REVISTA ELECTRONICĂ MATEINFO.RO

ISBN 2065-6432

NOIEMBRIE 2014

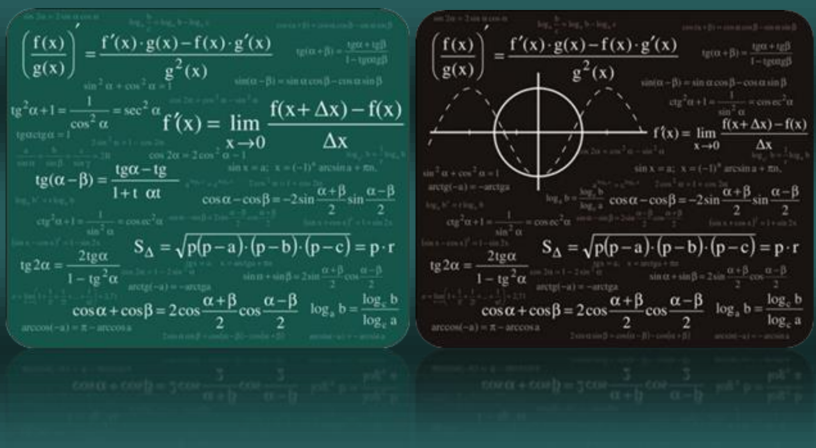
REVISTĂ LUNARĂ

DIN FEBRUARIE 2009

DE PESTE 4 ANI ÎN FIECARE LUNĂ

WWW.MATEINFO.RO

revistaelectronica@mateinfo.ro



Articole :

1. Solutions and hints of some problems from the Octogon Mathematical Magazine (III)

by D.M. Bătineţu-Giurgiu, Bucharest, Romania

Neculai Stanciu, Buzău, Romania

and

Titu Zvonaru, Comăneşti, Romania

PP. 19942. In all triangle ABC holds $\sum \frac{r_a^2}{r_a - r} \geq \frac{(4R+r)^2}{2(2R-r)}$.

Mihály Bencze

By well-known formulas and *Bergström*'s inequality we have

$$\sum \frac{r_a^2}{r_a - r} \overset{BERGSTROM}{\geq} \frac{\left(\sum r_a\right)^2}{\sum r_a - 3r} = \frac{(4R+r)^2}{4R+r-3r} = \frac{(4R+r)^2}{2(2R-r)}, \text{ q.e.d.}$$

PP. 20041. In all triangle ABC holds

1). $\sum \frac{a^2}{c(b+c)^2} \geq \frac{9}{8s}$

2). $\sum \frac{a^2}{(s-b)(s-c)(b+c)^2} \geq \frac{9}{4r(4R+r)}$

3). $\sum \frac{a^2}{(r_a + r_b r_c)(b+c)^2} \geq \frac{9}{4(4R+r+s^2)}$

4). $\sum \frac{a^2}{(b+c)^2 \sin^4 \frac{A}{2}} \geq \frac{18R^2}{8R^2+r^2-s^2}$

5). $\sum \frac{a^2}{(b+c)^2 \cos^6 \frac{A}{2}} \geq \frac{72R^3}{(4R+r)^3 - 3s^2(2R+r)}$

Mihály Bencze

By well-known formulas and *Bergström*'s inequality we have

1) $$\sum\frac{a^2}{c(b+c)^2}=\sum\frac{\left(\frac{a}{b+c}\right)^2}{c}\overset{BERGSTROM}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^2}{\sum a}\overset{NESBITT}{\geq}\frac{\left(\frac{3}{2}\right)^2}{a+b+c}=$$

$$=\frac{9}{4}\cdot\frac{1}{2s}=\frac{9}{8s};$$

2) $$\sum\frac{a^2}{(s-b)(s-c)(b+c)^2}=\sum\frac{\left(\frac{a}{b+c}\right)^2}{(s-b)(s-c)}\overset{BERGSTROM}{\geq}\frac{\left(\sum\frac{a}{b+c}\right)^2}{\sum(s-b)(s-c)}\overset{NESBITT}{\geq}$$

$$\overset{NESBITT}{\geq}\frac{9}{4}\cdot\frac{1}{\sum(s-b)(s-c)}=\frac{9}{4r(4R+r)};$$

3) As above we deduce that

$$\sum\frac{a^2}{(r_a+r_br_c)(b+c)^2}\geq\frac{9}{4}\cdot\frac{1}{r_a+r_b+r_c+r_ar_b+r_br_c+r_cr_a}=\frac{9}{4}\cdot\frac{1}{4R+r+s^2};$$

4) As above we deduce that

$$\sum\frac{a^2}{(b+c)^2\sin^4\frac{A}{2}}\geq\frac{9}{4}\cdot\frac{1}{\sum\sin^4\frac{A}{2}}=\frac{9}{4}\cdot\frac{8R^2}{8R^2+r^2-s^2}=\frac{18R^2}{8R^2+r^2-s^2};$$

5) As above we deduce that

$$\sum\frac{a^2}{(b+c)^2\cos^6\frac{A}{2}}\geq\frac{9}{4}\cdot\frac{1}{\sum\cos^6\frac{A}{2}}=\frac{9}{4}\cdot\frac{32R^3}{(4R+r)^3-2s^2(2R+r)}=$$

$$=\frac{72R^3}{(4R+r)^3-2s^2(2R+r)},$$

and the proof is complete.

PP.20150. If $a,b,c>0$ then $\sum\frac{a}{(a+b+c)^2+bc}\geq\frac{9}{10(a+b+c)}$.

Mihály Bencze

By *Bergström*'s inequality and AM-GM inequality we obtain

$$\sum\frac{a}{(a+b+c)^2+bc}=\sum\frac{a^2}{a(a+b+c)^2+abc}\geq\frac{(a+b+c)^2}{(a+b+c)^2(a+b+c)+3abc}\geq$$

$$\geq\frac{(a+b+c)^2}{(a+b+c)^3+\frac{(a+b+c)^3}{9}}=\frac{9}{10(a+b+c)},$$

and we are done.We have equality if and only if $a=b=c$, and the proof is complete.

PP.20153. If $x,a,b,c>0$ and $a+b+c=\frac{1}{x+1}$, then
$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{x(a+b)+(x+1)c}+\frac{x^2(a+b+c)^2}{(x+1)a+x(b+c)}\geq\frac{1}{2}$.

Mihály Bencze

By *Bergström*'s inequality we deduce that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{x(a+b)+(x+1)c}+\frac{x^2(a+b+c)^2}{(x+1)a+x(b+c)}\geq\frac{(a+b+c+x(a+b+c))^2}{2(x+1)(a+b+c)}=$$
$$=\frac{(a+b+c)(1+x)}{2}=\frac{1}{2},$$

and we are done.

PP.20160. Determine all $n\in N$ for which $\prod_{k=1}^{n}(2^{k-1}+1)$ is divisible by $n!$

Mihály Bencze

The product $\prod_{k=1}^{n}(2^{k-1}+1)$ it contains the factor 2 at first power (since for any $k\geq 2$, $2^{k-1}+1$ is odd). If $n\geq 4$, then $n!$ contains the factor 2 at the power at least 2. Hence $n\leq 3$. Finally, easily we get $n\in\{1,2,3\}$.

PP.20167. Solve the following equation $\frac{2-\sqrt{3}}{1-\cos 2x}+\frac{2+\sqrt{3}}{1+\cos 2x}=8-\frac{2}{\sin 2x}$.

Mihály Bencze

The given equation is written successively

$$\frac{1-\cos 30^0}{1-\cos 2x}+\frac{1+\cos 30^0}{1+\cos 2x}=4-\frac{2\sin 30^0}{\sin 2x}$$
$$\Leftrightarrow\frac{1-\cos 30^0}{1-\cos 2x}-1+\frac{1+\cos 30^0}{1+\cos 2x}-1=\frac{2(\sin 2x-\sin 30^0)}{\sin 2x}$$
$$\Leftrightarrow\frac{\cos 2x-\cos 30^0}{1-\cos 2x}+\frac{\cos 30^0-\cos 2x}{1+\cos 2x}=\frac{2(\sin 2x-\sin 30^0)}{\sin 2x}$$
$$\Leftrightarrow\frac{(\cos 2x-\cos 30^0)(1+\cos 2x-1+\cos 2x)}{1-\cos^2 2x}=\frac{2(\sin 2x-\sin 30^0)}{\sin 2x}$$
$$\Leftrightarrow\frac{2\sin(x+15^0)\sin(15^0-x)\cos 2x}{\sin^2 2x}=\frac{2\sin(x-15^0)\cos(x+15^0)}{\sin 2x}$$

$\Leftrightarrow \sin(15^0 - x)[\sin(x+15^0)\cos 2x + \cos(x+15^0)\sin 2x] = 0$

$\Leftrightarrow \sin(15^0 - x)\sin(x+15^0+2x) = 0$.

Therefore we have the solutions

$x - 15^0 = k\cdot 180^0, 3x+15^0 = p\cdot 180^0$, i.e. $x = k\cdot 180^0 + 15^0$ and $x = p\cdot 60^0 - 5^0$ with $k, p \in Z$.

The solution is complete.

PP.20179. Let ABC be a triangle and $A(a), B(b), C(c)$. Prove that $\left|\sum a\left(b^2-c^2\right)\right| = 4sRr.$

Mihály Bencze

Using $\sum a(b^2-c^2) = (a-b)(b-c)(c-a)$, then

$\left|\sum a(b^2-c^2)\right| = |(a-b)(b-c)(c-a)| = \|AB\|\cdot\|BC\|\cdot\|CA\| = 4sRr$, q.e.d

PP.20197. If $a,b,c>0$ then $3^{n-1}\left(a^{4n}+b^{4n}+c^{4n}\right) \geq a^n b^n c^n (a+b+c)^n$.

Mihály Bencze

Solution 1. We can assume that $a \geq b \geq c$. By Chebyshev' s inequality we deduce that

$$a^{4n}+b^{4n}+c^{4n} = a^{4n-1}\cdot a + b^{4n-1}\cdot b + c^{4n-1}\cdot c \geq \frac{1}{3}(a+b+c)(a^{4n-1}+b^{4n-1}+c^{4n-1}) \geq$$

$\geq \frac{1}{3^2}(a+b+c)^2(a^{4n-2}+b^{4n-2}+c^{4n-2}) \geq \ldots \geq \frac{1}{3^n}(a+b+c)^n(a^{3n}+b^{3n}+c^{3n})$, and because by AM-GM inequality we have $\frac{a^{3n}+b^{3n}+c^{3n}}{3} \geq a^n b^n c^n$, we obtain the given inequality.

Solution 2. Let $f: R_+^* \to R_+^*,\ f(x) = x^{4n}$, which is convex on R_+^*, So,

$$f(a)+f(b)+f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \Leftrightarrow a^{4n}+b^{4n}+c^{4n} \geq 3\frac{(a+b+c)^{4n}}{3^{4n}} =$$

$$= 3^{1-4n}(a+b+c)^n(a+b+c)^{3n} \overset{AM-GM}{\geq} 3^{1-4n}(a+b+c)^n\left(3\cdot\sqrt[3]{abc}\right)^{3n} =$$

$$= 3^{1-n}a^n b^n c^n (a+b+c)^n.$$

Hence, $3^{n-1}(a^{4n}+b^{4n}+c^{4n}) \geq a^n b^n c^n (a+b+c)^n$.

The solution is complete.

PP.20200. In all triangle ABC holds $\sum \frac{tg^2\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}} \geq \frac{1}{2}$.

Mihály Bencze

By *Bergström*'s inequality we deduce that

$$\sum \frac{tg^2\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}} \geq \frac{\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)^2}{2\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)} = \frac{1}{2}\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right),$$

but $tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2} \geq \sqrt{3}$ (see the item 2.33. from *Bottema*, Geometric Inequalities, Groningen, 1969).

Hence

$$\sum \frac{tg^2\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}} \geq \frac{\sqrt{3}}{2},$$

which is an inequality stronger than inequality to prove. The solution is complete.

PP.20209. In all triangle ABC holds $\sum \frac{\cos\frac{A-B}{2}}{\sin^2\frac{C}{2}} \geq 12$.

Mihály Bencze

Using the inequality $x^2+y^2+z^2 \geq \frac{1}{3}(x+y+z)^2$, we have that

$$\sum \frac{\cos\frac{A-B}{2}}{\sin^2\frac{C}{2}} \geq \frac{1}{3}\left(\sum \frac{\sqrt{\cos\frac{A-B}{2}}}{\sin\frac{C}{2}}\right)^2,$$

so it is enough to prove the inequality

$$\sum \frac{\sqrt{\cos\frac{A-B}{2}}}{\sin\frac{C}{2}} \geq 6.$$

Because $\cos\frac{A-B}{2} \leq 1$, we obtain

$$\sum\frac{\sqrt{\cos\frac{A-B}{2}}}{\sin\frac{C}{2}}=\sum\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\cdot\frac{1}{\sqrt{\cos\frac{A-B}{2}}}\geq\sum\frac{2\cos\frac{C}{2}\cos\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}=$$

$$=\sum\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\sin C}=\sum\frac{\sin A+\sin B}{\sin C}=\frac{\sin A}{\sin B}+\frac{\sin B}{\sin A}+\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}+$$

$+\frac{\sin C}{\sin A}+\frac{\sin A}{\sin C}\geq 2+2+2=6$ (we used well-known $\frac{x}{y}+\frac{y}{x}\geq 2$, true for any $x,y>0$).

The proof is complete.

PP.20212. Compute $\int_0^1\frac{e^{x^n}dx}{e^{x^n}+e^{(1-x)^n}}$.

Mihály Bencze

Let $I=\int_0^1\frac{e^{x^n}}{e^{x^n}+e^{(1-x)^n}}dx$ where we make the changes of variable $x=u(t)=1-t$, $u'(t)=-1, u(1)=0, u(0)=1$. Therefore, $I=\int_0^1\frac{e^{(1-x)^n}}{e^{(1-x)^n}+e^{x^n}}dx$, and then

$2I=\int_0^1\frac{e^{x^n}+e^{(1-x)^n}}{e^{x^n}+e^{(1-x)^n}}dx=\int_0^1 dx=1$, i.e. $I=\frac{1}{2}$, and we are done.

