a)+f(b)+f(c)}{3}+f\left(\frac{a+b+c}{3}\right)\ge\frac{2}{3}\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{b+c}{2}\right)+f\left(\frac{c+a}{2}\right)\right]$$

we get

$$\frac{1}{3}\sum\frac{1}{a}+\frac{3}{\sum a}\ge\frac{2}{3}\sum\frac{2}{a+b}$$

or

$$\sum\frac{1}{a}+\frac{9}{\sum a}\ge 4\sum\frac{1}{a+b}$$

and we only have to prove that

$$4\sum \frac{1}{a+b+2c} \geq \frac{9}{\sum a}.$$

But this follows immediately from the **AM-HM Inequality**. We have

$$\left(\sum \frac{1}{a+b+2c}\right)\left(\sum (a+b+2c)\right) \geq 9$$

so that $4\sum \frac{1}{a+b+2c} \geq \frac{9}{\sum a}$ and the problem is solved.

O 283. Prove that for all positive real numbers $x_1, x_2, \ldots, x_n$ the following inequality holds

$$\sum_{i=1}^{n} \frac{x_i^3}{x_1^2 + \ldots + x_{i-1}^2 + x_{i+1}^2 + \ldots + x_n^2} \geq \frac{x_1 + \ldots + x_n}{n-1}$$

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Ioan Viorel Codreanu, Satulung, Maramures, Romania

Let $s = \sum_{i=1}^{n} x_i^2$. Using the **Bergström Inequality** we get

$$\sum_{i=1}^{n} \frac{x_i^3}{s - x_i^2} = \sum_{i=1}^{n} \frac{\left(x_i^2\right)^2}{x_i\left(s - x_i^2\right)} \geq \frac{\left(\sum_{i=1}^{n} x_i^2\right)^2}{s\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^3}.$$

And so it is enough to prove that

$$(n-1)s^2 \geq s\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{i=1}^{n} x_i^3\right)\left(\sum_{i=1}^{n} x_i\right).$$

Using the **Cauchy-Schwarz Inequality** we get

$$\left(\sum_{i=1}^{n} x_i^3\right)\left(\sum_{i=1}^{n} x_i\right) = \left(\sum_{i=1}^{n} \left(x_i\sqrt{x_i}\right)^2\right)\left(\sum_{i=1}^{n} \sqrt{x_i}^2\right) \geq \left(\sum_{i=1}^{n} x_i^2\right)^2 = s^2.$$

And so it is enough to prove that

$$n\left(\sum_{i=1}^{n} x_i^2\right) \geq \left(\sum_{i=1}^{n} x_i\right)^2.$$

But this follows immediately from the **Cauchy-Schwarz Inequality**.

5. A generalization for J 10

by Marin Chirciu and Daniel Vacaru, Pitesti

Prove the inequality

$$\frac{1}{r_a^2+n r_a\cdot r_b}+\frac{1}{r_b^2+n\cdot r_c\cdot r_a}+\frac{1}{r_c^2+n\cdot r_a\cdot r_b}\geq\frac{27}{(n+1)\cdot(4R+r)^2}$$

.

One has, by Bergström, $\sum\left(\frac{1}{r_a^2+n\cdot r_b\cdot r_c}\right)\geq\frac{9}{\sum r_a^2+n\cdot\sum r_a r_b}$. But

$$(n+1)\cdot(r_a+r_b+r_c)^2\geq 3\cdot(\sum r_a^2+n\cdot\sum r_a\cdot r_c)\Leftrightarrow(n+1)\cdot(\sum r_a^2+2\sum r_b r_c)\geq 3\cdot(\sum r_a^2+n\sum r_b\cdot r_c)$$

$$(n+1-3)\cdot\sum r_a^2\geq(3n-2n-2)\cdot\sum r_a\cdot r_b$$

which is fulfilled for all $n\geq 2$.

The inequality

$$(n+1)\cdot(r_a+r_b+r_c)^2\geq 3\cdot(\sum r_a^2+n\sum r_b\cdot r_c)\Leftrightarrow\sum r_a^2+n\sum r_b\cdot r_c\leq\frac{(n+1)\cdot(r_a+r_b+r_c)^2}{3}\Leftrightarrow$$

$$\frac{1}{\sum r_a^2+n\sum r_b\cdot r_c}\geq\frac{3}{(n+1)\cdot(r_a+r_b+r_c)^2}\Rightarrow\frac{9}{\sum r_a^2+\sum r_b r_c}\geq\frac{27}{(n+1)\cdot(r_a+r_b+r_c)^2}$$

One has

$$r_a+r_b+r_c=\frac{S}{s-a}+\frac{S}{s-b}+\frac{S}{s-c}=S\cdot\left(\frac{ab+bc+ca-s^2}{(s-a)\cdot(s-b)\cdot(s-c)}\right)=S\cdot\left(\frac{r^2+s^2-4Rr-p^2}{(s-a)\cdot(s-b)\cdot(s-c)}\right)$$

$$S\cdot s\cdot\left(\frac{r^2+4R\cdot r}{S^2}\right)=4R+r$$

One must prove that

$$r^2+p^2+4Rr=\frac{S^2}{s^2}+s^2+\frac{abc}{s}=\frac{s\cdot(s-a)\cdot(s-b)\cdot(s-c)}{s^2}+s^2+\frac{abc}{s}=ab+bc+ca$$

. The equality apper when $r_a=r_b=r_c\Leftrightarrow a=b=c$.

One obtain the desired inequality.

For $n = 2$, one obtain the problem J 10 from Mathproblems, nr.3/2013, proposed by Dumitru M. Bătineţu – Giurgiu, Bucharest, and Neculai Stanciu, Buzău.