REVISTA ELECTRONICĂ MATEINFO.RO

ISSN 2065-6432

IANUARIE 2015

REVISTĂ LUNARĂ

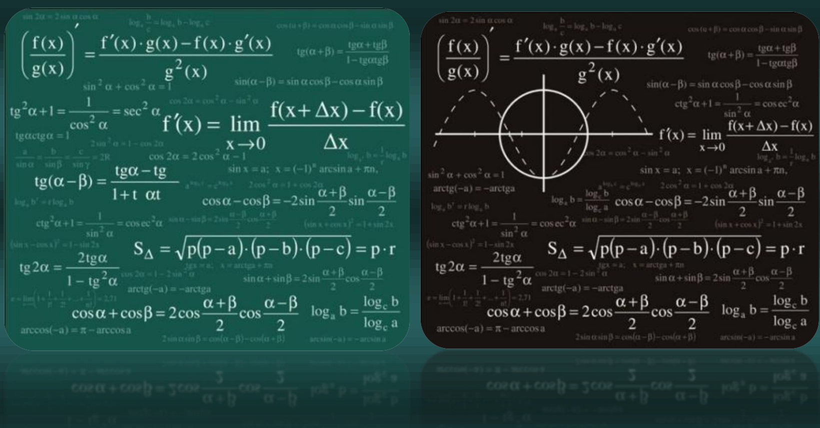
DIN FEBRUARIE 2009

ÎN LUNA FEBRUARIE 2015

ÎMPLINIM 5 ANI DE APARIŢII LUNARE

WWW.MATEINFO.RO

revistaelectronica@mateinfo.ro



Articole:

1. Solutions and hints of some problems from the Octogon Mathematical Magazine (V)

by D.M. Bătineţu-Giurgiu, Bucharest, Romania

and

Neculai Stanciu, Buzău, Romania

PP.20706. If $a,b,c>0$, then $2\cdot\sqrt{\sum(a+b)^2}\geq\sqrt{\sum a^2}+\sum a$.

Solution. Applying the inequality $\sqrt{2(x^2+y^2)}\geq x+y$, we have

$\sqrt{\sum(a+b)^2}=\sqrt{2(\sum a^2+\sum ab)}\geq\sqrt{\sum a^2}+\sqrt{\sum ab}$, so it suffices to show that

$2\cdot\sqrt{\sum a^2}+2\cdot\sqrt{\sum ab}\geq\sqrt{\sum a^2}+\sum a$

$\Leftrightarrow\sqrt{\sum a^2}+2\cdot\sqrt{\sum ab}\geq\sum a$, add squaring we obtain

$\sum a^2+4\sum ab+4\cdot\sqrt{(\sum a^2)(\sum ab)}\geq\sum a^2+2\sum ab$, evidently true, and the proof is complete.

PP.20714. For all $n\in N$ the expression $\frac{4^{n+2}-4}{3}-\frac{(n+1)(3n+8)}{2}$ is divisible by 9. (enunciation correction)

Solution. Since $2(4^{n+2}-4)-3(n+1)(3n+8)=2^{2n+5}-8-9n^2-33n-24=$
$=2^{2n+5}+9n^2+3n+4-18(n^2+2n+2)$. So it suffices to prove that:
$2^{n+5}+9n^2+3n+4$ is divisible by 18. Using mathematical induction for $n=1$ we have $2^7+9+3+4=144=18\cdot 8$. We assume that $2^{n+5}+9n^2+3n+4$ is divisible by 18.
We have $2^{2n+7}+9(n+1)^2+3(n+1)+4=2^{2n+7}+9n^2+21n+16=$
$=2^{2n+7}+36n^2+12n+16-27n^2+9n=2^2(2^{2n+5}+9n^2+3n+4)-9n(3n-1)$ which is divisible by 18 (from the hypothesis of induction and the fact that n and $3n-1$ has different parities, so their product is even).

PP.20728. In all triangle ABC holds $\sum \frac{s + r ctg\frac{A}{2}}{ctg\frac{B}{2} + ctg\frac{C}{2}} \le \left(\frac{s}{r}\right)^2$.

Solution. Something is missing from the statement, because LHS has degree one and RHS has degree zero.

PP.20732. In all triangle ABC holds $\sum (a+b)^4 + 4abc\sum a \ge 4\sum ab(a+b)^2$.

Solution. Since $(a+b)^2 \ge 4ab$, we have

$$\sum (a+b)^4 = \sum (a+b)^2(a+b)^2 \ge \sum 4ab(a+b)^2,$$

which is stronger that given inequality, and we are done.

PP.20736. If $x, y, z > 0$, $x + y + z = 1$, then $\sum x^3 + \sum x^2 \ge 3xyz + \sum xy$.

Solution. The given inequality is written successively:

$$\sum x^3 + (\sum x)(\sum x^2) \ge 3xyz + (\sum x)(\sum xy)$$

$$\Leftrightarrow \sum x^3 + \sum x^3 + \sum x^2 y + \sum xy^2 \ge 3xyz + \sum x^2 y + \sum xy^2 + 3xyz$$

$\Leftrightarrow \sum x^3 \ge 3xyz$, true by AM-GM inequality.

PP.20736. If $x, y, z > 0$, then:

$$3\prod (x^2 + 3y^2 + z^2 + 3xy + 3yz + zx) \ge 4\left(\sum x\right)^2\left(\sum x^2 + 3\sum xy\right)^2.$$

Solution. Applying the inequality $a^2 + ab + b^2 \ge \frac{3(a+b)}{4}$, we obtain:

$$x^2 + 3y^2 + z^2 + 3xy + 3yz + zx = (x+y)^2 + (x+y)(y+z) + (y+z)^2 \ge$$

$\ge \frac{3(x+y+y+z)^2}{4} = \frac{3(x+2y+z)^2}{4}$, and then the inequality from the statement is written as follows:

$$3^4 \cdot \frac{1}{4^3}\prod (x+2y+z)^2 \ge 4\left(\sum x\right)^2\left(\sum x^2 + 3\sum xy\right)^2$$

$\Leftrightarrow \prod (x+2y+z) \ge \frac{16}{9}\sum x\left(\sum x^2 + 3\sum xy\right)$, i.e. PP.21301, which we solved (see the solution from this Octogon Mathematical Magazine).

The proof is complete.

PP.20743. In all triangle ABC holds $\frac{9}{2\sum m_a} \le \sum \frac{1}{m_a + m_b} < \frac{5}{\sum m_a}$.

Solution. The left inequality yields by Harald Bergström' s inequality. Indeed,

$$\sum\frac{1}{m_a+m_b}\geq\frac{(1+1+1)^2}{\sum(m_a+m_b)}=\frac{9}{2\sum m_a}.$$

For the right inequality we prove the following strengthening:

$$\sum\frac{1}{m_a+m_b}+\frac{4\prod(m_a+m_b-m_c)}{\left(\sum m_a\right)\left(\prod(m_a+m_b)\right)}<\frac{5}{\sum m_a} \qquad (1)$$

Because m_a, m_b, m_c can be the sides of triangle, we can denote $m_a=y+z$, $m_b=x+z$, $m_c=x+y$, with $x,y,z>0$. So, the inequality (1) becomes:

$$\sum\frac{1}{2x+y+z}+\frac{16xyz}{\left(\sum x\right)\left(\prod(2x+y+z)\right)}\leq\frac{5}{2\sum x} \qquad (2)$$

We have:

$$\prod(2x+y+z)=2\sum x^3+7\sum x^2y+7\sum xy^2+16xyz;$$

$$\sum(2x+y+z)(x+2y+z)=5\sum x^2+11\sum xy, \text{ and}$$

$$\left(\sum x\right)\left(\sum(2x+y+z)(x+2y+z)\right)=5\sum x^3+16\sum x^2y+16\sum xy^2+33xyz.$$

After clearing the denominators the inequality (2) becomes:

$10\sum x^3+32\sum x^2y+32\sum xy^2+66xyz+32xyz\leq10\sum x^3+35\sum x^2y+35\sum xy^2+$ $+80xyz\Leftrightarrow\sum x^2y+\sum xy^2\geq6xyz$, which follows immediately by AM-GM inequality, because $\sum x^2y\geq3xyz$, $\sum xy^2\geq3xyz$.