PP.20219. In all triangle ABC holds $\sum\frac{ab}{\sqrt{(s-a)(s-b)}}\geq 2s\left(4-\frac{R}{r}\right)$.

Mihály Bencze

By AM-GM inequality we have $\sqrt{(s-a)(s-b)}\leq\frac{s-a+s-b}{2}=\frac{c}{2}$.

Applying the well-known $x^2+y^2+z^2\geq xy+yz+zx$, yields that

$$\sum\frac{ab}{\sqrt{(s-a)(s-b)}}\geq 2\sum\frac{ab}{c}=2\cdot\frac{\sum a^2b^2}{abc}\geq 2\cdot\frac{abc(a+b+c)}{abc}=4s.$$

Threfore, it suffices to prove that

$$4s\geq 2s\left(4-\frac{R}{r}\right)\Leftrightarrow\frac{R}{r}\geq 2\text{, true.}$$

The proof is complete.

PP.20222. In all triangle ABC holds $\sum\sqrt{\frac{b+c-a}{a}} \leq \sqrt{\frac{s^2+r^2+4Rr}{2Rr}}$.

Mihály Bencze

Using $s^2+r^2+4Rr = ab+bc+ca$, $2Rr = \dfrac{abc}{a+b+c}$ and squaring the given inequality we obtain

$$\sum\frac{b+c-a}{a} + 2\sum\sqrt{\frac{(b+c-a)(c+a-b)}{ab}} \leq \frac{\sum a\sum ab}{abc}$$

$$\Leftrightarrow \sum\left(\frac{b+c+a}{a}-2\right) + 2\sum\sqrt{\frac{2(s-a)2(s-b)}{ab}} \leq \frac{\sum a\sum ab}{abc}$$

$\Leftrightarrow \sum a\sum\frac{1}{a} - 6 + 4\sum\sin\frac{A}{2} \leq \sum a\sum\frac{1}{a} \Leftrightarrow \sum\sin\frac{A}{2} \leq \frac{3}{2}$, i.e. the item 2.9. from *Bottema*, Geometric Inequalities, Groningen, 1969. The solution is complete.

PP.20223. If $a,b,c \geq 1$, then $3\sum\sqrt{a-1} \geq \sum\sqrt{a(1+bc)}$.

Mihály Bencze

For $a=b=c=1$ is evidently that the inequality to prove is not true. We are done.

PP.20224. In all triangle ABC holds $\sum\cos(A-B) \leq \frac{s^2-7R^2-4Rr-r^2}{R^2}$.

Mihály Bencze

The statement, of problem 20224 is not true. The statement of problem 2024 is ''In all triangle ABC holds $\sum\cos(A-B) \leq \dfrac{s^2-7R^2-4Rr-r^2}{R^2}$'' for e.g. if ABC is an equilateral triangle with the lengths of sides equal with 1 we have

$s=\dfrac{3}{2}$, $R=\dfrac{1}{\sqrt{3}}$, $r=\dfrac{1}{2\sqrt{3}}$, and then $s^2-7R^2-4Rr-r^2 = \dfrac{9}{4}-\dfrac{7}{3}-\dfrac{2}{3}-\dfrac{1}{12} < 0$, while LHS is equal with 3. We propose the following statement

''In all triangle ABC holds $\sum\cos(A-B) \leq \dfrac{s^2-7R^2-Rr-r^2}{R^2}$''

Indeed, since $\sum\cos(A-B) = \dfrac{s^2+r^2+2Rr-2R^2}{2R^2}$, the last inequality it is write

$$2s^2-14R^2-2Rr-2r^2 \geq s^2+r^2+2Rr-2R^2 \Leftrightarrow s^2 \geq 12R^2+3r^2+4Rr.$$

By Gerretsen' s inequality we have $s^2 \geq 16R^2 - 5r^2$, so it remains to show that $2R^2 \geq 4r^2 + 2Rr$, which yields by $R^2 \geq 4r^2$ and $R^2 \geq 2Rr$ ($R \geq 2r$, is well-known Euler' s inequality). Our proof is complete.

PP.20226. In all acute triangle ABC holds $\sum \sin(A-B)\sin(A-C) \geq 0$.

Mihály Bencze

WLOG that $A \geq B \geq C$. Because $A+B+C=180^0$ we deduce that $C \leq 60^0$.
The inequality to prove is written successively

$$\sin(A-B)\sin(A-C)+\sin(B-C)\sin(B-A)+\sin(C-A)\sin(C-B) \geq 0$$

$$\Leftrightarrow \sin(A-B)\sin(A-C)-\sin(B-C)\sin(A-B)+\sin(A-C)\sin(B-C) \geq 0$$

$$\Leftrightarrow \sin(A-B)\cdot 2\sin\frac{A-C-B+C}{2}\cos\frac{A-C+B-C}{2}+\sin(A-C)\sin(B-C) \geq 0$$

$$\Leftrightarrow 2\sin(A-B)\sin\frac{A-B}{2}\cos\frac{180^0-3C}{2}+\sin(A-C)\sin(B-C) \geq 0$$

$$\Leftrightarrow 2\sin(A-B)\sin\frac{A-B}{2}\sin\frac{3C}{2}+\sin(A-C)\sin(B-C) \geq 0,$$

which is true since $A \geq B \geq C$ and $C \leq 60^0$.

PP.20238. If $a, b > 0$ then $\frac{(3a-b)^2}{(3a+b)^2+4b^2} + \frac{(3b-a)^2}{(3b+a)^2+4a^2} \geq \frac{2}{5}$.

Mihály Bencze

Since $\frac{(3a-b)^2}{(3a+b)^2+4b^2} - \frac{1}{5} = \frac{1}{5}\cdot\frac{36a(a-b)}{(3a+b)^2+4b^2}$ and

$$\frac{(3b-a)^2}{(3b+a)^2+4a^2} - \frac{1}{5} = \frac{1}{5}\cdot\frac{36b(b-a)}{(3b+a)^2+4a^2},$$

the inequality to prove becomes

$$(a-b)\left(\frac{a}{(3a+b)^2+4b^2} - \frac{b}{(3b+a)^2+4a^2}\right) \geq 0$$

$$\Leftrightarrow (a-b)(4a^3+9ab^2+6a^2b+a^3-4b^3-9a^2b-6ab^2-b^3) \geq 0$$

$$\Leftrightarrow (a-b)[5(a^3-b^3)-3ab(a-b)] \geq 0 \Leftrightarrow (a-b)^2(5a^2+2ab+5b^2) \geq 0,$$

which is true because $5a^2+2ab+5b^2=(a+b)^2+4(a^2+b^2)>0$. The solution is complete.

PP.20244. If $a, b > 0$, then $\frac{(3a+b)^2}{2a^2+(a+b)^2} + \frac{(a+b)^2}{2a^2+b^2} \leq 4$.

Mihály Bencze

We have $\frac{(3a+b)^2}{2a^2+(a+b)^2} - \frac{8}{3} = \frac{(a-b)(3a+5b)}{3\left(2a^2+(a+b)^2\right)}$, and

$$\frac{(a+b)^2}{2a^2+b^2} - \frac{4}{3} = -\frac{(a-b)(5a-b)}{3\left(2a^2+b^2\right)}.$$

Then, the given inequality becomes

$$\frac{(a-b)(5a-b)}{3(2a^2+b^2)} - \frac{(a-b)(3a+5b)}{3\left(2a^2+(a+b)^2\right)} \geq 0,$$

which after some algebra becomes

$(a-b)^2\left[(3a+b)^2+5b^2\right] \geq 0$, true. We have equality iff $a=b$.

PP.20256. Solve in R the equation
$\log_2\left(2^{\sin x}+2^{tgx}\right)+2^{1+\sin x+tgx} = 4^{x+1}-4^{\sin x}-4^{tgx}+x+1$.

Mihály Bencze

Considering the injective function $f(t)=\left(2^t\right)^2+t, t>0$, the given equation is equivalent with $f\left(\log_2\left(2^{\sin x}+2^{tgx}\right)\right)=f(x+1)$, and it remains to solve the equation

$$2^{\sin x}+2^{tgx}=2^{x+1},$$

which we let this the last equations like an exercise to readers (a solution is $x=0$).

PP.20257. Solve in R the equation $\log_7\frac{3^x+5^x}{4^x+2^x+2} = 5^x+3^x-4^x-2^x-2$.

Mihály Bencze

If we consider the injective function $f(t)=7^t-t$, $t>0$, the given equation is written as follows $f\left(\log_7(3^x+5^x)\right)=f\left(\log_7(4^x+2^x+2)\right)$, therefore it remaint to solve the equation $3^x+5^x=4^x+2^x+2$. For $x>1$ we have $3^x+5^x>4^x+2^x+2$.
We found $x=1$, but what happens for $x<1$ (we let this like an exercise to readers).

PP.20261. Solve in R the equation
$2^{3^x-1}-2\log_2(x+1) = -3^x+x^2+2x+2$.

Mihály Bencze

Using the injective function $f(t)=2^t+t, t>0$, then the equation to solve is written as $f\left(\log_2(x+1)^2\right)=f(3^x-1)$, so $2^{3^x-1}=(x+1)^2$, with the solutions $x_1=0$ and $x_2=1$.

PP.20262. Solve in R the equation
$2^{(2^x-1)^2} - \log_2(\sqrt{x}+1) = -4^x + 2^{x+1} + \sqrt{x}$.

Mihály Bencze

Written the equation to solve as below

$$2^{(2^x-1)^2} + (2^x-1)^2 = \sqrt{x} + 1 + \log_2(\sqrt{x}+1),$$

and than using the injectivity of the function $f(t) = 2^t + t$, $t > 0$, we obtain that

$$(2^x-1)^2 = \log_2(\sqrt{x}+1).$$

Hence, $2^x + \sqrt{x} = \sqrt{\log_2(\sqrt{x}+1)} + \sqrt{x} + 1$, and then from the injectivity of the function $g(t) = 2^t + \sqrt{t}, t > 0$, we deduce that $2^x = \sqrt{x} + 1 \Leftrightarrow 2^x - 1 = \sqrt{x}$. The last equation has the solutions $x_1 = 0$ and $x_2 = 1$ (because $\sqrt{x}$ is strictly concave and $2^x - 1$ is strictly convex). The solution is complete.

PP.20265. Solve in R the following system: $\begin{cases} x^2+1 = \sqrt{3y+1} \\ y^2+1 = \sqrt{3z+1} \\ z^2+1 = \sqrt{3x+1} \end{cases}$.

Mihály Bencze

Since $z^2 + 1 \geq 1$, we deduce $x \geq 0$ and analogous $y, z > 0$. If $x > 1$then $\sqrt{3y+1} > 2$, so $y > 1$ and then $z > 1$. Therefore $x, y, z > 1$ or $x, y, z \leq 1$. By squaring and adding the equations of system we obtain

$$x^4 + 2x^2 - 3x + y^4 + 2y^2 - 3y + z^4 + 2z^2 - 3z = 0$$

$$\Leftrightarrow x(x-1)(x^2+2x+3) + y(y-1)(y^2+2y+3) + z(z-1)(z^2+2z+3) = 0 \qquad (1)$$

We have $x^2 + 2x + 3 > 0$, $y^2 + 2y + 3 > 0$, $z^2 + 2z + 3 > 0$.

If $x, y, z > 1$, then $x(x-1) > 0, y(y-1) > 0, z(z-1) > 0$ and (1) has no solution.

If $x, y, z \leq 1$, then $x(x-1) < 0, y(y-1) < 0, z(z-1) < 0$ and (1) has no solution.

Hence $(x, y, z) \in \{(0,0,0); (1,1,1)\}$. We are done.

PP.20267. Solve in R the following equation
$\log_6 \frac{3x+2}{2^x+3^x} = 9^x + 4^x + 2 \cdot 6^x - 9x^2 - 12x - 4$.

Mihály Bencze

The equation to solve is written as follows

$$\log_6(3x+2) + (3x+2)^2 = \log_6(2^x+3^x) + (2^x+3^x)^2.$$

Using the injectivity of the function $f(t)=\log_6 t+t^2$, $t>0$, we obtain that $2^x+3^x=3x+2$. The last equation has only two solutions, namely $x_1=0$ and $x_2=1$ (since the function 2^x+3^x is strictly convex and the function $3x+2$ is linear). The solution is complete.

PP.20269. Solve in R the following equation
$\log_2(\cos(\cos x))+\cos^2(\cos x)=a\cdot\cos x+4^{a\cdot\cos x}$ where $a>0$.

Mihály Bencze

Considering the injective function $f(t)=2^{2t}+t$, $t>0$, the given equation is written as $f(\log_2(\cos(\cos x)))=f(a\cos x)$, and it remains to solve the equation $2^{a\cos x}=\cos(\cos x)$, which we propose like an exercise to the readers (a solution is $\cos x=0$).

PP.20270. If $a>1$ then solve in R the equation
$a^{2(x-2)}-\log_a((1-a)x+4a-3)=$
$(1-a)^2x^2+\left(-4a^2+7a-4\right)x+16a^2-24a+11.$

Mihály Bencze

Considering the injective function $f(t)=a^{2t}+t, t>0$,the given equation is written as $f(\log_a((1-a)x+4a-3))=f(x-2)$, and it remains to solve the equation $a^{x-2}=(1-a)x+4a-3$, which we propose like an exercise to the readers (a solution is $x=3$).

PP.20271. Solve in R the equation
$\log_2\frac{\sqrt{x+7}-\sqrt{x-1}}{x^2-2x+2}+2\sqrt{x^2+6x-7}=x^4-4x^3+8x^2-10x+5.$

Mihály Bencze

The given equation is equivalent with

$$\log_2\frac{\sqrt{x+7}-\sqrt{x-1}}{x^2-2x+2}+2\sqrt{x^2+6x-7}=x^4-4x^3+8x^2-10x-2.$$

Considering the injective function $f(t)=t^2+\log_2 t$, $t>0$, the given equation is written as $f(\sqrt{x+7}-\sqrt{x-1})=f(x^2-2x+2)$.
Therefore it remains to solve the equation

$$\sqrt{x+7}-\sqrt{x-1}=x^2-2x+2\Leftrightarrow\sqrt{x+7}=\sqrt{x-1}+(x-1)^2+1.$$

Denoting $\sqrt{x-1}=t\geq 0$, we obtain successively

$$t^4+t+1=\sqrt{t^2+8} \Leftrightarrow (t^4+t+1)^2=t^2+8 \Leftrightarrow t^8+2t^5+2t^4+2t-7=0 \Leftrightarrow$$
$$\Leftrightarrow (t-1)(t^7+t^6+t^5+3t^4+5t^3+5t^2+5t+7)=0.$$

The last equation has only one positive solution $t=1$.
Hence the given equation has only one solution $x=2$.