The proof is complete.

PP.20744. If $x,y,z>0$ and $x^2+y^2+z^2\leq1$, then $\sum\frac{1}{\sqrt{1+x^2}}\geq\frac{9}{4}$.

Solution. We will prove a stronger inequality, i.e. we prove that: $\sum\frac{1}{\sqrt{1+x^2}}\geq\frac{3\sqrt{3}}{2}$.

Indeed, by Hölder's inequality we obtain:

$$\left(\sum\frac{1}{\sqrt{1+x^2}}\right)\left(\sum\frac{1}{\sqrt{1+x^2}}\right)\left(\sum(1+x^2)\right)\geq27\Leftrightarrow\left(\sum\frac{1}{\sqrt{1+x^2}}\right)^2\geq\frac{27}{3+x^2+y^2+z^2},$$

and from hypothesis $x^2+y^2+z^2\leq1$, we get $\left(\sum\frac{1}{\sqrt{1+x^2}}\right)^2\geq\frac{27}{4}\Leftrightarrow\sum\frac{1}{\sqrt{1+x^2}}\geq\frac{3\sqrt{3}}{2}$.

The proof is complete.

PP.20755. If $x,y,z\in N$ such that $x^2+y^2+z^2=2002$, then $x+y+z\leq70$.

Solution. We can assume that $x\leq y\leq z$. Because we have:

$$y+z\leq\sqrt{2(y^2+z^2)}\leq\sqrt{2\cdot2002}<64, \text{ so if } x\leq6, \text{ then } x+y+z<70.$$

If $x=7$, then $y+z\le\sqrt{2(y^2+z^2)}\le\sqrt{2\cdot(2002-49)}<63$, so $x+y+z<70$.
If $x\ge 8$, after some algebra we get the solutions (9,20,39), (9,25,36), (15,16,39), so in all these cases $x+y+z\le 70$, and the solution is complete.

PP.20762. If $a,b,c>0$ then:

$$4\prod(a^2+ab+b^2)\le(a-b)^2(b-c)^2(c-a)^2+9\sum a^2b^2(a+b)^2 .$$

Solution. We have:

$$\prod(a^2+ab+b^2)=\sum a^4b^2+\sum a^4b^2+\sum a^4bc+\sum a^3b^3+2\sum a^3b^2c+$$
$$+2\sum a^3bc^2+\sum a^2b^2c^2 ;\ (a-b)^2(b-c)^2(c-a)^2=\sum a^4b^2+\sum a^2b^4-$$
$$-2\sum a^3b^3-2\sum a^4bc-6a^2b^2c^2+2\sum a^3b^2c+2\sum a^3bc^2 .$$

The inequality from the statement is written as follows:

$$\sum a^4b^2+\sum a^2b^4+2\sum a^3b^3\ge\sum a^4bc+\sum a^3b^2c+\sum a^3bc^2+3a^2b^2c^2 \qquad (1)$$

By AM-GM inequality we obtain:

$$a^4b^2+a^4c^2\ge 2a^4bc\Rightarrow\sum a^4b^2+\sum a^2b^4\ge 2\sum a^4bc \qquad (2)$$
$$a^4bc+ab^4c+abc^4\ge 3a^2b^2c^2\Rightarrow\sum a^4bc\ge 3a^2b^2c^2 \qquad (3)$$
$$a^3b^3+a^3b^3+b^3c^3\ge 3a^2b^3c$$
$$b^3c^3+b^3c^3+c^3a^3\ge 3ab^2c^3\Rightarrow\sum a^3b^3\ge\sum a^3bc^2 \qquad (4)$$
$$c^3a^3+c^3a^3+a^3b^3\ge 3a^3bc^2$$

and similar $\sum a^3b^3\ge\sum a^3b^2c$ (5)

Adding up the inequalities (2), (3), (4) and (5) yields (1).
Remark. With Muirhead's inequality, because $(3,3,0)\succ(3,2,1)$ we obtain:

$$\sum_{sym}a^3b^3\ge\sum_{sym}a^3b^2c,$$ whics means, using cyclic summation,\

$$2\sum a^3b^3\ge\sum a^3b^2c+\sum a^3bc^2 .$$

The proof is complete.

PP.20765. In all triangle ABC holds:

$$\sum(1-\cos A-\cos 2A-\cos(B-C))^2=\left(\frac{s^2-4Rr-r^2}{R^2}\right)^2-\left(\frac{s^2+r^2+4Rr}{2R^2}\right)^2 .$$

Solution. We have:

$$1-\cos A-\cos 2A-\cos(B-C)=1-\cos 2A-\cos A-\cos(B-C)=$$
$$=2\sin^2 A-2\cos\frac{A+B-C}{2}\cos\frac{A-B+C}{2}=2\sin^2 A-2\sin B\sin C=$$

$=2\cdot\frac{a^2}{4R^2}-2\cdot\frac{bc}{4R^2}=\frac{a^2}{2R^2}-\frac{bc}{2R^2}$, thus by $\sum a^2=2(s^2-r^2-4Rr)$ and $\sum ab=s^2+r^2+4Rr$ we obtain that:

$$\sum(1-\cos A-\cos 2A-\cos(B-C))^2=\sum\left(\frac{a^2-bc}{2R^2}\right)^2=$$

$$=\left(\frac{1}{2R^2}\right)^2\sum(a^4-2a^2bc+b^2c^2)=\frac{1}{4R^4}[\left(\sum a^2\right)^2-2\sum a^2b^2-2\sum a^2bc+\left(\sum bc\right)^2-$$

$$-2\sum a^2bc]=\frac{\left(\sum a^2\right)^2}{4R^4}+\frac{1}{4R^4}[-2\left(\sum ab\right)^2+4\sum a^2bc-4\sum a^2bc+\left(\sum ab\right)^2]=$$

$$=\frac{\left(\sum a^2\right)^2}{4R^4}-\frac{\left(\sum ab\right)^2}{4R^4}=\frac{4(s^2-r^2-4Rr)^2}{4R^4}-\frac{(s^2+r^2+4Rr)^2}{4R^4}=$$

$$=\left(\frac{s^2-4Rr-r^2}{R^2}\right)^2-\left(\frac{s^2+r^2+4Rr}{2R^2}\right)^2$$, and we are done.

PP.20768. Prove that the following three statements

1) a,b,c are in geometrical progression

2) $\left(\sum a^2\right)^2=\left(\sum ab\right)^2$

3) $\left(\sum ab\right)^3=abc\left(\sum a\right)^3$

are equivalent, where $a,b,c\in R$.

Solution. If we take $a=1,b=2,c=4$ (in geometrical progression) then $\left(\sum a^2\right)^2=21^2$ and $\left(\sum ab\right)^2=14^2$, so (1) and (2) are not equivalents.

We prove that $(3)\Leftrightarrow(1)$. Indeed.

$$\left(\sum ab\right)^3-abc\left(\sum a\right)^3=\sum a^3b^3+3abc\left(\sum a^2b+\sum ab^2\right)+6a^2b^2c^2-$$
$$-\sum a^4bc-3abc\left(\sum a^2b+\sum ab^2\right)-6a^2b^2c^2=\sum a^3b^3-\sum a^4bc=$$
$$=a^3b^3-a^4bc+b^3c^3-abc^4+a^3c^3-ab^4c=a^3b(b^2-ac)+bc^3(b^2-ac)-$$
$$-ac(b^4-a^2c^2)=(b^2-ac)(a^3b-ab^2c+bc^3-a^2c^2)=(b^2-ac)(a^2-bc)(ab-c^2).$$

So $\left(\sum ab\right)^3=abc\left(\sum a\right)^3\Leftrightarrow a^2=bc$ or $b^2=ac$ or $c^2=ab$, and we are done.

PP.20771. If $a+b+c=\sqrt{\frac{3k+1}{k(k+1)}}$ and $a^2+b^2+c^2=\frac{1}{k}$, then

$$\sum_{k=1}^{n}\left((a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2\right)=\frac{n(n+2)}{(n+1)^2}.$$

Solution. Since $ab+bc+ca=\frac{(a+b+c)^2-(a^2+b^2+c^2)}{2}=\frac{1}{k+1}$, we have:

$$(a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2=$$
$$=a^4+b^4+c^4-2abc(a+b+c)+b^2c^2+c^2a^2+a^2b^2=$$
$$=(a^2+b^2+c^2)^2-2\sum a^2b^2-2abc\sum a+\left(\sum bc\right)^2-2abc\sum a=$$
$$=\frac{1}{k^2}-2\left(\sum ab\right)^2+4abc\sum a+\left(\sum bc\right)^2=\frac{1}{k^2}-\frac{1}{(k+1)^2}.$$

Therefore,

$$\sum_{k=1}^{n}\left((a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2\right)=\sum_{k=1}^{n}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)=1-\frac{1}{(n+1)^2}=$$
$=\frac{n(n+2)}{(n+1)^2}$, and we are done.

PP.20785. If $a_i>0$ $(i=1,2,...,n)$, $k\in\{1,2,...,n\}$ such that

$$\sum_{cyclic}a_1a_2...a_k=1\text{, then }\sum_{cyclic}\frac{a_1a_2...a_k}{\left((1+a_1^k)(1+a_2^k)...(1+a_k^k)\right)^{\frac{1}{k}}}\le\frac{n}{n+1}.$$

Solution. By Hölder's inequality, we obtain that:

$$(1+a_1^k)(1+a_2^k)...(1+a_k^k)\ge\left(1+\sqrt[k]{a_1^ka_2^k...a_k^k}\right)^k=(1+a_1a_2...a_k)^k,$$

so is suffices to show that

$$\sum_{cyclic}\frac{a_1a_2...a_k}{1+a_1a_2...a_k}\le\frac{n}{n+1}\Leftrightarrow\sum_{cyclic}\left(\frac{1+a_1a_2...a_k}{1+a_1a_2...a_k}-\frac{1}{1+a_1a_2...a_k}\right)\le\frac{n}{n+1}$$
$$\Leftrightarrow n-\sum_{cyclic}\frac{1}{1+a_1a_2...a_k}\le\frac{n}{n+1}\Leftrightarrow\sum_{cyclic}\frac{1}{1+a_1a_2...a_k}\ge\frac{n^2}{n+1}.$$

But, by Bergström's inequality (or AM-HM inequality) we have that:

$$\sum_{cyclic}\frac{1}{1+a_1a_2...a_k}\ge\frac{n^2}{n+\sum_{cyclic}a_1a_2...a_k}=\frac{n^2}{n+1}\text{, and the proof is complete.}$$

PP.20793. If $x,y,z>0$, $n\in N$ and $(xy)^{n-1}+(yz)^{n-1}+(zx)^{n-1}=1$, then

$$x^{2n+1}+y^{2n+1}+z^{2n+1}\ge xyz.$$

Solution. By well-known inequality $\sum a^2\ge\sum ab$, we have:

$$x^{2n-2}+y^{2n-2}+z^{2n-2}=(x^{n-1})^2+(y^{n-1})^2+(z^{n-1})^2\ge(xy)^{n-1}+(yz)^{n-1}+(zx)^{n-1}=1.$$

By Chebyshev' s inequality and AM-GM inequality yields that:

$$x^{2n+1}+y^{2n+1}+z^{2n+1}=x^{2n-2}\cdot x^3+y^{2n-2}\cdot y^3+z^{2n-2}\cdot z^3\ge$$

$\geq \frac{1}{3}\left(x^{2n-2}+y^{2n-2}+z^{2n-2}\right)\left(x^3+y^3+z^3\right) \geq \frac{1}{3}\cdot 1\cdot 3xyz = xyz$, and the proof is complete.

PP.20795. If $a,b,c>0$ then prove that the inequalities $\sum \frac{a}{2a+b+c} \leq \frac{3}{4}$ and $\sum \frac{a}{b+c} \geq \frac{3}{2}$ are equivalent.

Solution. We have successively:

$$\sum \frac{a}{2a+b+c} \leq \frac{3}{4} \Leftrightarrow \frac{3}{2} - \sum \frac{a}{2a+b+c} \geq \frac{3}{2} - \frac{3}{4} \Leftrightarrow \sum \left(\frac{1}{2} - \frac{a}{2a+b+c}\right) \geq \frac{3}{4}$$

$\Leftrightarrow \sum \frac{b+c}{2(2a+b+c)} \geq \frac{3}{4} \Leftrightarrow \sum \frac{b+c}{(a+b)+(a+c)} \geq \frac{3}{2} \Leftrightarrow \sum \frac{x}{y+z} \geq \frac{3}{2}$, where we denote $b+c=x, a+c=y, a+b=z$, and the proof is complete.

PP.20809. If $x,y,z>0$, then $\frac{1}{4}\sum \frac{1}{x+y} \leq \sum \frac{x}{3y^2+2yz+3z^2} \leq \frac{\sum x^2}{8xyz}$.

Solution. By Bergström's inequality we obtain:

$$\sum \frac{x}{3y^2+2yz+3z^2} = \sum \frac{x^2}{3xy^2+2xyz+3xz^2} \geq \frac{\left(\sum x\right)^2}{3\sum x^2y + 6xyz + 3\sum xy^2} =$$

$$= \frac{\left(\sum x\right)^2}{3(x+y)(y+z)(z+x)} = \frac{4\sum x^2 + 8\sum xy}{12(x+y)(y+z)(z+x)} \geq \frac{3\sum x^2 + 9\sum xy}{12(x+y)(y+z)(z+x)} =$$

$$= \frac{x^2 + \sum xy + y^2 + \sum xy + z^2 + \sum xy}{4(x+y)(y+z)(z+x)} = \frac{(x+y)(x+z)+(y+x)(y+z)+(z+x)(z+y)}{4(x+y)(y+z)(z+x)} =$$

$$= \frac{1}{4}\sum \frac{1}{x+y}.$$

For the right inequality we apply AM-GM inequality, i.e.

$3y^2+2yz+3z^2 \geq 8yz$, and then $\sum \frac{x}{3y^2+2yz+3z^2} \leq \sum \frac{x}{8yz} = \frac{\sum x^2}{8xyz}$.

The proof is complete.

PP.20811. In all triangle ABC holds

$$\frac{5s^2+r^2+4Rr}{8s(s^2+r^2+2Rr)} \leq \sum \frac{a}{3b^2+2bc+3c^2} \leq \frac{s^2-r^2-4Rr}{16sRr}.$$

Solution. Using the facts that $\sum a^2 = 2(s^2-r^2-4Rr)$, $abc=4sRr$ and $\sum ab = s^2+r^2+4Rr$, we obtain that:

$5s^2+r^2+4Rr=\left(\sum a\right)^2+\sum ab=\sum a^3+3\sum ab$;

$2s(s^2+r^2+2Rr)=2s(s^2+r^2+4Rr)-4Rrs=\sum a\sum ab-abc=$

$=(a+b)(b+c)(c+a)$, and now the inequalities from the statement yields using PP.20809.