PP.20274. Solve in R the equation
$$2^{x^2-x}+3^{x^2-x}+\log_5\left(2^{x^2-x}+3^{x^2-x}\right)=2\cdot 5^{x^2-x}+\log_5 2+x^2-x.$$

Mihály Bencze

We considering the injective function $f(t)=t+\log_5 t$, $t>0$, the given equation is written as $f\left(2^{x^2-x}+3^{x^2-x}\right)=f(2\cdot 5^{x^2-x})$, and it remains to solve the equation $2^{x^2-x}+3^{x^2-x}=2\cdot 5^{x^2-x}$, which has the solutions $x_1=0$ and $x_2=1$.

PP.20275. Solve in R the equation
$$\log_3\frac{9^x+4^x}{5^x+2\cdot 6^x}=-81^x+2\cdot 36^x+4\cdot 30^x+25^x-16^x.$$

Mihály Bencze

We considering the injective function $f(t)=3^{2t}+t$, $t>0$, the given equation is written as $f\left(\log_3(9^x+4^x)\right)=f\left(\log_3(5^x+2\cdot 6^x)\right)$, and it remains to solve the equation $9^x+4^x=5^x+2\cdot 6^x$, which has a solution $x=2$ (we ask the readers there are others?).

PP.20277. Solve in R the equation
$$\log_3\left(x+\sqrt{3+\sqrt{x}}\right)+x^2+2x\sqrt{3+\sqrt{x}}+\sqrt{x}=7.$$

Mihály Bencze

The equation to solve is written as
$$\log_3\left(x+\sqrt{3+\sqrt{x}}\right)+\left(x+\sqrt{3+\sqrt{x}}\right)^2=\log_3 3+3^2,$$
and because the function $f(t)=\log_3 t+t^2$ is injective (since is increasing for $t>0$) we obtain that
$$x+\sqrt{3+\sqrt{x}}=3 \Leftrightarrow \sqrt{3+\sqrt{x}}=3-x.$$
Denoting $\sqrt{x}=t$ we have to solve the equation $\sqrt{3+t}=3-t^2$, with $t^2\le 3$ and $t>0$.
After squaring the last equation yields that

$$t^4-6t^2-t+6=0 \Leftrightarrow (t-1)(t+2)(t^2-t-3)=0,$$

and because $t_1=-2$ and $t_2=\frac{1+\sqrt{13}}{2}$ doesn't satisfy the relation $t^2\le 3$, and $t_3=\frac{1-\sqrt{13}}{2}<0$, we deduce that the equation has one solution, and this is $x=1$.
The solution is complete.

PP.20278. Solve in R the equation
$\log_2\left(x^{\log_2\frac{3}{5}}+x^{\log_2\frac{4}{5}}\right)+x^{2\log_2\frac{3}{5}}+x^{2\log_2\frac{4}{5}}+2x^{\log_2\frac{12}{25}}=1.$

Mihály Bencze

We consider the function $f(t)=\log_2 t+t^2$, $t>0$, which is strictly increasing so is injective. The equation to solve is written as follows
$f\left(x^{\log_2\frac{3}{5}}+x^{\log_2\frac{4}{5}}\right)=f(1)$, with $x>0$. Hence $x^{\log_2\frac{3}{5}}+x^{\log_2\frac{4}{5}}=1$, which has only one solution $x=4$, because the function $h(t)=t^{\log_2\frac{3}{5}}+t^{\log_2\frac{4}{5}}$, $t>0$, is strictly decreasing (since the exponents are negative). We are done.

PP.20279. Solve in R the equation
$\log_2^2\left(x+\frac{4}{x}\right)+\log_2\left(x+\frac{4}{x}\right)-2^{-x^4+7x^2+4x-18}=$
$=x^8-14x^6-8x^5+84x^4+56x^3-229x^2-141x+306-\frac{4}{x}.$

Mihály Bencze

Using the injective function $f(t)=2^t+t^2+t, t>0$, the given equation is written like that

$$f\left(\log_2\left(x+\frac{4}{x}\right)\right)=f\left(-x^4+7x^2+4x-18\right),$$

so it remains to solve the equation

$$x+\frac{4}{x}=2^{-x^4+7x^2+4x-18}.$$

Since $x+\frac{4}{x}\ge 4$, it must to have $2^{-x^4+7x^2+4x-18}\ge 2^2$, i.e.

$$x^4-7x^2-4x-20\le 0 \Leftrightarrow (x-2)^2(x^2+4x+5)\le 0,$$

which that happens only if $x=2$, which verify the equation.
Hence, we get the solution is $x=2$. The solution is complete.

PP.20280. Solve in R the equation
$x^n - nx + n + 1 - \frac{x}{x^2+1} = 4^{\frac{x}{x^2+1}} - \log_4(x^n - nx + n + 1)$, where $n \geq 1$, $n \in N$.

Mihály Bencze

Considering the injective function $f(t) = 4^t + t$, $t > 0$, the equation

$x^n - nx + n - 1 - \frac{x}{x^2+1} = 4^{\frac{x}{x^2+1}} - \log_4(x^n - nx + n + 1)$, $n \geq 1, n \in N$ becomes

$$f\left(\frac{x}{x^2+1}\right) = f\left(\log_4(x^n - nx + n + 1)\right).$$

So we must to solve the equation

$$\frac{x}{x^2+1} = \log_4(x^n - nx + n + 1).$$

By AM-GM inequality we obtain

$\frac{x}{x^2+1} \leq \frac{x}{2x} = \frac{1}{2}$ and $x^n + n - 1 = x^n + \underbrace{1 + 1 + ... + 1}_{n-1} \geq n \cdot \sqrt[n]{x^n \cdot 1 \cdot 1 \cdot ... \cdot 1} = nx \Leftrightarrow$

$\Leftrightarrow x^n - nx + n + 1 \geq 2 \Leftrightarrow \log_4(x^n - nx + n + 1) \geq \frac{1}{2}$.

Hence $x = 1$, and we are done.

PP.20281. Solve in R the equation $\log_2\left(\frac{\log_2\left(x+\frac{1}{x}\right)}{3x^2-2x^3}\right) = 2^{3x^2-2x^3} - x - \frac{1}{x}$.

Mihály Bencze

We have $x > 0$. We consider the function $f(t) = \log_2 t + 2^t$ $(t > 0)$ which is increasing so is injective and the equation to solve becomes

$$f\left(\log_2\left(x + \frac{1}{x}\right)\right) = f(3x^2 - 2x^3).$$

Therefore, we must to solve the equation

$$\log_2\left(x + \frac{1}{x}\right) = 3x^2 - 2x^3.$$

Since $x + \frac{1}{x} \geq 2$, we have $\log_2\left(x + \frac{1}{x}\right) \geq 1$. Also we have $3x^2 - 2x^3 \leq 1$, because the last inequality is equivalent with $(x-1)^2(2x+1) \geq 0$.

Hence, we obtain the only one solution, and this is $x = 1$.
The solution is complete.

PP.20287. Solve in R the equation
$\log_2(\log_2(1+\cos 4x)) + 2\sin x \sin 5x = 2^{2^{1+\cos 6x}}$.

Mihály Bencze

Using the the injective function $f(t) = 2^{2^t} + t \quad (t > 0)$, and the fact $2\sin x \sin 5x = \cos 6x - \cos 4x$, the given equation is wrriten as follows

$$f(\log_2(\log_2(1+\cos 4x))) = f(1+\cos 6x),$$

so it remains to solve the equation

$$1+\cos 4x = 2^{2^{1+\cos 6x}},$$

which we let it to readers as exercise.

PP.20288. Solve in R the equation

$$\log_2^2\left(x+\tfrac{4}{x}\right) - 2^{-x^2+4x-2} = x^4 - 8x^3 + 20x^2 - 17x + 4 - \tfrac{4}{x}.$$

Mihály Bencze

Written the equation to be solved as below

$$\log_2^2\left(x+\frac{4}{x}\right) + x + \frac{4}{x} = (-x^2+4x-2)^2 + 2^{-x^2+4x-2},$$

and using the injectivity of function $f(t) = 2^t + t^2, t > 0$, we obtain

$$\log_2\left(x+\frac{4}{x}\right) = -x^2+4x-2.$$

We have

$$x+\frac{4}{x} \geq 4 \text{ and } -x^2+4x-2 \leq 2 \Leftrightarrow (x-2)^2 \geq 0.$$

Hence, we find one solution, and this is $x = 2$. We are done.

PP.20289. Solve in R the equation

$$(x^2+1)\left(4^{\frac{x}{x^2+1}} - \log_4(x^4-4x+5)\right) = x^6 + x^4 - 4x^3 + 5x^2 - 5x + 5.$$

Mihály Bencze

The given equation is writting as

$$4^{\frac{x}{x^2+1}}+\frac{x}{x^2+1}=\log_4\left(x^4-4x+5\right)+x^4-4x+5,$$

and like in the solution of PP.20280 we obtain the solution $x=1$.

PP.20291. Solve in R the equation

$$\log_4\frac{\sqrt[4]{x-1}+\sqrt[4]{3-x}}{2+\sqrt[4]{x-2}}+\sqrt{x-1}+2\sqrt[4]{-x^2+4x-3}\left(1+2\sqrt[4]{x-1}+2\sqrt[4]{3-x}\right)+$$
$$+\sqrt{3-x}+6\sqrt{-x^2+4x-3}=20+4\sqrt[4]{x-2}+25\sqrt{x-2}+x+8\sqrt[4]{(x-2)^3}.$$

Mihály Bencze

The equation to be solved is equivalent with the following

$$\log_4\frac{\sqrt[4]{x-1}+\sqrt[4]{3-x}}{2+\sqrt[4]{x-2}}+\sqrt{x-1}+2\cdot\sqrt[4]{-x^2+4x-3}(1+2\sqrt{x-1}+2\sqrt{3-x})+\sqrt{3-x}+$$
$$+6\sqrt{-x^2+4x-3}=16+36\cdot\sqrt[4]{x-2}+25\sqrt{x-2}+x+8\cdot\sqrt[4]{(x-2)^3}.$$

Evidently $x\in[2,3]$.

We consider the injective function

$$f(t)=\log_4 t+t^2+t^4,\ t>0,$$

and the equation from above is written as

$$f(\sqrt[4]{x-1}+\sqrt[4]{3-x})=f(2+\sqrt[4]{x-2}),$$

so we must to solve the equation

$$\sqrt[4]{x-1}+\sqrt[4]{3-x}=2+\sqrt[4]{x-2}.$$

Applying the inequality of meens we obtain

$$2+\sqrt[4]{x-2}=\sqrt[4]{(x-1)\cdot1\cdot1\cdot1}+\sqrt[4]{(3-x)\cdot1\cdot1\cdot1}\le\frac{x-1+1+1+1+3-x+1+1+1}{4}=2,$$

which yields $x=2$. Hence the given equation has the solutions $x=2$, and we are done.

PP.20302. Solve in R the equation

$$\log_2\left(x^2+2^x\right)+\left(x^2-1\right)2^{x+1}+x^4+x^2+2^x=4^x+x+1.$$

Mihály Bencze

Using the injective function $f(t)=\log_2 t+t^2+t$, $t>0$, the given equation is written as follows $f\left(x^2+2^x\right)=f(2^{x+1})$. Hence, $x^2+2^x=2^{x+1}\Leftrightarrow x^2=2^x$, with solutions $x_1=2$ and $x_2=4$. The solution is complete.

PP.20309. Solve the equation $8^x + (x^3 + x^2 + x)\, 2^x = (x^2 + x + 1)\, 4^x + x^3$.

Mihály Bencze

The equation to solve, i.e. $8^x + (x^3 + x^2 + x)\cdot 2^x = (x^2 + x + 1)\cdot 4^x + x^3$,
is written successively as follows

$$(x^2 + x + 1)\cdot 2^x \cdot (2^x - x) - (2^x - x)(2^{2x} + x\cdot 2^x + x^2) = 0$$

$$\Leftrightarrow (2^x - x)(x^2\cdot 2^x + x\cdot 2^x + 2^x - 2^{2x} - x\cdot 2^x - x^2) = 0$$

$$\Leftrightarrow (2^x - x)[x^2(2^x - 1) - 2^x(2^x - 1)] = 0 \Leftrightarrow (2^x - 1)(2^x - x)(2^x - x^2) = 0.$$

Hence we obtain the solutions $x_1 = 0, x_2 = 2, x_3 = 4$. The solution is complete.

PP.20306. Solve in R the equation
$\log_3 \frac{\cos 5x + 8\cos^3 x}{\cos x} = 8^{\cos x} + 3\cos x - 2^{\cos 5x + 8\cos^3 x} - \cos 5x - 8\cos^3 x + 1.$

Mihály Bencze

Considering the injective function $f(t) = \log_3 t + 2^t + t$, $t > 0$, the equation to solve, i.e. $\log_3 \frac{\cos 5x + 8\cos^3 x}{\cos x} = 8^{\cos x} + 3\cos x - 2^{\cos 5x + 8\cos^3 x} - \cos 5x - 8\cos^3 x + 1$, it is written successively

$$f(\cos 5x + 8\cos^3 x) = f(3\cos x) \Leftrightarrow \cos 5x + 8\cos^3 x = 3\cos x$$

$$\Leftrightarrow \cos 5x + 4\cos x\cdot 2\cos^2 x - 3\cos x = 0 \Leftrightarrow \cos 5x + 4\cos x(1 + \cos 2x) - 3\cos x = 0$$

$$\Leftrightarrow \cos 5x + \cos x + 4\cos x\cos 2x = 0 \Leftrightarrow 2\cos 3x\cos 2x + 4\cos x\cos 2x = 0$$

$$\Leftrightarrow \cos 2x(\cos 3x + 2\cos x) = 0 \Leftrightarrow \cos 2x\left(\cos x(4\cos^2 x - 3) + 2\cos x\right) = 0$$

$$\Leftrightarrow \cos x\cos 2x(4\cos^2 x - 1) = 0 \Leftrightarrow \cos x\cos 2x\left(2(1 + \cos 2x) - 1\right) = 0$$

$$\Leftrightarrow \cos x\cos 2x(2\cos 2x + 1) = 0.$$

Hence, $\cos x = 0$, or $\cos 2x = 0$, or $\cos 2x = -\frac{1}{2}$.

We obtain the solutions $x \in \left\{\pm\frac{\pi}{2} + 2k\pi \middle| k \in Z\right\} \cup \left\{\pm\frac{\pi}{4} + k\pi \middle| k \in Z\right\} \cup \left\{\pm\frac{\pi}{6} + k\pi \middle| k \in Z\right\}$.

The solutions is complete.

PP.20312. Solve in R the equation
$\sqrt{6 - 11x + 6x^2 - x^3} + \sqrt{12 - 19x + 8x^2 - x^3} = \sqrt{15 - 23x + 9x^2 - x^3}.$

Mihály Bencze

We have

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3),\ x^3-8x^2+9x-12=(x-1)(x-3)(x-4) \text{ and}$$

$$x^3-9x^2+23x-15=(x-1)(x-3)(x-5).$$

The condition of existence is

$$x\in\{(-\infty,1]\cup[2,3]\}\cap\{(-\infty,1)\cup[3,4]\}\cap\{(-\infty,1)\cup[3,5]\}=(-\infty,1]\cup\{3\}.$$

The equation is

$$\sqrt{-(x-1)(x-2)(x-3)}+\sqrt{-(x-1)(x-3)(x-4)}=\sqrt{-(x-1)(x-3)(x-5)},$$

so we obtain the solutions $x_1=1, x_2=3$ and remains to solve the equation

$$\sqrt{2-x}+\sqrt{4-x}=\sqrt{5-x}\Leftrightarrow 2\sqrt{(2-x)(4-x)}=x-1,$$

which has no solutions in condition $x<1$, and we are done.

PP.20314. Solve the equation
$x^6-21x^5+175x^4-735x^3+1624x^2-1764x+720=0.$

Mihály Bencze

The equation to solve is written as follows

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)=0,$$

so it has the solutions $x_1=1, x_2=2, x_3=3, x_4=4, x_5=5, x_6=6$. The solution is complete.

PP.20316. Solve in R the following system:
$\sqrt[4]{2-x}+\sqrt[4]{15+y}=\sqrt[4]{2-y}+\sqrt[4]{15+z}=\sqrt[4]{2-z}+\sqrt[4]{15+x}=3.$

Mihály Bencze

From given equation, i.e.

$$\sqrt[4]{2-x}+\sqrt[4]{15+y}=\sqrt[4]{2-y}+\sqrt[4]{15+z}=\sqrt[4]{2-z}+\sqrt[4]{15+x}=3,$$

we have

$$\sqrt[4]{2-x}=3-\sqrt[4]{15+y},$$

which yields that $x=2-\left(3-\sqrt[4]{15+y}\right)^4$.

Denoting $f(t)=2-\left(3-\sqrt[4]{15+t}\right)^4$ we obtain that $x=f(y), y=f(z), z=f(x)$.

So $x=(f\circ f\circ f)(x)$, and because the function f is increasing yields that the fixed points of the function $f\circ f\circ f$ is the same with the fixed points of the function f.

Hence, we have to solve the equation

$$x=2-\left(3-\sqrt[4]{15+x}\right)^4\Leftrightarrow\left(3-\sqrt[4]{15+x}\right)^4=2-x,$$

and denoting $x=t^4-15$, then

$(3-t)^4 = 17 - t^4 \Leftrightarrow t^4 - 6t^3 + 27t^2 - 54t + 32 = 0$
$\Leftrightarrow (t-1)(t-2)(t^2 - 3t + 16) = 0$.
Hence $t_1 = 1, t_2 = 2$. Finally we get $x = y = z = 1$ and $x = y = z = -14$.
The solution is complete.

PP.20319. Solve in R the equation
$x^4 + 3 + 2 \cdot 3^{x^4-5x^2+3} + 3 \cdot 9^{x^4-5x^2+3} = 5x^2 + \frac{x(2x^2+3x+4)}{(x^2+2)^2} + \log_3 \frac{x}{x^2+2}$.

Mihály Bencze

Considering the injective function $f(t) = t + 2 \cdot 3^t + 3 \cdot 3^{2t}$, $t > 0$, the equation

$$x^4 + 3 + 2 \cdot 3^{x^4-5x^2+3} + 3 \cdot 9^{x^4-5x^2+3} = 5x^2 + \frac{x(2x^2+3x+4)}{(x^2+2)^2} + \log_3 \frac{x}{x^2+2},$$

is written

$$f\left(\log_3 \frac{x}{x^2+2}\right) = f(x^4 - 5x^2 + 3).$$

Therefore, it remains to solve the equation

$$\log_3 \frac{x}{x^2+2} = x^4 - 5x^2 + 3, \ x > 0.$$

If $x \in (1,2)$, then $x^4 - 5x^2 + 3 < -1$ and $\frac{x}{x^2+2} > \frac{1}{3} \Leftrightarrow \log_3 \frac{x}{x^2+2} > -1$;

If $x \in (0,1) \cup (2,\infty)$, then $x^4 - 5x^2 + 3 > -1$ and $\frac{x}{x^2+2} < \frac{1}{3} \Leftrightarrow \log_3 \frac{x}{x^2+2} < -1$.

Yields that the only possible $x_1 = 1$ and $x_2 = 2$ which are indeed solutions.
This completes the proof.

PP.20321. Solve in R the equation $2x^4 - 9x^2 - 3x + 10 = \log_2 \frac{3x}{x^2+2}$.

Mihály Bencze

For to solve the equation

$$2x^4 - 9x^2 - 3x + 10 = \log_2 \frac{3x}{x^2+2},$$

we consider the injective function $f(t) = \log_2 t + 2t + 2t^2, t > 0$.
Then the equation to solve is written as

$$f(x^2+2) = f(3x).$$

So, $x^2 - 3x + 2 = 0$, and we get the solutions $x_1 = 1, x_2 = 2$.
The proof is complete.

PP.20322. Solve in R the equation
$\log_6 \frac{x^3+11x}{x^2+1} = 34 - 22x - 243x^2 - 2x^3 + 42x^4 - 3x^6$.

Mihály Bencze

The given equation is equivalent with

$$\log_6 \frac{x^3+11x}{x^2+1} = 121 - 22x - 135x^2 - 2x^3 + 42x^4 - 3x^6.$$

Considering the injective function

$$f(t) = \log_6 t + 2t + 3t^2, t > 0,$$

the equation to solve is written as

$$f(x^3+11x) = f(6x^2+6).$$

So we must to solve the equation $x^3 + 11x = 6x^2 + 6$.
We get the solutions $x_1 = 1, x_2 = 2, x_3 = 3$. The proof is complete.

PP.20323. Solve in R the equation
$x^2 + 7 + \log_2 \frac{x^2-5x+8}{\sqrt{x-2}} + \log_3 \frac{x^2-5x+8}{2\sqrt{x-2}} = 5x + 2\sqrt{x-2}$.

Mihály Bencze

Considering the injective function $f(t) = \log_2 t + \log_3 t + t$, the equation

$$x^2 + 7 + \log_2 \frac{x^2-5x+8}{\sqrt{x-2}} + \log_3 \frac{x^2-5x+8}{2\sqrt{x-2}} = 5x + 2\sqrt{x-2},$$

becomes

$$f(x^2-5x+8) = f(2\sqrt{x-2}).$$

So we must to solve the equation

$$x^2 - 5x + 8 = 2\sqrt{x-2},$$

which by squaring becomes

$$x^4 - 10x^3 + 41x^2 - 84x + 72 = 0 \Leftrightarrow (x-3)^2(x^2-4x+8) = 0.$$

Hence $x = 3$, and we are done.

PP.20325. Solve in R the equation
$3x^3 + \log_2\left(x^2 + \frac{1}{3x}\right) + \log_3\left(x^2 + \frac{1}{3x}\right) + 1 = 3x$.

Mihály Bencze

Considering the injective function $f(t) = \log_2 t + \log_3 t + t$, the equation

$$3x^3 + \log_2\left(x^2 + \frac{1}{3x}\right) + \log_3\left(x^2 + \frac{1}{3x}\right) + 1 = 3x,$$

Becomes

$$f(3x^3+1)=f(3x).$$

So it remains to solve the equation $3x^3+1=3x$, which has all roots real for e.g. $x_1\in\left(0,\frac{1}{2}\right), x_2\in\left(\frac{1}{2},1\right)$. So, you find the solutions using Cardano' s formulas.

PP.20333. Prove that $\prod_{k=1}^{n} k! < 3^{\frac{n(n+1)(2n+1)}{24}}$.

Mihály Bencze

We prove the given inequality, i.e.

$$\prod_{k=1}^{n} k! < 3^{\frac{n(n+1)(2n+1)}{24}},$$

by mathematical induction (MI).

First, is easily demonstrated by (MI) that

$$(1)\ 3^{2k+1} > (k+1)^4.$$

Second we prove by (MI) that

$$(2)\ k! < 3^{\frac{k^2}{4}}.$$

For $k=1,2$ it is verify that (2) is true.

We suppose that $k! < 3^{\frac{k^2}{4}}$ is true and by (1) yields that

$$(k+1)! = k!(k+1) < 3^{\frac{k^2}{4}}(k+1) < 3^{\frac{k^2}{4}}\cdot 3^{\frac{2k+1}{4}} = 3^{\frac{(k+1)^2}{4}}.$$

Using (2) and the identity $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, we obtain what we must to prove.

The proof is complete.

PP.20351. In all triangle ABC holds

$$4r^2(2R-r)+3s^2(2R+r)\le(4R+r)^3.$$

Mihály Bencze

Using the inequality $3s^2\le(4R+r)^2$ (see for e.g. the item 5.5. from Bottema) to show that

$$4r^2(2R-r)+3s^2(2R+r)\le(4R+r)^3,$$

it suffices to prove

$$4r^2(2R-r)+(4R+r)^2(2R+r)\le(4R+r)^3$$

$\Leftrightarrow 4r^2(2R-r)\le(4R+r)^2(4R+r-2R-r)\Leftrightarrow 16R^3+8R^2r+2r^3\ge 3Rr^2$,
which is true because $8R^2r\ge 16Rr^2\Leftrightarrow R\ge 2r$. The proof is complete.

PP.20353. In all triangle ABC holds $\left(s^2+r^2+10Rr\right)(4R+r)\le 8Rs^2$.

Mihály Bencze

Since $s^2+r^2+4Rr=\sum ab$ and $Rr=\frac{abc}{2\sum a}$ the given inequality becomes

$$\left(\sum ab+6\cdot\frac{abc}{2\sum a}\right)(4R+r)\le 2R\left(\sum a\right)^2$$

$$\Leftrightarrow 2R\left[\left(\sum a\right)^2-2\sum ab-\frac{6abc}{\sum a}\right]\ge r\left(\sum ab+\frac{3abc}{\sum a}\right)$$

$\Leftrightarrow \frac{R}{2r}\ge\frac{1}{4}\cdot\frac{\sum a\sum ab+3abc}{\left(\sum a\right)^3-2\sum a\sum ab-6abc}$, and because $R\ge 2r$, it is enough to prove that

$$4\left(\sum a\right)^3-8\sum a\sum ab-24abc\ge\sum a\sum ab+3abc$$

$$\Leftrightarrow 4\left(\sum a\right)^3-9\sum a\sum ab\ge 27abc$$

$$\Leftrightarrow 4\sum a^3+12\sum a^2b+12\sum ab^2+24abc-9\sum a^2b-9\sum ab^2-27abc\ge 27abc$$

$\Leftrightarrow 4\sum a^3+3\sum a^2b+3\sum ab^2\ge 30abc$, which is true because
$\sum a^3\ge 3abc$, $\sum a^2b\ge 3abc$, $\sum ab^2\ge 3abc$, and we are done.

PP.20356. Solve in N the following system: $\begin{cases}2^x=2+y!\\2^y=2+z!\\2^z=2+x!\end{cases}$.

Mihály Bencze

If $y>3$, then $y!=M_4$, so $y!+2=M_4+2$, and from the first equation we get $x=1$, then $y!=0$, contradiction. Therefore $y\le 3$. We obtain the solutions $(x,y,z)\in\{(2,2,2);(3,3,3)\}$, and we are done.

PP.20381. In all triangle ABC holds $\sum\frac{ab}{m_a^2m_b^2}\le\frac{s^2+r^2-8Rr}{s^2r^2}$.

Mihály Bencze

We have $m_a^2\ge s(s-a)\Leftrightarrow 2b^2+2c^2-a^2\ge(b+c)^2-a^2\Leftrightarrow(b-c)^2\ge 0$.

Denoting by F the are of triangle ABC, and using the well-known formulas

$$F = sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R}, \sum ab = s^2 + r^2 + 4Rr,$$

we obtain

$$\sum \frac{ab}{m_a^2 m_b^2} \le \sum \frac{ab}{s(s-a)s(s-b)} = \sum \frac{ab(s-c)}{s \cdot s \cdot (s-a)(s-b)(s-c)} = \sum \frac{ab(s-c)}{s \cdot F^2} =$$

$$= \frac{1}{F^2} \sum \left(ab - \frac{abc}{s} \right) = \frac{1}{F^2} \left(s^2 + r^2 + 4Rr - \frac{3abc}{s} \right) =$$

$$= \frac{1}{F^2} \left(s^2 + r^2 + 4Rr - \frac{12Rsr}{s} \right) = \frac{s^2 + r^2 - 8Rr}{s^2 r^2}, \text{ q.e.d.}$$

PP.20387. Solve in R the following system:

$$\begin{cases} x^4 + 41y^2 + 72 = 10z^3 + 84x \\ y^4 + 41z^2 + 72 = 10x^3 + 84y \\ x^4 + 41x^2 + 72 = 10y^3 + 84z \end{cases}.$$

Mihály Bencze

Adding the equations of the system we obtain

$$(x-3)^2(x^2 - 4x + 8) + (y-3)^2(y^2 - 4y + 8) + (z-3)^2(z^2 - 4z + 8) = 0,$$

and because

$$x^2 - 4x + 8 = (x-2)^2 + 4 > 0,$$

we have the solution $x = y = z = 3$. The solution is complete.