PP.20819. If $a,b,c>0$, then $\sum\frac{a^2}{b^2}\sum\frac{a^4}{b^4}\sum\frac{a^8}{b^8}\sum\frac{a^{16}}{b^{16}}\ge\left(\sum\frac{a}{b}\right)^2\left(\sum\frac{a}{c}\right)^2$.

Solution. We take $n=4$ in PP.20820, or:

$\sum\frac{a^2}{b^2}\ge\sum\frac{a}{c}$; $\sum\frac{a^4}{b^4}\ge\sum\frac{a^2}{c^2}\ge\sum\frac{a}{b}$; $\sum\frac{a^8}{b^8}\ge\sum\frac{a^4}{c^4}\ge\sum\frac{a^2}{b^2}\ge\sum\frac{a}{c}$; and

$\sum\frac{a^{16}}{b^{16}}\ge\sum\frac{a^8}{c^8}\ge\sum\frac{a^4}{b^4}\ge\sum\frac{a^2}{c^2}\ge\sum\frac{a}{b}$; which by multiplying yields the inequality from the statement.

PP.20820. If $a,b,c>0$, then $\prod_{k=1}^{2n}\left(\sum\left(\frac{a}{b}\right)^{2^k}\right)\ge\left(\sum\frac{a}{b}\right)^n\left(\sum\frac{a}{c}\right)^n$.

Solution. Using the inequality $x^2+y^2+z^2\ge xy+yz+zx$, we obtain:

$$\sum\left(\frac{a}{b}\right)^{2^k}=\sum\left(\left(\frac{a}{b}\right)^{2^{k-1}}\right)^2\ge\left(\frac{a}{b}\cdot\frac{b}{c}\right)^{2^{k-1}}+\left(\frac{b}{c}\cdot\frac{c}{a}\right)^{2^{k-1}}+\left(\frac{c}{a}\cdot\frac{a}{b}\right)^{2^{k-1}}=\sum\left(\frac{a}{c}\right)^{2^{k-1}};$$

$$\sum\left(\frac{a}{c}\right)^{2^k}=\sum\left(\left(\frac{a}{c}\right)^{2^{k-1}}\right)^2\ge\left(\frac{a}{c}\cdot\frac{b}{a}\right)^{2^{k-1}}+\left(\frac{b}{a}\cdot\frac{c}{b}\right)^{2^{k-1}}+\left(\frac{c}{b}\cdot\frac{a}{c}\right)^{2^{k-1}}=\sum\left(\frac{a}{b}\right)^{2^{k-1}}, \text{ so:}$$

$\sum\left(\frac{a}{b}\right)^{2^{2n}}\ge\sum\frac{a}{b}$ and $\sum\left(\frac{a}{b}\right)^{2^{2n+1}}\ge\sum\frac{a}{c}$, and by mathematical induction easily yields the desired result. The proof is complete.

PP.20824. In all triangle ABC holds $\sum\frac{m_a^{k+1}}{m_b+m_c-m_a}\ge\sum m_a^k$ for all $k\in N$.

Solution. We have:

$$\sum\frac{m_a^{k+1}}{m_b+m_c-m_a}-\sum m_a^k=\sum\left(\frac{m_a^{k+1}}{m_b+m_c-m_a}-m_a^k\right)=$$

$$=\sum\left(\frac{m_a^k(m_a-m_b)}{m_b+m_c-m_a}+\frac{m_a^k(m_a-m_c)}{m_b+m_c-m_a}\right)=\sum\frac{m_a^k(m_a-m_b)}{m_b+m_c-m_a}+\sum\frac{m_a^k(m_a-m_c)}{m_b+m_c-m_a}=$$

$$=\sum\frac{m_a^k(m_a-m_b)}{m_b+m_c-m_a}+\sum\frac{m_a^k(m_b-m_a)}{m_c+m_a-m_b}=$$

$$=\sum(m_a-m_b)\cdot\frac{m_a^k(m_c+m_a-m_b)-m_b^k(m_b+m_c-m_a)}{(m_b+m_c-m_a)(m_c+m_a-m_b)}=$$

$$=\sum(m_a-m_b)\cdot\frac{(m_a-m_b)(m_a^k+m_b^k)+m_c(m_a^k-m_b^k)}{(m_b+m_c-m_a)(m_c+m_a-m_b)}=$$

$$=\sum\frac{(m_a-m_b)^2(m_a^k+m_b^k)+m_c(m_a-m_b)(m_a^k-m_b^k)}{(m_b+m_c-m_a)(m_c+m_a-m_b)}\geq 0,$$ because the expressions m_a-m_b and $m_a^k-m_b^k$ have the same sign. The proof is complete.

PP.20825. In all triangle ABC holds $5s^2<3r^2+2Rr$.

Solution. The inequality from the enunciation is not true, for. e.g. if triangle ABC is equilateral with the length of side equal with 1 we should have

$$5\cdot\frac{9}{4}<3\cdot\left(\frac{\sqrt{3}}{6}\right)^2+2\cdot\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{6}\Leftrightarrow\frac{45}{4}<\frac{1}{4}+\frac{1}{3},$$ which is not true.

PP.20830. In all triangle ABC holds $\sum\frac{a^2}{s-a}\geq 4s$.

Solution. By the inequality of Harald Bergström we have:

$$\sum\frac{a^2}{s-a}\geq\frac{(a+b+c)^2}{3s-a-b-c}=\frac{4s^2}{s}=4s,$$ and we are done.

PP.20835. In all triangle ABC holds $\sum\left(\frac{m_a}{\cos\frac{A}{2}}\right)^2\geq 18Rr$.

Solution. Using $m_a\geq\sqrt{s(s-a)}$, $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$, $Rr=\frac{abc}{4s}$ it suffices to show that:

$$\sum\left(\frac{\sqrt{s(s-a)}\cdot\sqrt{bc}}{\sqrt{s(s-a)}}\right)^2\geq\frac{9abc}{2s}\Leftrightarrow\sum a\sum bc\geq 9abc.$$

The last inequality yields from $\sum a\geq 3\cdot\sqrt[3]{abc}$, $\sum bc\geq 3\cdot\sqrt[3]{a^2b^2c^2}$ (AM-GM inequality) by multiplying. The proof is complete.

PP.20836. In all triangle ABC holds $\sum m_a \le \frac{3}{2}\sqrt{\frac{R(s^2+r^2+Rr)}{2r}}$.

Solution. Let F be the area of triangle ABC. We use:

$$\left(\sum x\right)^2 \le 3\sum x^2 \,;\, \sum m_a^2 = \frac{3(a^2+b^2+c^2)}{4} \,;\, ab+bc+ca = s^2+r^2+4Rr.$$

It suffices to show that:

$$\frac{9(a^2+b^2+c^2)}{4} \le \frac{9}{4}\cdot\frac{R}{2r}(ab+bc+ca) \Leftrightarrow a^2+b^2+c^2 \le \frac{abc}{4F}\cdot\frac{s}{2F}(ab+bc+ca)$$

$$\Leftrightarrow 16F^2(a^2+b^2+c^2) \le abc(a+b+c)(ab+bc+ca)$$

$$\Leftrightarrow (2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4)(a^2+b^2+c^2) \le abc(a+b+c)(ab+bc+ca)$$

$$\Leftrightarrow \sum a^6 + \sum a^3b^2c + \sum a^3bc^2 \ge 3a^2b^2c^2 + \sum a^4b^2 + \sum a^2b^4 \qquad (1)$$

By Schur's inequality we have

$\sum a^6 + 3a^2b^2c^2 \ge \sum a^4b^2 + \sum a^2b^4$, and from AM-GM inequality we obtain $\sum a^3b^2c \ge 3a^2b^2c^2$, $\sum a^3bc^2 \ge 3a^2b^2c^2$, which by adding up yields the inequality (1), and the proof is complete.