PP.20392. Solve in R the following system: $\begin{cases} x^2 - 9y + 22 = 2\sqrt{z-4} \\ y^2 - 9z + 22 = 2\sqrt{x-4} \\ z^2 - 9x + 22 = 2\sqrt{y-4} \end{cases}$.

Mihály Bencze

We prove first that for any $a \in R$ we have $a^4 - a^2 - 2a + 2 \ge 0$. Indeed,

$$a^4 - a^2 - 2a + 2 = a^2(a-1)(a+1) - 2(a-1) =$$

$$= (a-1)(a^3 + a^2 - 2) = (a-1)^2(a^2 + 2a + 2) \ge 0,$$

with equality for $a = 1$.

Adding the equations of the system we obtain

$$\sum (x^2 - 9x + 22 - 2\sqrt{x-4}) = 0 \Leftrightarrow \sum [x^2 - 8x + 16 - x + 4 - 2\sqrt{x-4} + 2] = 0$$

$\Leftrightarrow \sum[(x-4)^2-(x-4)-2\sqrt{x-4}+2]=0$

$\Leftrightarrow \sum\left((\sqrt{x-4})^4-(\sqrt{x-4})^2-2\sqrt{x-4}+2\right)=0$.

Therefore, by above we deduce that

$$\sqrt{x-4}=\sqrt{y-4}=\sqrt{z-4}=1.$$

Hence $x=y=z=5$. The solution is complete.

PP.20394. Solve in R the following equation
$\log_3\left(\frac{x}{6}+\frac{3}{2x}\right)=-819+6x-226x^2+216x^3-28x^4-x^6$.

Mihály Bencze

If we consider the injective function

$$f(t)=\log_3 t+t^3+t^2+t,$$

the equation

$$\log_3\left(\frac{x}{6}+\frac{3}{2x}\right)=-819+6x-226x^2+216x^3-28x^4-x^6,$$

it is written in the form

$f(x^2+9)=f(6x)$. So, $x^2+9=6x$, i.e. $x=3$, and we are done.

PP.20395. Solve in R the following equation
$2^{\cos^2 x}-2^{\sin^2 x}+\left(3-\sin^2 x\cos^2 x\right)\cos 2x=0$.

Mihály Bencze

We have

$(3-\sin^2 x\cos^2 x)\cos 2x=\left(3(\sin^2 x+\cos^2 x)^2-\sin^2 x\cos^2 x\right)\cos 2x$

$=(3\sin^4 x+3\cos^4 x+5\sin^2 x\cos^2 x)(\cos^2 x-\sin^2 x)$

$=3\cos^6 x-3\sin^6 x-2\sin^4 x\cos^2 x+2\sin^2 x\cos^4 x$

$=3\cos^6 x+2\cos^4 x(1-\sin^2 x)-3\sin^6 x-2\sin^4 x(1-\sin^2 x)$

$=\cos^6 x+2\cos^4 x-\sin^6 x-2\sin^4 x$.

If we consider the injective function $f(t)=2^t+t^3+2t^2$, then the given equation, i.e.

$$2^{\cos^2 x}-2^{\sin^2 x}+(3-\sin^2 x\cos^2 x)\cos 2x=0,$$

becomes $f(\cos x)=f(\sin x)$.

Therefore, $\cos x=\sin x$, which yields $x=\frac{\pi}{4}+k\pi$, $k\in Z$.

PP.20399. Solve in R the following equation
$(x-6)\sqrt{x-7}=-12+13x-x^2$.

Mihály Bencze

The condition of existence is LHS $\geq 0 \Leftrightarrow x\geq 7$, and also RHS $\geq 0 \Leftrightarrow x\in[1,12]$. Because 7 and 12 no verify the given equation the solutions is in interval $(7,12)$. Squaring the equation we deduce

$$x^4-27x^3+212x^2-432x+396=0$$

$$\Leftrightarrow (x-11)(x^3-16x^2+36x-36)=0,$$

and because

$$x^3-16x^2+36x-36=(x-12)(x+2)(x-6)-180<0, \text{ for any } x\in(6,12),$$

we obtain $x=11$. The solution is complete.

PP.20403. Solve in R the following system: $\begin{cases} 1+tg\left(x+\frac{\pi}{4}\right)=3ctgy \\ 1+tg\left(y+\frac{\pi}{4}\right)=3ctgz \\ 1+tg\left(z+\frac{\pi}{4}\right)=3ctgx \end{cases}$.

Mihály Bencze

Denoting $a=tgx$, $b=tgy$, $c=tgz$ the given system becomes

$$\begin{cases} 1+\dfrac{a+1}{1-a}=\dfrac{3}{b} \\ 1+\dfrac{b+1}{1-b}=\dfrac{3}{c} \\ 1+\dfrac{c+1}{1-c}=\dfrac{3}{a} \end{cases} \Leftrightarrow \begin{cases} b=\dfrac{3(1-a)}{2} \\ c=\dfrac{3(1-b)}{2} \\ a=\dfrac{3(1-c)}{2} \end{cases}.$$

We obtain $c=\dfrac{3\left(1-\dfrac{3-3a}{2}\right)}{2}=\dfrac{3(3a-1)}{4}$, which yields

$$a=\dfrac{3\left(1-\dfrac{9a-3}{4}\right)}{2}\Leftrightarrow a=\dfrac{3}{5}.$$

Therefore, the system has the solution $x=y=z=arctg\dfrac{3}{5}$. We are done.

PP.20407. Solve in R the following system
$\sqrt[4]{1-x}+\sqrt[4]{16+y}=\sqrt[4]{1-y}+\sqrt[4]{16+z}=\sqrt[4]{1-z}+\sqrt[4]{16+x}=3.$

Mihály Bencze

If we make the substitutions $x\to x-1, y\to y-1, z\to z-1$, we obtain the system from the problem PP.20316. Proceed as there we obtain the solutions
$x=y=z=0$, $x=y=z=-15$. The solution is complete.

PP.20412. Solve the equation
$(x^2-6x+5)^5+(x^2-9x+14)^5=(2x^2-15x+19)^5$.

Mihály Bencze

Denoting $x^2-6x+5=a$ and $x^2-9x+14=b$, the given ecuation, i.e.

$$(x^2-6x+5)^5+(x^2-9x+14)^5=(2x^2-15x+19)^5 ,$$

becomes successively

$$a^5+b^5=(a+b)^5 \Leftrightarrow a^5+b^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

$$\Leftrightarrow 5ab(a^3+b^3+2a^2b+2ab^2)=0 \Leftrightarrow 5ab(a+b)(a^2-ab+b^2+2ab)=0$$

$$\Leftrightarrow 5ab(a+b)\left[\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}\right]=0.$$

Square brackets is canceled if $a=b=0$, what does not happen.
Hence $(x^2-6x+5)(x^2-9x+14)(2x^2-15x+19)=0$, so we obtain solutions
$x\in\left\{1,5,2,7,\frac{15-3\sqrt{7}}{4},\frac{15+3\sqrt{7}}{4}\right\}$, and we are done.

PP.20430. Solve the equation
$(1+\sin 2x)(1+\sin x+\cos x+\sin x\cos x)=8\left(5+2\sqrt{6}\right)$.

Mihály Bencze

The given equation has no solution, because LHS$\le(1+1)(1+1+1+1)=8$.

PP.20453. Solve in R the equation $9^x+15^x+25^x+3\cdot 49^x=3\left(21^x+35^x\right)$.

Mihály Bencze

The equation $9^x+15^x+25^x+3\cdot 49^x=3(21^x+35^x)$ is writing $(3^x-5^x)^2+3(7^x-3^x)(7^x-5^x)=0$, and because 7^x-3^x respectively 7^x-5^x have the same sign, follows that $(7^x-3^x)(7^x-5^x)$ is positive. Therefore, we get the only one solution, $x=0$.

PP.20454. Solve in R the equation
$x^5+6^x+\log_6\frac{x^5+6^x}{x^2\cdot 3^x+x^3\cdot 2^x}=x^2\left(3^x+x\cdot 2^x\right).$

Mihály Bencze

Writting the equation on the form

$$x^5+6^x+\log_6(x^5+6x)=x^2\cdot 3^x+x^3\cdot 2^x+\log_6(x^2\cdot 3^x+x^3\cdot 2^x),$$

and using the fact that the function $f(t)=\log_6 t+t$, we obtain

$$x^5+6^x=x^2\cdot 3^x+x^3\cdot 2^x \Leftrightarrow (2^x-x^2)(3^x-x^3)=0.$$

Hence we obtain the solutions
$x_1=2, x_2=3, x_3=4$. The solution is complete.

PP.20469. Let ABC be a triangle. If $\lambda\geq 1$ then $\sum\left(tg\frac{A}{4}\right)^\lambda\geq 3\left(2-\sqrt{3}\right)^\lambda.$

Mihály Bencze

Let $f:(0,\pi)\to R_+^*,\ f(x)=\left(tg\frac{x}{4}\right)^\lambda$, convex on $(0,\pi)$, and then by Jensen's inequality we have

$$f(A)+f(B)+f(C)\geq 3f\left(\frac{A+B+C}{3}\right)\Rightarrow tg^\lambda\frac{A}{4}+tg^\lambda\frac{B}{4}+tg^\lambda\frac{C}{4}\geq 4\cdot tg^\lambda\frac{A+B+C}{12}=$$

$$=3\cdot tg^\lambda\frac{\pi}{12}=3(2-\sqrt{3})^\lambda, \text{ q.e.d.}$$

PP.20479. In all triangle ABC holds $s^2\geq 7r^2+14Rr$.

Mihály Bencze

The given inequality is not true in all triangle, for e.g. if we take an equilateral triangle with the length side 1, then $s=\frac{3}{2}, R=\frac{\sqrt{3}}{3}, r=\frac{\sqrt{3}}{6}$, and we have

$$s^2\geq 7r^2+14Rr \Leftrightarrow \frac{9}{4}\geq\frac{7}{12}+\frac{14}{6}\Leftrightarrow 27\geq 7+28,$$ false, and this completes the proof.

PP.20524. If $a, b > 0$ then $\frac{2\left(a^2+2ab\right)}{9a^2+(a+2b)^2} + \frac{b}{2(b+2a)} \le \frac{1}{2}$.

Mihály Bencze

The inequality to prove is written

$$\frac{1}{3} - \frac{2(a^2+2ab)}{9a^2+(a+2b)^2} + \frac{1}{6} - \frac{b}{2(b+2a)} \ge 0$$

$$\Leftrightarrow \frac{4(a-b)^2}{9a^2+(a+2b)^2} + \frac{a-b}{b+2a} \ge 0 \Leftrightarrow (a-b)[4ab+8a^2-4b^2-8ab+10a^2+4ab+4b^2] \ge 0$$

$\Leftrightarrow (a-b)18a^2 \ge 0$, which is true with the condition $a \ge b > 0$. The solution is complete.

PP.20526. If $a, b > 0$ then $1 - \frac{1}{\sqrt{2}} \le \frac{a+b}{\sqrt{a^2+b^2}} \le \sqrt{2}$.

Mihály Bencze

The inequality from the statements is equivalent with

(i) $a+b \le \sqrt{2(a^2+b^2)}$ and (ii) $\left(\sqrt{a^2+b^2}\right)\left(\sqrt{2}-1\right) \le \sqrt{2}(a+b)$.

(i) $\Leftrightarrow a^2+2ab+b^2 \le 2(a^2+b^2) \Leftrightarrow a^2-2ab+b^2 \ge 0 \Leftrightarrow (a-b)^2 \ge 0$, true.

(ii) $(3-2\sqrt{2})(a^2+b^2) \le 2(a^2+2ab+b^2) \Leftrightarrow a^2+b^2 \le 2\sqrt{2}(a^2+b^2)+4ab$, true, and we are done.

PP.20530. If $a, b, c > 0$ then $\sum \sqrt{a(a+b)^3} + \sum b\sqrt{a^2+b^2} \le 3\sqrt{2}\sum a^2$.

Mihály Bencze

After multiply with $\sqrt{2}$ the inequality to prove becomes

$$\sum (a+b)\sqrt{(2a)(a+b)} + \sum \sqrt{(2b^2)(a^2+b^2)} \le 6\sum a^2 .$$

By AM-GM inequality we obtain that

$$\sum (a+b)\sqrt{(2a)(a+b)} + \sum \sqrt{(2b^2)(a^2+b^2)} \le \sum \frac{(a+b)(2a+a+b)}{2} +$$

$$+\sum \frac{2b^2+a^2+b^2}{2} = \frac{1}{2}\sum (3a^2+b^2+4ab+a^2+3b^2) = \frac{1}{2}\sum (4a^2+4b^2+4ab) =$$

$= \sum 2a^2 + \sum 2a^2 + 2\sum ab = 4\sum a^2 + 2\sum ab$, so it is suffices to prove that

$$4\sum a^2 + 2\sum ab \le 6\sum a^2 \Leftrightarrow \sum ab \le \sum a^2 \text{, which is true.}$$

The equality occurs iff $a=b=c$. Q.E.D.

PP.20531. In all triangle ABC holds $\sum \frac{a^2}{bc} \geq \frac{2(2R-r)}{R}$.

Mihály Bencze

Because $\sum \frac{a^2}{bc} = \frac{\sum a^3}{abc} = \frac{2s(s^2-3r^2-6Rr)}{4Rrs} = \frac{s^2-3r^2-6Rr}{2Rr}$,

then the given inequality, i.e. $\sum \frac{a^2}{bc} \geq \frac{2(2R-r)}{R}$, is equivalent with

$\frac{s^2-3r^2-6Rr}{2Rr} \geq \frac{4R-2r}{R} \Leftrightarrow s^2-3r^2-6Rr \geq 8Rr-4r^2 \Leftrightarrow s^2 \geq 14Rr-r^2$.

By Gerretsen inequality we have $s^2 \geq 16Rr-5r^2$, so it is suffices to show that $16Rr-5r^2 \geq 14Rr-r^2 \Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r$, i.e. Euler's inequality. Q.E.D.