PP.20848. If $a,b,c>0$, then $\left(\sum\left(\frac{a}{b}\right)^2\right)\left(\sum\left(\frac{a}{b}\right)^4\right) \ge \left(\sum\frac{a}{b}\right)\left(\sum\frac{a}{c}\right)$.

Solution. Applying the well-known inequality $x^2+y^2+z^2 \ge xy+yz+zx$, we obtain

$$\sum\left(\frac{a}{b}\right)^2 \ge \frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b} = \sum\frac{a}{c};$$

$$\sum\left(\frac{a}{b}\right)^4 \ge \left(\frac{a}{b}\cdot\frac{b}{c}\right)^2+\left(\frac{b}{c}\cdot\frac{c}{a}\right)^2+\left(\frac{c}{a}\cdot\frac{a}{b}\right)^2 = \sum\left(\frac{a}{c}\right)^2 \ge \frac{a}{c}\cdot\frac{b}{a}+\frac{b}{a}\cdot\frac{c}{b}+\frac{c}{b}\cdot\frac{a}{c} = \sum\frac{a}{b},$$

and by multiplying we get the desired result.

PP. 20855. If $x_k>0$ $(k=1,2,\ldots n)$, and $\sum_{k=1}^{n} x_k \ge n$, then $\sum_{k=1}^{n} x_k^m \ge n$ for all $m\in N$.

Solution. By Chebyshev' s inequality we obtain:

$$\sum_{k=1}^{n} x_k^m = \sum_{k=1}^{n} x_k^{m-1}\cdot x_k \ge \frac{1}{n}\sum_{k=1}^{n} x_k \sum_{k=1}^{n} x_k^{m-1} \ge \sum_{k=1}^{n} x_k^{m-1} \ge \ldots \ge \sum_{k=1}^{n} x_k \ge n,$$ and we are done.

PP.20861. Prove that for all $n\ge 2$ exist $a_1,a_2,\ldots,a_n \in N$ such that $\sum_{k=1}^{n}\frac{1}{a_k} = \frac{3}{2}$.

Solution. For $n=2$ we have $\frac{1}{1}+\frac{1}{2}=\frac{3}{2}$; for $n=3$ we have $\frac{1}{1}+\frac{1}{4}+\frac{1}{4}=\frac{3}{2}$ and so on using the fact $\frac{1}{2^{n-1}}=\frac{1}{2^n}+\frac{1}{2^n}$, easily follows the conclusion, and we are done.

PP.20865. Prove that:

$$\left((a^3-2a)^2+(2a^2-1)^2\right)\left((a^6-2a^2)^2+(2a^4-1)^2\right)\left((a^9-2a^3)^2+(2a^6-1)^2\right)=$$
$$=(a^6+1)(a^{12}+1)(a^{18}+1) \text{ for all } a\in C.$$

Solution. We have:

$$(a^3-2a)^2+(2a^2-1)^2=a^6-4a^4+4a^2+4a^4-4a^2+1=a^6+1;$$
$$(a^6-2a^2)^2+(2a^4-1)^2=a^{12}-4a^8+4a^4+4a^8-4a^4+1=a^{12}+1;$$
$$(a^9-2a^3)^2+(2a^6-1)^2=a^{18}-4a^{12}+4a^6+4a^{12}-4a^6+1=a^{18}+1.$$

By above we obtain the desired result and we are done.

PP.20866. Prove that:

$$\left((a^3-2a)^2+(2a^2-1)^2\right)\cdot\left((a^5-2a^3+2a)^2+(2a^4-2a^2+1)^2\right)\cdot$$
$$\cdot\left((a^7-2a^5+2a^3-2a)^2+(2a^6-2a^4+2a^2-1)^2\right)=(a^6+1)(a^{10}+1)(a^{14}+1)$$

for all $a\in C$.

Solution. We have:

$$(a^3-2a)^2+(2a^2-1)^2=a^6-4a^4+4a^2+4a^4-4a^2+1=a^6+1;$$
$$(a^5-2a^3+2a)^2+(2a^4-2a^2+1)^2=a^{10}+4a^6+4a^2-4a^8+4a^6-8a^4+4a^8+4a^4+1-$$
$$-8a^6+4a^4-4a^2=a^{10}+1;$$
$$(a^7-2a^5+2a^3-2a)^2+(2a^6-2a^4+2a^2-1)^2=a^{14}+4a^{10}+4a^6+4a^2-4a^{12}+4a^{10}-$$
$$-4a^8-8a^8+8a^6-8a^4+4a^{12}+4a^8+4a^4+1-8a^{10}+8a^8-4a^6-8a^6+4a^4-4a^2=$$
$$=a^{14}+1.$$

From the above we obtain the desired result.

PP.20867. If $a,b,c,d\in C$, then:

$$\sum\left((a+b)^3+(a+c)^3+(a+d)^3+(a-b)^3+(a-c)^3+(a-d)^3\right)=6\left(\sum a\right)\left(\sum a^2\right).$$

Solution. We have:

$$\sum\left((a+b)^3+(a+c)^3+(a+d)^3+(a-b)^3+(a-c)^3+(a-d)^3\right)=$$
$$=\sum(6a^3+6ab^2+6ac^2+6ad^2)=6\left(\sum(a^3+ab^2+ac^2+ad^2)\right)=6\left(\sum a\right)\left(\sum a^2\right),$$ and we are done.

PP.20869. If $x_i>0$ $(i=1,2,...,n)$ and $k\in N$, then $\prod_{cyclic}\frac{x_1^{2k+2}+x_2^{2k+2}}{x_1^{2k}+x_2^{2k}}\geq\prod_{i=1}^{n}x_i^2$.

Solution. We have $\dfrac{a^{2k+2}+b^{2k+2}}{a^{2k}+b^{2k}} \geq ab \Leftrightarrow (a-b)(a^{2k+1}-b^{2k+1}) \geq 0$, true.

Yields that $\prod_{cyclic} \dfrac{x_1^{2k+2}+x_2^{2k+2}}{x_1^{2k}+x_2^{2k}} \geq \prod_{cyclic} x_1 x_2 = \prod_{i=1}^{n} x_i^2$, and the proof is complete.

PP.20870. If $x_i > 0\ (i=1,2,...,n)$ and $k \in N$, then:

$$2\sum_{i=1}^{n} x_i^{2k+2} \geq \sum_{cyclic} x_1 x_2 (x_1^{2k}+x_2^{2k}) \geq \sum_{cyclic} x_1^2 x_2^2 (x_1^{2k-2}+x_2^{2k-2}) \geq ... \geq$$
$$\geq \sum_{cyclic} x_1^k x_2^k (x_1^2+x_2^2) \geq 2\sum_{cyclic} x_1^{k+1} x_2^{k+1} .$$

Solution. For $t \geq 2, t \in N$ and $x, y > 0$ we have:
$x^t + y^t \geq xy(x^{t-2}+y^{t-2}) \Leftrightarrow (x^{t-1}-y^{t-1})(x-y) \geq 0$, true because $x^{t-1}-y^{t-1}$ and $x-y$ have the same sign. Repeatedly applying this inequality we deduce that:

$$2\sum_{i=1}^{n} x_i^{2k+2} = \sum_{cyclic}(x_1^{2k+2}+x_2^{2k+2}) \geq \sum_{cyclic} x_1 x_2 (x_1^{2k}+x_2^{2k}) \geq \sum_{cyclic} x_1^2 x_2^2 (x_1^{2k-2}+x_2^{2k-2}) \geq ... \geq$$
$$\geq ... \geq \sum_{cyclic} x_1^k x_2^k (x_1^2+x_2^2) \geq 2\sum_{cyclic} x_1^{k+1} x_2^{k+1}$$, and we are done.