PP.20564. Solve in R the equation $\sum_{k=1}^{n} (\cos kx)^2 = n$.

Mihály Bencze

Since, $\cos^2 kx \leq 1$ and, $\sum_{k=1}^{n} \cos^2 kx = n$, we deduce that

$\cos^2 kx = 1 \Leftrightarrow \cos kx = \pm 1$. Therefore, $kx \in \pi Z \Leftrightarrow x \in \frac{\pi}{k} Z \Leftrightarrow x = \frac{n\pi}{k}, k = \overline{1,n}, n \in Z$.

PP.20566. Solve in R the equation
$\left(\sqrt[5]{x+2} - \sqrt[5]{2x+1} + \sqrt[5]{4x+7}\right)^5 = 3x+8.$

Mihály Bencze

Denoting $x+2=a^5$, $2x+1=b^5$, $4x+7=c^5$, the given equation, i.e.
$\left(\sqrt[5]{x+2}-\sqrt[5]{2x+1}-\sqrt[5]{4x+7}\right)^5 = 3x+8$, is written successively
$(a-b+c)^5 = a^5-b^5+c^5 \Leftrightarrow (a-b+c)^5 - c^5 = a^5-b^5$
$\Leftrightarrow (a-b)[(a-b+c)^4 + c(a-b+c)^3 + c^2(a-b+c)^2 + c^3(a-b+c) + c^4 - a^4 - a^3b -$
$-a^2b^2 - ab^3 - b^4] = 0$. We obtain $a-b=0 \Leftrightarrow x+2 = 2x+1$, so $x_1 = 1$ and it remains to solve $[a-(b-c)]^4 + c[a-(b-c)^3] + c^2[a-(b-c)]^2 + c^3[a-(b-c)] + c^4 - a^4 -$
$-a^3b - a^2b^2 - ab^3 - b^4 = 0 \Leftrightarrow -4a^3(b-c) + 6a^2(b-c)^2 - 4a(b-c)^3 + (b-c)^4 +$

$+a^3c-3a^2c(b-c)+3ac(b-c)^2-c(b-c)^3+a^2c^2-2ac^2(b-c)+c^2(b-c)^2+ac^3-$
$-c^3(b-c)+c^4-a^3b-a^2b^2-ab^3-b^4=0$. Since,
$a^3c+a^2c^2+ac^3+c^4-a^3b-a^2b^2-ab^3-b^4=-(b-c)(b^3+b^2c+bc^2+c^3)-$
$-a^2(b-c)(b+c)-a^3(b-c)$, we obtain $b=c \Leftrightarrow 2x+1=4x+7$, which yields $x_2=3$,
and it remains $4a^3-6a^2b+6a^2c+4ab^2-8abc+4ac^2-b^3+3b^2c-3bc^2+c^3+3a^2c-$
$-3abc+3ac^2+b^2c-2bc^2+c^3+2ac^2-bc^2+c^3+c^3+b^3+b^2c+bc^2+c^3+ab^2+$
$+abc+ac^2+a^2b+a^2c+a^3=0$
$\Leftrightarrow 5a^3+5c^3+10a^2c+10ac^2-5a^2b-10abc-5bc^2+5ab^2+5b^2c=0$
$\Leftrightarrow 5(a+c)(a^2-ac+c^2+2ac-ab-bc+b^2)=0$ (1)

Hence, $a=-c \Leftrightarrow x+2+4x+7=0$, so we obtain $x_3=-\frac{9}{5}$.

The second bracket from (1) is written $(a-b)^2+(b-c)^2+(a+c)^2=0$, so $a=b, b=c$ and $a=-c$, i.e. $a=b=c=0$, which does not exists.

Therefore, we obtain $x\in\left\{1,3,-\frac{9}{5}\right\}$, and the solution is complete.

PP.20568. Solve in R the equation
$512\sin^9 x+54\sin 3x+57\sqrt{3}=216\sin x+54\sqrt{3}\cos 2x.$

Mihály Bencze

Using the well-known formulas $\cos 2x=1-2\sin^2 x$ and respectively $\sin 3x=3\sin x-4\sin^3 x$, then the given equation is written as follows

$(8\sin^3 x)^3=(6\sin x-\sqrt{3})^3$. So, $8\sin^3 x=6\sin x-\sqrt{3} \Leftrightarrow \sin 3x=\frac{\sqrt{3}}{2}$.

Hence, $x=(-1)^k\frac{\pi}{9}+\frac{k\pi}{3}$, where k is positive integer. We are done.

PP.20574. Solve in R the following system:

$$\begin{cases}\sqrt{x+1}+\sqrt{y^2+4y+7}=\sqrt{z+1}\\ \sqrt{y+1}+\sqrt{z^2+4z+7}=\sqrt{x+1}\\ \sqrt{z+1}+\sqrt{x^2+4x+7}=\sqrt{y+1}\end{cases}.$$

Mihály Bencze

Adding up the equations of the system to solve we obtain

$$\sqrt{x^2+4x+7}+\sqrt{y^2+4y+7}+\sqrt{z^2+4z+7}=0$$

$$\Leftrightarrow \sqrt{(x+2)^2+3}+\sqrt{(y+2)^2+3}+\sqrt{(z+3)^2+7}=0,$$

which doesn't solutions in real numbers, so the system doesn't solutions in the set of real numbers. The solution is complete.

PP.20590. Solve in R the following system:

$$\begin{cases} 2\left(\sqrt{2}x+2y\right)=x^2+z^2+6 \\ 2\left(\sqrt{2}y+2z\right)=y^2+x^2+6 \\ 2\left(\sqrt{2}z+2x\right)=z^2+y^2+6 \end{cases}.$$

Mihály Bencze

Adding up the equations of the system to solve we obtain $\sum 2x^2+18=2\sqrt{2}\sum x+4\sum x \Leftrightarrow \sum\left(x^2-(\sqrt{2}+2)x+3\right)=0$. Because the equation $x^2-(\sqrt{2}+2)x+3=0$, has the discriminate negative, we get that the system doesn't real solutions, and we are done.

PP.20601. Compute $\int\limits_0^{\frac{\pi}{2}} \frac{(\sin^3 x+\sin^2 x+\sin x)(\sin x+\cos x)}{2-\sin x\cos x}dx$.

Mihály Bencze

Let $I=\int\limits_0^{\frac{\pi}{2}} \frac{(\sin^3 x+\sin^2 x+\sin x)(\sin x+\cos x)}{2-\sin x\cos x}dx$, where we make the changes of variable $x=\frac{\pi}{2}-t$, and we deduce that

$I=\int\limits_0^{\frac{\pi}{2}} \frac{(\cos^3 x+\cos^2 x+\cos x)(\sin x+\cos x)}{2-\sin x\cos x}dx$. Therefore,

$$2I=\int\limits_0^{\frac{\pi}{2}} \frac{(\sin^3 x+\cos^3 x+\sin^2 x+\cos^2 x+\sin x+\cos x)(\sin x+\cos x)}{2-\sin x\cos x}dx=$$

$$=\int\limits_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)^2(\sin^2 x+\cos^2 x-\sin x\cos x)}{2-\sin x\cos x}dx+\int\limits_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{2-\sin x\cos x}dx=$$

$$=\int_0^{\frac{\pi}{2}}(\sin x+\cos x)^2dx+\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{2-\sin x\cos x}dx=\int_0^{\frac{\pi}{2}}(1+2\sin x\cos x)dx+\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{2-\sin x\cos x}dx=$$

$$=x\Big|_0^{\frac{\pi}{2}}+\sin^2 x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{2-\sin x\cos x}dx=\frac{\pi}{2}+1+J \quad (1)$$

Where $J=\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{2-\sin x\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{2(\sin x+\cos x)}{4-2\sin x\cos x}dx=2\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{3+(\sin x-\cos x)^2}dx$,
and we make the change of variable

$t=u(x)=\sin x-\cos x, u'(x)=\cos x+\sin x, u(0)=-1, u\left(\frac{\pi}{2}\right)=1$, so $J=2\int_{-1}^{1}\frac{t}{3+t^2}dt=0$,

which yields that $2I=\frac{\pi}{2}+1$. Hence, $I=\frac{1}{2}+\frac{\pi}{4}$.

PP.20608. In all triangle ABC holds $\sum\frac{1}{(a^2+bc)(b^2+ca)}\le\frac{3s^2-r^2-4Rr}{8s^2Rr(s^2+r^2+2Rr)}$.

Mihály Bencze

Denoting with F the area of triangle ABC, and using the well-known formula $\sum ab=s^2+r^2+4Rr$, we obtain

$$3s^2-r^2-4Rr=2s^2+s^2-r^2-4Rr=\frac{\left(\sum a\right)^2}{2}+\frac{4s^2-2(s^2+r^2+4Rr)}{2}$$

$$=\frac{\left(\sum a\right)^2}{2}+\frac{\left(\sum a\right)^2-2\sum ab}{2}=\frac{\left(\sum a^2\right)+\sum a^2}{2}=\sum a^2+\sum ab.$$

$$8s^2Rr(s^2+r^2+2Rr)=8s^2\cdot\frac{abc}{4F}\cdot\frac{F}{s}(s^2+r^2+4Rr-2Rr)=$$

$$=2sabc\left(\sum ab-\frac{abc}{2s}\right)=abc\sum a\sum ab-a^2b^2c^2$$

$$=abc\sum a^2b+abc\sum ab^2+2a^2b^2c^2.$$

Since,

$$\sum\frac{1}{(a^2+bc)(b^2+ca)}=\frac{\sum a^2+\sum ab}{2a^2b^2c^2+\sum a^3b^3+abc\sum a^3},$$

then the inequality to prove, i.e.

$$\sum\frac{1}{(a^2+bc)(b^2+ca)}\le\frac{3s^2-r^2-4Rr}{8s^2Rr(s^2+r^2+2Rr)},$$

becomes

$$\frac{\sum a^2+\sum ab}{abc\sum a^2b+abc\sum ab^2+2a^2b^2c^2}\geq\frac{\sum a^2+\sum ab}{2a^2b^2c^2+\sum a^3b^3+abc\sum a^3}$$

$\Leftrightarrow abc\sum a^3+\sum a^3b^3\geq abc\sum a^2b+abc\sum ab^2$

$\Leftrightarrow abc\left(\sum a^3+3abc-\sum a^2b-\sum ab^2\right)+\left(\sum a^3b^3-3a^2b^2c^2\right)\geq 0$, which is true, because the first bracket is positive by Schur inequality and the second bracket is also positive by AM-GM inequality.

PP.20611. In all triangle ABC holds $\sum\frac{a(a^{n+1}-1)}{(a-1)(b+c-a)}\geq$
$\geq\frac{a^{n+1}-1}{a-1}+\frac{b^{n+1}-1}{b-1}+\frac{c^{n+1}-1}{c-1}$ when $a,b,c\in(0,1)\cup(1,+\infty)$ and $n\in N$.

Mihály Bencze

Denoting $x=\frac{a^{n+1}-1}{a-1}, y=\frac{b^{n+1}-1}{b-1}, z=\frac{c^{n=1}-1}{c-1}$ we have $x+y>0$ and $x-y=a^n-b^n+a^{n-1}-b^{n-1}+...+a-b=(a-b)E_{ab}$, with $E_{ab}>0$.

We have

$$\sum\frac{ax}{b+c-a}-\sum x=\sum\frac{x(a-b-c+a)}{b+c-a}=\sum\left(\frac{x(a-b)}{b+c-a}+\frac{x(a-c)}{b+c-a}\right)=$$

$$=\sum\frac{x(a-b)}{b+c-a}+\sum\frac{y(b-a)}{c+a-b}=\sum(a-b)\left(\frac{x}{b+c-a}-\frac{y}{c+a-b}\right)=$$

$$=\sum(a-b)\cdot\frac{x(a-b)+y(a-b)+c(x-y)}{(b+c-a)(c+a-b)}=\sum\frac{(a-b)^2(x+y)+c(a-b)^2E_{ab}}{(b+c-a)(c+a-b)}\geq 0.$$

We have equality if and only if $a=b=c$. The solution is complete.

PP.20618. The sides of triangle ABC are natural numbers. Prove that $\sqrt{\sum\frac{(b+c)^2w_a^2}{bc}}\in N$.

Mihály Bencze

Since, $w_a=\frac{2bc\cos\frac{A}{2}}{b+c}$ and $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$, we obtain that

$$\sum\frac{(b+c)^2w_a^2}{bc}=\sum\frac{(b+c)^2}{bc}\cdot\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=4s\sum(s-a)=4s^2.$$

Hence the statement is true in a weaker condition, i.e. the perimeter of given triangle to be natural number. The solution is complete.

PP.20620. Let ABC be a triangle. Prove that the inequalities $\sum am_a^2 \geq 9sR$ and $s^2+5r^2 \geq 16Rr$ are equivalent.

Mihály Bencze

By usual formulas we obtain

$$4\sum am_a^2 = \sum a(2b^2+2c^2-a^2) = 2\sum a^2b + 2\sum ab^2 + 4abc - 4abc - \sum a^3$$

$$= 2(a+b)(b+c)(c+a) - 16Rrs - \sum a^3 = 4s(s^2+r^2+2Rr) - 16Rrs - 2s(s^2-3r^2-6Rr)$$

$= 2s(s^2+5r^2+2Rr)$, and then we have that

$\sum am_a^2 \geq 9srR \Leftrightarrow 2s(s^2+5r^2+2Rr) \geq 36srR \Leftrightarrow s^2+5r^2 \geq 16Rr$, and we are done.

The inequality $s^2+5r^2 \geq 16Rr$, is well-known inequality of Gerretsen. The solution is complete.

PP.20621. In all triangle ABC holds $\sum a\cos 2A \geq 2s\left(\frac{r}{R}-1\right)$.

Mihály Bencze

Since $\sum a^3 = 2s(s^2-3r^2-6Rr)$, the inequality to prove is written successively

$$\sum a(1-2\sin^2 A) \geq 2s\left(\frac{r}{R}-1\right) \Leftrightarrow 2s - 2\sum \frac{a^3}{4R^2} \geq 2s\left(\frac{r}{R}-1\right)$$

$$\Leftrightarrow 1 - \frac{s^2-3r^2-6Rr}{2R^2} \geq \frac{r}{R} - 1 \Leftrightarrow 4R^2 - s^2 + 3r^2 + 6Rr \geq 2Rr$$

$\Leftrightarrow s^2 \leq 4R^2+3r^2+4Rr$, which is true (see for e.g. the item 5.8. from Bottema). The solution is complete.