PP.20875. Prove that the equation $x_1^5+x_2^5+x_3^5+x_4^5+x_5^5=0$ have infinitely many solutions in Z.

Solution. We think something is missing from the statement, because as it is too easy. We have infinitely many solutions on form $(k,-k,0,0,0)$, where $k \in Z$.

PP.20876. Prove that the equation $x_1^7+x_2^7+x_3^7+x_4^7+x_5^7+x_6^7+x_7^7+x_8^7=0$ have infinitely many solutions in Z.

Solution. We think something is missing from the statement, because as it is too easy. We have infinitely many solutions on form $(k,-k,0,0,0,0,0,0)$, where $k \in Z$.

PP.20880. If $x_i > 0\ (i=1,2,...,n)$ and $k \in N^*$, then:

$$2\sum_{i=1}^{n} x_i^{2k+1} \geq \sum_{cyclic} x_1 x_2 (x_1^{2k-1}+x_2^{2k-1}) \geq \sum_{cyclic} x_1^2 x_2^2 (x_1^{2k-3}+x_2^{2k-3}) \geq ... \geq$$
$$\geq \sum_{cyclic} x_1^k x_2^k (x_1+x_2).$$

Solution. For $t \geq 2, t \in N$ and $x, y > 0$ we have:

$x^t + y^t \ge xy(x^{t-2} + y^{t-2}) \Leftrightarrow (x^{t-1} - y^{t-1})(x-y) \ge 0$, true because $x^{t-1} - y^{t-1}$ and $x - y$ have the same sign. Repeatedly applying this inequality like in the solution of PP.20870 we deduce the inequality from the statement.

PP.20892. If $a, b, c > 0$, then $3\sqrt{6}(\sum a^2 - \sum ab) \ge \left(\sum |a-b|\right)^2$.

Solution. Because $3\sqrt{6} > \frac{16}{3}$ and $\sum a^2 - \sum ab \ge 0$, we prove that

(*) $\frac{16}{3}(\sum a^2 - \sum ab) \ge \left(\sum |a-b|\right)^2$, which is stronger than the given inequality.

WLOG, we assume that $a \le b \le c$; let $x, y \ge 0$ such that $b = a + x, c = a + x + y$.

Because

$\sum a^2 - \sum ab = \frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) = \frac{1}{2}\left(x^2 + y^2 + (x+y)^2\right) = x^2 + xy + y^2$,

then (*) is equivalent with

$16(x^2 + xy + y^2) \ge 3(x + y + x + y)^2 \Leftrightarrow 4(x-y)^2 \ge 0$, evidently true, and the proof is complete.

PP.20895. Prove that the sum:

$(n^2 + 2n + 1)^3 + (n^2 + 8n + 16)^3 + (9n^2 + 42n + 49)^3 + (9n^2 + 48n + 64)^3$

is divisible by $2n^2 + 10n + 13$ for all $n \in N$.

Solution. Since $a^{2k+1} + b^{2k+1} = M(a+b)$, we have:

$(n^2 + 2n + 1)^3 + (9n^2 + 48n + 64)^3 = M(10n^2 + 50n + 65) = M\left(5 \cdot (2n^2 + 10n + 13)\right)$,

and $(n^2 + 8n + 16)^3 + (9n^2 + 42n + 49)^3 = M(10n^2 + 50n + 65) = M\left(5 \cdot (2n^2 + 10n + 13)\right)$,

which by adding yields to conclusion, and we are done.

2. Other solutions for some problems from math journal Mathematical Reflections_5_2014

By Nela Ciceu, Roşiori, Bacău, Romania
and
Roxana Mihaela Stanciu, Buzău, Romania

J313. Solve in real numbers the system of equations

$$x(y+z-x^3) = y(z+x-y^3) = z(x+y-z^3) = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

We write the system as follows

$$\begin{cases} xy + xz = 1 + x^4 \\ xy + yz = 1 + y^4 \\ yz + xz = 1 + z^4 \end{cases}.$$

Adding up the equations of the system and applying AM-GM inequality and then well-known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

we obtain

$$2(xy + yz + zx) = 1 + x^4 + 1 + y^4 + 1 + z^4 \geq 2(x^2 + y^2 + z^2) \geq 2(xy + yz + zx).$$

So, we have equality all over, i.e.

$$x = y = z \text{ and } x^2 = y^2 = z^2 = 1, \text{ i.e. } x = y = z = 1 \text{ or } x = y = z = -1.$$

J315. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \geq \sqrt{5} + 2.$$

Proposed by Cosmin Pohoata, Columbia University, USA

Solution:

We prove first that if $xy \geq 0$, then $\sqrt{1+x} + \sqrt{1+y} \geq 1 + \sqrt{1+x+y}$, (1).

Indeed, by successive squaring we have

$$\sqrt{1+x}+\sqrt{1+y}\ge 1+\sqrt{1+x+y}$$

$$\Leftrightarrow 1+x+1+y+2\sqrt{(1+x)(1+y)}\ge 1+1+x+y+2\sqrt{1+x+y}$$

$$\Leftrightarrow (1+x)(1+y)\ge 1+x+y$$

$\Leftrightarrow xy\ge 0$, true.

Applying (1) it suffices to prove that

$$\sqrt{1+4a}+1+\sqrt{1+4b+4c}\ge 2+\sqrt{5}$$

$$\Leftrightarrow \sqrt{1+4a}+\sqrt{5-4a}\ge 1+\sqrt{5}$$

$$\Leftrightarrow 1+4a+5-4a+2\sqrt{(1+4a)(5-4a)}\ge 6+2\sqrt{5}$$

$$\Leftrightarrow (1+4a)(5-4a)\ge 5$$

$\Leftrightarrow a(1-a)\ge 0$, true.

We have equality if and only if one of variables is 1 and the other two variables are 0.

J316. Solve in prime numbers the equation

$$x^3+y^3+z^3+u^3+v^3+w^3=53353.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

We assume that $x\le y\le z\le u\le v\le w$. By the reasons of parity, an odd numbers of these 6 numbers are even (i.e. equals with 2). We deduce easily that $y\le 19$ and $w\le 37$.

By method "trial and error" (or using the computer) we obtain the solution

$$2^3+3^3+5^3+7^3+13^3+37^3=53353.$$

J317. In triangle ABC, the angle-bisector of angle A intersects line BC at D and the circumference of triangle ABC at E. The external angle-bisector of angle A intersects line BC at F and the circumference of triangle ABC at G. Prove that $DG\perp EF$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution:

Suppose that the triangle ABC is not isosceles at A. The point D is the middle point of the arc BC which does not contain the point A, so it belongs to the perpendicular bisector of BC. From similar reasons the point G belongs to the perpendicular bisector of BC.

We have: $FD\perp GE$ and $EA\perp FG$. So, D is the orthocenter of ΔFEG. Hence, $GD\perp FE$.

J318. Determine the functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(x-y)-xf(y)\le 1-x$ for all real numbers x and y.

Proposed by Marcel Chirita, Bucharest, Romania

Solution:

We note that the function $f(x)=1$ satisfies the relation from the statement.

For $x=0$ we obtain $f(-y)\le 1$, so $f(x)\le 1$ for any real number x, (1).

If we take $x=2y$, with $x\ge 1$ (i.e. $y\ge \frac{1}{2}$), then

$$f(y)-2f(y)\le 1-2y \Leftrightarrow (2y-1)(1-f(y))\le 0.$$

Since $2y\ge 1$, yields that $1-f(y)\le 0 \Leftrightarrow f(y)\ge 1$.