PP.20623. Solve in R the following system: $\begin{cases} (x+y-1)^2+1=2yz \\ (y+z-1)^2+1=2zx \\ (z+x-1)^2+1=2xy \end{cases}$.

Mihály Bencze

Adding up the equations of system we obtain

$$2x^2+2y^2+2z^2+2xy+2yz+2zx-4x-4y-4z+6 = 2xy+2yz+2zx$$

$$\Leftrightarrow x^2+y^2+z^2-2x-2y-2z+3=0 \Leftrightarrow (x-1)^2+(y-1)^2+(z-1)^2=0.$$

Hence $x=y=z=1$, and we are done.

PP.20633. If $a_k > 1$ $(k = 1, 2, ..., n)$, then
$\prod_{cyclic} \left(\frac{n-1}{a_2^{n-1}+a_3^{n-1}+...+a_n^{n-1}}\right)^{\log_{a_1} \sqrt[n]{n-1}} \leq \frac{1}{(n-1)^{n-1}}$.

Mihály Bencze

By AM-GM inequality we have

$$\sum_{k=1}^{n} a_k^3 + 3\sum_{cyclic}\frac{1}{a_1+a_2+a_3} \geq 4\cdot\sqrt[4]{\left(\sum_{k=1}^{n} a_k^3\right)\left(\sum_{cyclic}\frac{1}{a_1+a_2+a_3}\right)^3} \quad (1)$$

By J. Radon's inequality we have

$$\sum_{k=1}^{n} a_k^3 \geq \frac{\left(\sum_{k=1}^{n} a_k\right)^3}{n^2} \quad (2)$$

Also by H. Bergström's inequality we have

$$\sum_{cyclic}\frac{1}{a_1+a_2+a_3} \geq \frac{n^2}{\sum_{cyclic}(a_1+a_2+a_3)} = \frac{n^2}{3\sum_{k=1}^{n} a_k} \quad (3)$$

By (1), (2) and (3) we obtain

$$\sum_{k=1}^{n} a_k^3 + 3\sum_{cyclic}\frac{1}{a_1+a_2+a_3} \geq 4\cdot\sqrt[4]{\frac{\left(\sum_{k=1}^{n} a_k\right)^3}{n^2}\cdot\frac{n^6}{3^3\left(\sum_{k=1}^{n} a_k\right)^3}} =$$

$$= 4\cdot\sqrt[4]{\frac{n^4}{3^3}} = \frac{4n}{3}\sqrt[4]{3}, \text{ q.e.d.}$$

PP.20636. If $x, y, z > 0$ then $\sum \frac{x}{\sqrt{(x+2y)(x+2z)}} \geq 1$.

Mihály Bencze

By AM-GM inequality we have

$$\sqrt{(x+2y)(x+2z)} \leq \frac{x+2y+x+2z}{2} = x+y+z, \text{ so}$$

$$\sum\frac{x}{\sqrt{(x+2y)(x+2z)}} \geq \sum\frac{x}{x+y+z} = 1, \text{ and we are done.}$$

PP.20637. If $x, y, z > 0$ then $\sum \frac{x((x^2-yz))}{\sqrt{(x+2y)(x+2z)}} \geq \sum x^2 - \sum xy$.

Mihály Bencze

By AM-GM inequality we have

$$\sqrt{(x+2y)(x+2z)} \leq \frac{x+2y+x+2z}{2} = x+y+z = \sum x.$$

Writing other two similar inequalities yields that it suffices to prove that

$$\sum x(x^2-yz) \geq (\sum x)(\sum x^2 - \sum xy).$$

The last relation yields from well-known identity

$\sum x^3 - 3xyz = (\sum x)(\sum x^2 - \sum xy)$, and the proof is complete.

PP.20650. Prove that $\sum_{k=1}^{n} \left(\frac{F_k}{F_{n+2}-F_k-1}\right)^2 \geq \frac{n}{(n-1)^2}$, where F_k denote the k^{th} Fibonacci number.

Mihály Bencze

By well-known identity $X_n = \sum_{k=1}^{n} F_k = F_{n+2} - 1$, the inequality to prove becomes

$$\sum_{k=1}^{n}\left(\frac{F_k}{X_n - F_k}\right)^2 \geq \frac{n}{(n-1)^2}.$$

By *H. Bergström's* inequality we have

(1) $\sum_{k=1}^{n}\left(\frac{F_k}{X_n - F_k}\right)^2 \geq \frac{1}{n}\left(\sum_{k=1}^{n}\frac{F_k}{X_n - F_k}\right)^2$, but

$$S_n = \sum_{k=1}^{n}\frac{F_k}{X_n - F_k} \Leftrightarrow S_n + n = \sum_{k=1}^{n}\left(\frac{F_k}{X_n - F_k} + 1\right) = X_n \sum_{k=1}^{n}\frac{1}{X_n - F_k} \overset{BERGSTROM}{\geq}$$

$$\geq X_n \cdot \frac{n^2}{\sum_{k=1}^{n}(X_n - F_k)} = X_n \cdot \frac{n^2}{nX_n - X_n} = \frac{n^2}{n-1} \Leftrightarrow (2)\ S_n \geq \frac{n^2}{n-1} - n = \frac{n}{n-1}.$$

From (1) and (2) we get the inequality to prove, and we are done.

2. A GENERALIZATION AND SOLUTIONS OF THE PROBLEM 11670 FROM AMM

by D.M. BĂTINEŢU-GIURGIU, ''Matei Basarab'' National College, Bucharest,

NECULAI STANCIU, ''George Emil Palade'' General School, Buzău,

and

TITU ZVONARU, Comăneşti, ROMANIA

11670. *Proposed by Miranda Bakke, Benson Wu, and Bogdan Suceavă, California State University, Fullerton, CA.* Prove that if $n \geq 3$ and $a_1, \ldots, a_n > 0$, then

$$\frac{(n-1)}{4}\sum_{k=1}^{n} a_k \geq \sum_{1\leq j<k\leq n} \frac{a_j a_k}{a_j + a_k},$$

with equality if and only if all a_j are equal.

Let $M_{<} = \{(i,j) | i < j, i, j = \overline{1,n}\}$ and $M_{>} = \{(i,j) | i > j, i, j = \overline{1,n}\}$.

Proposition. If $n \in N^* - \{1\}, a \in R_+, b \in R^*, x_k \in R_+^*, k = \overline{1,n}, X_n = \sum_{k=1}^{n} x_k$, then:

$$\sum_{(i,j)\in M_{<}} (2aX_n + b(x_i + x_j)) = \sum_{(i,j)\in M_{>}} (2aX_n + b(x_i + x_j)) = (n-1)(na+b)X_n, \forall n \in N^*.$$

Proof. We observe that in $M_{<}$ every $k = \overline{1,n}$ is in $(k-1)$ pairs (i,k) with $i < k$ and is in $(n-k)$ pairs (k,j) with $k < j$, so k is in $(n-1)$ pairs (i,j) with $i \neq j, i, j = \overline{1,n}$.
Also in $M_{>}$ every $k = \overline{1,n}$ is in $(k-1)$ pairs (k,i) with $k > i$ and is in $(n-k)$ pairs (j,k) with $j > k$, so k is in $(n-1)$ pairs (i,j) with $i \neq j, i, j = \overline{1,n}$.
Therefore:

$$\sum_{(i,j)\in M_{<}} (2aX_n + b(x_i + x_j)) = \sum_{(i,j)\in M_{>}} (2aX_n + b(x_i + x_j)) =$$

$$= \sum_{(i,j)\in M_{<}} (aX_n + bx_i + aX_n + bx_j)) = (n-1)\sum_{k=1}^{n} (aX_n + bx_k) =$$

$$= (n-1)(naX_n + b\sum_{k=1}^{n} x_k) = (n-1)(na+b)X_n,$$

and the proof is complete.

Theorem. If $n \in N^* - \{1\}, a \in R_+, b \in R^*, x_k \in R_+^*, k = \overline{1,n}, X_n = \sum_{k=1}^{n} x_k$, such that $aX_n + b \max_{1 \le k \le n} x_k \in R_+^*$, then:

$$(n-1)(an+b)X_n \ge 4 \cdot \sum_{(i,j)\in M_<} \frac{a^2 X_n^2 + ab(x_i + x_j)X_n + b^2 x_i x_j}{2aX_n + b(x_i + x_j)}, \ \forall n \in N^*,$$

$$(n-1)(an+b)X_n \ge 4 \cdot \sum_{(i,j)\in M_>} \frac{a^2 X_n^2 + ab(x_i + x_j)X_n + b^2 x_i x_j}{2aX_n + b(x_i + x_j)}, \ \forall n \in N^*$$

with equality if and only if $x_i = x_j, \forall i.j = \overline{1,n}$.

Proof. By the above proposition the inequality to prove is equivalent with:

$$\sum_{(i,j)\in M_<} (2aX_n + b(x_i + x_j)) \ge 4 \cdot \sum_{(i,j)\in M_<} \frac{a^2 X_n^2 + ab(x_i + x_j)X_n + b^2 x_i x_j}{2aX_n + b(x_i + x_j)}$$

$$\Leftrightarrow \sum_{(i.,j)\in M_<} \left(2aX_n + b(x_i + x_j) - 4 \cdot \frac{a^2 X_n^2 + ab(x_i + x_j)X_n + b^2 x_i x_j}{2aX_n + b(x_i + x_j)} \right) \ge 0$$

$$\Leftrightarrow \sum_{(i.,j)\in M_<} \frac{(2aX_n + b(x_i + x_j))^2 - 4(a^2 X_n^2 + ab(x_i + x_j)X_n + b^2 x_i x_j)}{2aX_n + b(x_i + x_j)} \ge 0$$

$$\Leftrightarrow \sum_{(i.,j)\in M_<} \frac{4a^2 X_n^2 + 4ab(x_i + x_j)X_n + b^2 (x_i + x_j)^2 - 4a^2 X_n^2 - 4ab(x_i + x_j)X_n - 4b^2 x_i x_j}{2aX_n + b(x_i + x_j)} \ge 0$$

$\Leftrightarrow b^2 \cdot \sum_{(i.,j)\in M_<} \frac{(x_i - x_j)}{2aX_n + b(x_i + x_j)} \ge 0$, which is true,

with equality if and only if $x_i = x_j, \forall i.j = \overline{1,n}$.

If we taking account that:

$$\sum_{(i,j)\in M_<} (2aX_n + b(x_i + x_j)) = \sum_{(i,j)\in M_>} (2aX_n + b(x_i + x_j)), \ \forall n \in N^*,$$

we also obtain that:

$$(n-1)(an+b)X_n \ge 4 \cdot \sum_{(i,j)\in M_>} \frac{a^2 X_n^2 + ab(x_i + x_j)X_n + b^2 x_i x_j}{2aX_n + b(x_i + x_j)}, \ \forall n \in N^*,$$

with equality if and only if $x_i = x_j, \forall i.j = \overline{1,n}$.

Observation. For $a = 0, b = 1$ we obtain the problem 11670 from AMM.

Solution 1:

In the set of the pairs $(i, j), i < j, i, j = \overline{1,n}$ the element $k \in \{1,2,..,n\}$ is in $(k-1)$ pairs with $i < k$ and is in $(n-k)$ pairs with $k < i$, i.e. is in $n-k+k-1 = n-1$ pairs, and then:

$$\sum_{\substack{i,j=1\\ i<j}}^{n}(x_i + x_j) = (n-1)\sum_{k=1}^{n} x_k \tag{1}$$

By the AM-HM inequality we have that:

$$\frac{x+y}{2} \geq \frac{2xy}{x+y}, \forall x, y \in R_+^* \tag{2}$$

Therefore:

$$x_i + x_j \geq 4 \cdot \frac{x_i x_j}{x_i + x_j}, \forall i, j = \overline{1,n} \tag{3}$$

Yields that:

$$\sum_{\substack{i,j=1\\ i<j}}^{n}(x_i + x_j) \geq 4 \cdot \sum_{\substack{i,j=1\\ i<j}}^{n}\frac{x_i x_j}{x_i + x_j}, \forall n \in N^* \tag{4}$$

By the above we deduce that:

$$(n-1)\sum_{k=1}^{n} x_k = \sum_{\substack{i.j=1\\ i<j}}^{n}(x_i + x_j) \geq 4 \cdot \sum_{\substack{i,j=1\\ i<j}}^{n}\frac{x_i x_j}{x_i + x_j}, \forall n \in N^*,$$

and because in AM-HM inequality we have equality if and only if $x_i = x_j, \forall i, j = \overline{1,n}$, the problem 11670 from AMM is proved.

Solution 2:

We use the AM-HM inequality, i.e.:

$$\frac{2ab}{a+b} \leq \frac{a+b}{2} \Leftrightarrow \frac{1}{a+b} \leq \frac{1}{4}\left(\frac{1}{a} + \frac{1}{b}\right), \text{ with equality iff } a = b.$$

Hence,

$$\sum_{1\leq j<k\leq n}\frac{a_j a_k}{a_j + a_k} \leq \frac{1}{4}\sum_{1\leq j<k\leq n} a_j a_k\left(\frac{1}{a_j} + \frac{1}{a_k}\right) = \frac{1}{4}\sum_{1\leq j<k\leq n}(a_j + a_k) =$$

$$= \frac{1}{4}((n-1)a_1 + (n-2)a_2 + ... + 2a_{n-2} + a_{n-1} + a_2 + ... + (n-3)a_{n-2} + (n-2)a_{n-1} + (n-1)a_n)$$

$$= \frac{n-1}{4}\sum_{k=1}^{n} a_k .$$

The equality holds iff all a_j are equal.

3. Other solutions for some problems from MR-4/2014

By Nela Ciceu, Roşiori, Bacău, Romania

and

Roxana Mihaela Stanciu, Buzău, Romania

J307. Prove that for each positive integer n there is a perfect square whose sum of digits is equal to 4^n.