By (1), we deduce that for $y\ge \frac{1}{2}$, $f(y)=1$, (2).

Now, let $y<\frac{1}{2}$. Evidently we can choose $x>0$ (e.g. $|y|+\frac{1}{2}$) such that $x-y\ge \frac{1}{2}$. For this x, by (2), we have $f(x-y)=1$ and the relation from the statement becomes $1-xf(y)\le 1-x \Leftrightarrow xf(y)\ge x \Leftrightarrow f(y)\ge 1$, and using (1) again, yields that $f(y)=1$.

In conclusion, the only function which satisfy is the constant function $f(x)=1$.

S313. Let a, b, c be nonnegative real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. Prove that

$$\sqrt{(a+b+1)(c+2)}+\sqrt{(b+c+1)(a+2)}+\sqrt{(c+a+1)(b+2)}\ge 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(a+b+1)(c+2)=(a+b+1)(1+1+c)\ge(\sqrt{a}+\sqrt{b}+\sqrt{c})^2=9, \text{ so}$$

$$\sqrt{(a+b+1)(c+2)}\ge 3,$$

and other two similar which by adding yields the given inequality.

S314. Let p, q, x, y, z be real numbers satisfying

$$x^2y+y^2z+z^2x=p \quad \text{and} \quad xy^2+yz^2+zx^2=q.$$

Evaluate $(x^3-y^3)(y^3-z^3)(z^3-x^3)$ in terms of p and q.

Proposed by Marcel Chirita, Bucharest, Romania

Solution:

We have:

$$p^3-q^3=x^6y^3+y^6z^3+x^3z^6+3x^4y^4z+3x^2y^5z^2+3x^5y^2z^2+3x^4yz^4+3xy^4z^4+$$
$$+3x^2y^2z^5+6x^3y^3z^3-x^3y^6-y^3z^6-x^6z^3-3x^2y^5z^2-3xy^4z^4-3x^4y^4z-$$
$$-3x^5y^2z^2-3x^2y^2z^5-3x^4yz^4-6x^3y^3z^3=x^6y^3-x^3y^6-z^3(x^6-y^6)+z^6(x^3-y^3)=$$
$$=(x^3-y^3)(x^3y^3-x^3z^3-y^3z^3+z^6)=(x^3-y^3)[(y^3-z^3)-z^3(y^3-z^3)]=$$
$$=-(x^3-y^3)(y^3-z^3)(z^3-x^3).$$

Hence, $(x^3-y^3)(y^3-z^3)(z^3-x^3)=q^3-p^3$.

S315. Consider triangle ABC with inradius r. Let M and M' be two points inside the triangle such that $\angle MAB=\angle M'AC$ and $\angle MBA=\angle M'BC$. Denote by d_a,d_b,d_c and d'_a,d'_b,d'_c the distances from M and M' to the sides BC,CA,AB, respectively. Prove that

$$d_ad_bd_cd'_ad'_bd'_c\le r^6.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution:

Vom folosi notatiile obisnuite intr-un triunghi.

Notam:
$$D=AM\cap BC, E=BM\cap CA, F=CM\cap AB,$$
$$D'=AM'\cap BC, E'=BM'\cap CA, F'=CM'\cap AB$$
$$x=\frac{BD}{DC}, y=\frac{CE}{EA}, z=\frac{AF}{FB}, x'=\frac{BD'}{D'C}, y'=\frac{CE'}{E'A}, z'=\frac{AF'}{F'B}$$

Cu teorema lui Van Aubel obtinem

$$\frac{AM}{MD}=\frac{1}{y}+z\Rightarrow\frac{AD}{MD}=\frac{yz+y+1}{y}, \text{ si atunci } \frac{d_a}{h_a}=\frac{MD}{AD}$$

Obtinem
$$d_a=\frac{y}{yz+y+1}\cdot\frac{2sr}{a} \text{ si similar } d'_a=\frac{y'}{y'z'+y'+1}\cdot\frac{2sr}{a}.$$

Aplicand teorema lui Steiner pentru perechile de drepte izogonale (AD, AD'), (BE,BE'), (CF,CF'), rezulta

$$xx'=\frac{c^2}{b^2}, yy'=\frac{a^2}{c^2}, zz'=\frac{b^2}{a^2}.$$

Putem scrie succesiv:

$$d_ad'_a=\frac{y}{yz+y+1}\cdot\frac{2sr}{a}\cdot\frac{y'}{y'z'+y'+1}\cdot\frac{2sr}{a}=$$
$$=\frac{a^2}{c^2}\cdot\frac{4s^2r^2}{a^2}\cdot\frac{1}{yy'zz'+yy'z+yz+yy'z'+yy'+y+y'z'+y'+1}$$

Deoarece
$$c^2(yy'zz'+yy'z+yz+yy'z'+yy'+y+y'z'+y'+1)=$$
$$=c^2\left(\frac{b^2}{c^2}+\frac{a^2}{c^2}\cdot z+\frac{a^2}{c^2}\cdot z'+yz+y'z'+\frac{a^2}{c^2}+y+y'+1\right)=$$
$$=b^2+a^2(z+z')+c^2(yz+y'z')+a^2+c^2(y+y')+c^2\ge$$
$$\ge a^2+b^2+c^2+2a^2\sqrt{zz'}+2c^2\sqrt{yy'zz'}+2c^2\sqrt{yy'}=$$
$$=a^2+b^2+c^2+2ab+2bc+2ac=4s^2$$

rezulta ca $\quad d_ad'_a\le r^2$

de unde obtinem inegalitatea dorita.

S317. Let ABC be an acute triangle inscribed in a circle of radius 1. Prove that

$$\frac{\tan A}{\tan^3 B}+\frac{\tan B}{\tan^3 C}+\frac{\tan C}{\tan^3 A}\geq 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

Since $R=1$, we have $\frac{4}{a^2}-1=\frac{4}{4R^2\sin^2 A}-1=\frac{\cos^2 A}{\sin^2 A}=\cot^2 A$.

Denoting $x=\cot A,\ y=\cot B$, $z=\cot C$, we have $x,y,z>0$ and we must to prove that

$$\frac{y^3}{x}+\frac{z^3}{y}+\frac{x^3}{z}\geq x^2+y^2+z^2.$$

Using *Cauchy-Buniakovski-Schwarz* inequality (''…**SQ** form'') and well-known inequality $x^2+y^2+z^2\geq xy+yz+zx$, we obtain

$$\frac{y^3}{x}+\frac{z^3}{y}+\frac{x^3}{z}=\frac{y^4}{xy}+\frac{z^4}{yz}+\frac{x^4}{zx}\geq\frac{(x^2+y^2+z^2)^2}{xy+yz+zx}\geq$$

$$\geq\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xy+yz+zx}=x^2+y^2+z^2.$$

We have equality if and only if $x=y=z$, i.e. the given triangle is equilateral.

Olympiad problems

O313. Find all positive integers n for which there are positive integers $a_0,a_1,\ldots,a_n$ such that $a_0+a_1+\cdots+a_n=5(n-1)$ and

$$\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}=2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

We use the inequality

$$(*)\left(\sum_{k=0}^{n}a_k\right)\left(\sum_{k=0}^{n}\frac{1}{a_k}\right)\geq(n+1)^2.$$

Since $\sum_{k=0}^{n}a_k=5(n-1)$ and $\sum_{k=0}^{n}\frac{1}{a_k}=2$, we obtain

$$10(n-1)\geq(n+1)^2 \text{ which yields that } n\in\{2,3,4,5,6\}.$$

We assume that $a_0\leq a_1\leq a_2\leq a_3\leq a_4\leq a_5\leq a_6$.