Proposed by Mihaly Bencze, Brasov, Romania

Solution:

It is well-known the fact that for any $n \equiv 0,1,4,7 (\bmod 9)$ there is a perfect square whose sum of digits is equal to n (see for e.g. the book ''Mathematical Olympiad Challenges'' , p. 243, by *Titu Andreescu* and *Răzvan Gelcă* or Crux Mathematicorum no. 3/2013, p. 128).

On the other hand we have

$4^{3p} - 1 = 64^p - 1 = M(64-1) = M9$, $4^{3p+1} - 4 = 4(4^{3p} - 1)$ and $4^{3p+2} - 7 = 16(4^{3p} - 1) + 9$.

The conclusion of the problem yields now easily

- for n of the form $3p$ we choose a perfect square with the sum of digits $9k+1$, where $k = \dfrac{4^{3p} - 1}{9}$;
- for n of the form $3p+1$ we choose a perfect square with the sum of digits $9k+4$, where $k = \dfrac{4^{3p+1} - 4}{9}$;
- for n of the form $3p+2$ we choose a perfect square with the sum of digits $9k+7$, where $k = \dfrac{4^{3p+2} - 7}{9}$.

J312. Let ABC be a triangle with circumcircle Γ and let P be a point in its interior. Let M be the midpoint of side BC and let lines AP, BP, CP intersect BC, CA, AB at X, Y, Z, respectively. Furthermore, let line YZ intersect Γ at points U and V. Prove that M, X, U, V are concyclic.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution:

We denote

$$x = \frac{XB}{XC}, y = \frac{YC}{YA}, z = \frac{ZA}{ZB}.$$

We have

$$BX = \frac{ax}{x+1}, \text{ and by Ceva's Theorem yields that}$$

$$xyz = 1.$$

If $YZ \| BC$, then $yz = 1$, i.e. $X \equiv M$ and we have nothing to prove.

We assume that $YZ \cap BC = \{N\}$ such that we have the order $N - B - C$.

Applying the Theorem of Menelaus in triangle ABC with the transversal $N - Z - Y$ we obtain

$$\frac{NB}{NC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1 \Leftrightarrow \frac{NB}{NB+a} = \frac{1}{yz}, \text{ so } NB = \frac{a}{yz-1}.$$

Using the power of point N with respect to circumcircle of the triangle ABC we obtain

$$NU \cdot NV = NB \cdot NC.$$

The concyclicity of the points M, X, U, V is successively equivalent to

$$NX \cdot NM = NU \cdot NV$$

$$\Leftrightarrow NX \cdot NM = NB \cdot NC$$

$$\Leftrightarrow (NB + BX)(NB + BM) = NB(NB + BC)$$

$$\Leftrightarrow \left(\frac{a}{yz-1} + \frac{ax}{x+1}\right)\left(\frac{a}{yz-1} + \frac{a}{2}\right) = \frac{a}{yz-1}\left(\frac{a}{yz-1} + a\right)$$

$$\Leftrightarrow \frac{(x+1+xyz-x)(2+yz-1)}{2(x+1)} = yz$$

$$\Leftrightarrow \frac{yz+1}{x+1} = yz \Leftrightarrow yz+1 = xyz + yz, \text{ true, and the proof is complete.}$$

O310. Let ABC be a triangle and let P be a point in its interior. Let X, Y, Z be the intersections of AP, BP, CP with sides BC, CA, AB, respectively. Prove that

$$\frac{XB}{XY} \cdot \frac{YC}{YZ} \cdot \frac{ZA}{ZX} \le \frac{R}{2r}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

With the notations of O310 we reformulated a problem proposed by Dan Ștefan Marinescu in RMT, No. 2/2001, page 51, which requires to demonstrate that

$$XY \cdot YZ \cdot ZX \ge 8 \cdot BX \cdot CY \cdot AZ \cdot \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \ (1).$$

Since $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2r}{R}$, the relation (1) it can write $\frac{XB}{XY} \cdot \frac{YC}{YZ} \cdot \frac{ZA}{ZX} \le \frac{R}{2r}$, i.e. exactly q.e.d.

S307. Let ABC be a triangle such that $\angle ABC - \angle ACB = 60^\circ$. Suppose that the length of the altitude from A is $\frac{1}{4}BC$. Find $\angle ABC$.

Proposed by Omer Cerrahoglu and Mircea Lascu, Romania

Solution:

We denote the altitude from A with h.

By the hypothesis $h = \frac{a}{4}$, Sine Law and well known formulas we deduce successively

$$\sin B = \frac{h}{c} \Leftrightarrow 4\sin B \sin C = \sin A \Leftrightarrow 2\cos(B-C) - 2\cos(B+C) = \sin(B+C)$$

$\Leftrightarrow \sin(B+C) + 2\cos(B+C) - 1 = 0$, (1).

Denoting $t = \tan\frac{B+C}{2}$, yields that $t > 0$, and by (1) we obtain the equation

$3t^2 - 2t - 1 = 0$, with only one convenient root, i.e. $t = 1$.

We deduce $B + C = 90^0$ which with the hypothesis $B - C = 60^0$, yields $B = 75^0$ (and $A = 90^0, C = 15^0$).

4. Metode de calcul pentru derivata unui determinant

Prof. Boer Elena Milena, Școala Gimnazială Vulcan, Jud. Brașov

Metoda 1.

Teoremă

Fie $a_{ij}: R \to R$ funcții derivabile pe R, $i, j \in \{1, 2, \dots, n\}$, iar $a : R \to R$,

$$a(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix}, x \in R$$

Atunci $a(x)$ este o funcție derivabila pe R și

$$a'(x) = \sum_{j=1}^{n} \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ \dots & \dots & \dots & \dots \\ a'_{j1}(x) & a'_{j2}(x) & \dots & a'_{jn}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix}, (\forall)\, x \in R$$

Demonstrație: Faptul ca funcția $a(x)$ este derivabilă pe R rezulta din aceea că dacă funcțiile $b_1, b_2, \dots, b_n$ sunt funcții derivabile pe R, atunci funcția $b_1 \cdot b_2 \cdot \dots \cdot b_n$ este derivabila pe R și

$$(b_1 \cdot b_2 \cdot \dots \cdot b_n)' = \sum_{j=1}^{n} b_1 \cdot b_2 \cdot \dots \cdot b_j \cdot \dots \cdot b_n$$

În continuare, conform formulei lui Leibniz, avem:

$$a(x) = \sum_{\varphi \in P_n} sgn(\varphi) a_{1\varphi(1)}(x) \cdot a_{2\varphi(2)}(x) \cdot \dots \cdot a_{n\varphi(n)}(x), \qquad (\forall) x \in R \qquad (1)$$

unde, $sgn(\varphi)$ reprezintă funcția signum (semn), adică $sgn(\varphi) = (-1)^{N(\varphi)}$, unde $N(\varphi)$ este numărul de inversări în permutația φ.

Din (1), prin derivare rezultă:

$$a'(x) = \sum_{j=1}^{n} \sum_{\varphi \in P_n} sgn(\varphi) a_{1\varphi(1)}(x) \cdot a_{2\varphi(2)}(x) \cdot \dots \cdot a'_{n\varphi(2)}(x) \cdot \dots \cdot a_{n\varphi(n)}(x), \quad (\forall) x \in R$$

adică tocmai relația de demonstrat.

Metoda 2

O altă metodă de aflare a derivatei unui determinant pentru o matrice A, este să folosim formula lui Jacobi bazată matricea reciprocă a matricei A și pe derivata acesteia.

Lemă: Fie A și B două matrici pătratice având dimensiunea n. Atunci:

(1)

$$\sum_k \sum_j \mathrm{A_{kj}} \mathrm{B_{k,j}} = Tr(A^t B)$$

A^t- fiind transpusa matricei A.

Demonstrație:

$$(A^t B)_{ij} = \sum_k (A^t)_{ik} B_{kj}$$

$$(A^t)_{ik} = A_{ki}$$

$$(A^t B)_{ij} = \sum_k A_{ki} B_{kj}$$

$$Tr(A^t B) = \sum_j (A^t B)_{jj} = \sum_j \left(\sum_k \mathrm{A_{kj}} \mathrm{B_{kj}} \right) = \sum_k \sum_j \mathrm{A_{kj}} \mathrm{B_{kj}}$$

Teoremă (formula lui Jacobi) : Fie o matrice pătratică A, având dimensiunea n. Vom nota cu det(A) determinantul asociat matricei A. Diferențiala lui det(A) este data de relația:

(2)

$$d\ (det(A)) = Tr(A^* dA)$$

Unde A^* reprezintă matricea adjunctă : $A^* = C^t$, C fiind complementul algebraic al matricei A.

Demonstrație:

Pe baza formulei lui Laplace

(3)

$$det(A) = \sum_{j} A_{ij}\,(A^*)^t_{ij}$$

Determinantul matricei A poate fi considerat ca fiind o funcție de elementele matricei A :

$$det(A) = F(\mathrm{A_{ij}})$$

Prin urmare, diferențiala determinantului poate fi scris sub forma:

$$d\,(det(A)) = \sum_{i}\sum_{j} \frac{\partial F}{\partial \mathrm{A_{ij}}} d\mathrm{A_{ij}}$$

Pe baza relației de mai sus:

(4)

$$\frac{\partial \det(A)}{\partial \mathrm{A_{ij}}} = \frac{\partial F}{\partial \mathrm{A_{ij}}} = \frac{\partial}{\partial \mathrm{A_{ij}}}\left[\sum_{k} A_{ik}(A^*)^t_{ik}\right] = \sum_{k} \frac{\partial [A_{ik}(A^*)^t_{ik}]}{\partial \mathrm{A_{ij}}}$$

Ținând seama de derivata produsului obținem:

(5)

$$\frac{\partial [A_{ik}(A^*)^t_{ik}]}{\partial \mathrm{A_{ij}}} = \frac{\partial A_{ik}}{\partial \mathrm{A_{ij}}}(A^*)^t_{ik} + A_{ik}\frac{\partial (A^*)^t_{ik}}{\partial \mathrm{A_{ij}}}$$

Dar, $\frac{\partial (A^*)^t_{ik}}{\partial \mathrm{A_{ij}}} = 0$, deoarece elementul ij al matricei adjunct, sigur nu conține nici un element de pe linia i sau coloana j a matricei originale A.

Pe de alta parte:

(6)

$$\frac{\partial \mathrm{A_{ik}}}{\partial \mathrm{A_{ij}}} = \delta_{jk}$$

Notam cu δ_{jk} simbolul lui Kronecker

$$\delta_{jk} = \begin{cases} i & dacă\ j = k \\ 0 & dacă\ j \neq k \end{cases}$$

Din relațiile (4) și (5) rezultă:

(7)

$$\frac{\partial F}{\partial \mathrm{A_{ij}}} = \frac{\partial \det(\mathrm{A})}{\partial \mathrm{A_{ij}}} = \sum_{\mathrm{k}} \delta_{kj} (A^*)_{ik}^t = (A^*)_{ij}^t$$

Pentru diferențiala determinantului A obținem astfel:

$$d\ (det(A)) = \sum_{\mathrm{i}} \sum_{\mathrm{j}} (A^*)_{ij}^t d\mathrm{A_{ij}}$$

Ținând cont de lema demonstrată mai sus

$$\sum_{\mathrm{i}} \sum_{\mathrm{j}} (A^*)_{ij}^t d\mathrm{A_{ij}} \quad = Tr[(A^*)^{TT} dA] = Tr(A^* dA)$$

Astfel teorema este demonstrată

Calculul derivatei unui determinant folosind ambele metode:

Enunț : Să se calculeze derivata determinantului pentru matricea :

$$A = \begin{pmatrix} x^2 & x+2 & x \\ 0 & x & x+1 \\ 1 & 1 & x^3 \end{pmatrix}$$

Rezolvare:

Metoda 1

$$f(x) = \begin{vmatrix} x^2 & x+2 & x \\ 0 & x & x+1 \\ 1 & 1 & x^3 \end{vmatrix}$$

$$f'(x) = \begin{vmatrix} 2x & 1 & 1 \\ 0 & x & x+1 \\ 1 & 1 & x^3 \end{vmatrix} + \begin{vmatrix} x^2 & x+2 & x \\ 0 & 1 & 1 \\ 1 & 1 & x^3 \end{vmatrix} + \begin{vmatrix} x^2 & x+2 & x \\ 0 & x & x+1 \\ 0 & 0 & 3x^2 \end{vmatrix} =$$
$$(2x^5 + x + 1 - x - 2x^2 - 2x) + (x^5 + x + 2 - x - x^2) + 3x^5 =$$
$$6x^5 - 3x^2 - 2x + 3$$

Metoda 2

$$A = \begin{pmatrix} x^2 & x+2 & x \\ 0 & x & x+1 \\ 1 & 1 & x^3 \end{pmatrix}, \quad B = \begin{pmatrix} 2x & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3x^2 \end{pmatrix}$$

În matricea B am derivat fiecare element al matricei A.

$$A^* = \begin{pmatrix} x^4 - x - 1 & x+1 & -x \\ -x^4 - 2x^3 & x^5 - x & -x^2 + x + 2 \\ 3x + 2 & -x^3 - x^2 & x^3 \end{pmatrix}$$

Se înmulțește fiecare element din A^* cu fiecare element corespunzator din B ($a^*{}_{ij} * b_{ij}$) și se obține derivata determinantului:

$2x(x^4 - x - 1) + 1(x + 1) + 1(-x) + 0(-x^4 - 2x^3) + 1(x^5 - x) + 1(-x^2 + x + 2) + 0(3x + 2) + 0(-x^3 - x^2) + x^3 \cdot 3x^2 = 6x^5 - 3x^2 - 2x + 3$

Bibliografie:

1. Bușneag D., Maftei I., "*Teme pentru cercurile și concursurile de matematică ale elevilor*", Editura Scrisul Românesc, Craiova, 1983.
2. Pop V., Lupșor V., "*Matematică pentru grupe de performanță, cls.a XI-a", Editura Dacia Educațional*, Cluj-Napoca, 2004
3. Bellmann, R „*Introduction to Matrix Analysis",* SIAM Philadelphia, 1997
4. Magnus, Jan R.; Neudecker, Heinz "*Matrix Differential Calculus with Applications in Statistics and Econometrics"*, Wiley & Sons ,1999