If $a_0 = 1$, then by (*) it must that

$$(5(n-1)-1) \geq n^2 \Leftrightarrow n^2 - 5n + 6 \leq 0 \Leftrightarrow 2 \leq n \leq 3.$$

1. For $n = 2$ we have $a_0 + a_1 + a_2 = 5$, $\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} = 2$. We obtain $2 \leq \frac{3}{a_0}$, so $a_0 = 1$. Yields the solution

$$1 + 2 + 2 = 5,\ 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

2. For $n = 3$ we have $a_0 + a_1 + a_2 + a_3 = 10$, $\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 2$. We obtain $2 \leq \frac{4}{a_0}$, so $a_0 \leq 2$. For $a_0 = 1$ yields the solution

$$1 + 3 + 3 + 3 = 10, 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 2.$$

3. For $n = 4$ we have $a_0 + a_1 + a_2 + a_3 + a_4 = 15$, $\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} = 2$. We obtain $2 \leq \frac{5}{a_0}$, so $a_0 = 2$. Yields the solution

$$2 + 2 + 2 + 3 + 6 = 15, \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2.$$

4. For $n = 5$ we have $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 20$, $\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} = 2$. We obtain $2 \leq \frac{6}{a_0}$, so $a_0 \leq 3$. We take $a_0 = 2$ and we deduce that $\frac{3}{2} \leq \frac{5}{a_1}$, so $a_1 \leq 3$. We take $a_1 = 2$ and yields the solution

$$2 + 2 + 4 + 4 + 4 + 4 = 20, \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 2.$$

5. For $n = 6$ we have $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 25$, $\frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} = 2$. We obtain $2 \leq \frac{7}{a_0}$, so $a_0 \leq 3$.

We take $a_0 = 3$ and we deduce that $\frac{5}{3} \leq \frac{6}{a_1}$, so $a_1 \leq 5$.

We take $a_1 = 3$ and yields the solution

$$3 + 3 + 3 + 4 + 4 + 4 + 4 = 25, \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 2.$$

In conclusion, the values of n are 2, 3, 4, 5, 6.

O315. Let a, b, c be positive real numbers. Prove that

$$(a^3+3b^2+5)(b^3+3c^2+5)(c^3+3a^2+5) \geq 27(a+b+c)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution:

Applying the inequalities $a^3+2=a^3+1+1\geq 3a$, $b^3+2\geq 3b$, $c^3+2\geq 3c$ (which yields by AM-GM inequality) and then by Hölder's inequality we obtain

$$(a^3+3b^2+5)(b^3+3c^2+5)(c^3+3a^2+5)\geq 27(a+b^2+1)(1+b+c^2)(a^2+1+c)\geq$$

$$\geq 27\left(\sqrt[3]{a\cdot 1\cdot a^2}+\sqrt[3]{b^2\cdot b\cdot 1}+\sqrt[3]{1\cdot c^2\cdot c}\right)^3=27(a+b+c)^3,$$ q.e.d.

3. Progresii aritmetice cu şiruri de numere naturale

Prof. Ciobîcă Constantin- Colegiul Vasile Lovinescu , Fălticeni
Prof. Ciobîcă Elena

1. Fie $(a_n)_{n\in N^*}$ un şir de numere reale în progresie aritmetică de raţie r şi şirurile $\left(b_p^m\right)_{p\in N^*}\in N; m=\overline{1,s}$ de numere naturale de raţii $r_m; m=\overline{1,s}$. Demonstraţi egalitatea:

$$\frac{1}{\sum_{k=1}^{n}\left(a_{b_k^1}+a_{b_k^2}+\ldots+a_{b_k^s}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+1}^1}+a_{b_{k+1}^2}+\ldots a_{b_{k+1}^s}\right)}+$$

$$+\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+1}^t}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+2}^t}\right)}+$$

$$+\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i}^{t}}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^{t}}\right)}=$$

$$=\frac{i+1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k}^{t}}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^{t}}\right)},\forall i\in N,\forall n\in N^{*},s\in N^{*}$$

Rezolvare:

$$\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+1}^{t}}\right)-\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k}^{t}}\right)=\sum_{k=1}^{n}\left(\sum_{t=1}^{s}\left(a_{b_{k+1}^{t}}-a_{b_{k}^{t}}\right)\right)=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[a_{1}+\left(b_{k+1}^{t}-1\right)r-a_{1}-\left(b_{k}^{t}-1\right)r\right]\right\}=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[\left(b_{k+1}^{t}-b_{k}^{t}\right)r\right]\right\}=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[b_{1}^{t}+kr_{t}-b_{1}^{t}-(k-1)r_{t}\right]r\right\}=\sum_{k=1}^{n}\left(\sum_{t=1}^{s}r\cdot r_{t}\right)=\sum_{k=1}^{n}\left(r_{1}+r_{2}+...+r_{s}\right)r=$$

$$=n\cdot\left(r_{1}+r_{2}+....+r_{s}\right)\cdot r.$$

$$\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k}^{1}}+a_{b_{k}^{2}}+...+a_{b_{k}^{s}}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+1}^{1}}+a_{b_{k+1}^{2}}+...a_{b_{k+1}^{s}}\right)}=$$

$$=\frac{1}{nr\left(r_{1}+r_{2}+....+r_{s}\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k}^{t}}\right)}-\frac{1}{\sum_{k=1}^{t}\left(\sum_{t=1}^{s}a_{b_{k+1}^{t}}\right)}\right]$$

$$\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+2}^{t}}\right)-\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+1}^{t}}\right)=\sum_{k=1}^{n}\left(\sum_{t=1}^{s}\left(a_{b_{k+2}^{t}}-a_{b_{k+1}^{t}}\right)\right)=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[a_1+\left(b_{k+2}^{t}-1\right)r-a_1-\left(b_{k+1}^{t}-1\right)r\right]\right\}=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[\left(b_{k+2}^{t}-b_{k+1}^{t}\right)r\right]\right\}=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[b_1^t+(k+1)r_t-b_1^t-kr_t\right]r\right\}=\sum_{k=1}^{n}\left(\sum_{t=1}^{s}r_t\cdot r\right)=\sum_{k=1}^{n}\left(r_1+r_2+\ldots+r_s\right)r=$$

$$=nr\left(r_1+r_2+\ldots.+r_s\right)$$

$$\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+1}^{t}}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+2}^{t}}\right)}=$$

$$=\frac{1}{nr\left(r_1+r_2+\ldots+r_s\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+1}^{t}}\right)}-\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+2}^{t}}\right)}\right]$$

$$\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^{t}}\right)-\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i}^{t}}\right)=\sum_{k=1}^{n}\left(\sum_{t=1}^{s}\left(a_{b_{k+i+1}^{t}}-a_{b_{k+i}^{t}}\right)\right)=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[a_1+\left(b_{k+i+1}^{t}-1\right)r-a_1-\left(b_{k+i}^{t}-1\right)r\right]\right\}=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[\left(b_{k+i+1}^{t}-b_{k+i}^{t}\right)r\right]\right\}=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[b_1^t-(k+i)r_t-b_1^t-(k+i-1)r_t\right]r\right\}=\sum_{k=1}^{n}\left[\sum_{t=1}^{s}r_t\cdot r\right]=nr\cdot\left(r_1+r_2+\ldots.+r_s\right)$$

$$\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i}^{t}}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^{t}}\right)}=$$

$$=\frac{1}{nr\left(r_1+r_2+\ldots.r_s\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i}^{t}}\right)}-\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^{t}}\right)}\right]$$

$$S=\frac{1}{nr(r_1+r_2+....+r_s)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_k^t}\right)}-\frac{1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^t}\right)}\right]$$

$$\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^t}\right)-\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_k^t}\right)=\sum_{k=1}^{n}\left[\sum_{t=1}^{s}\left(a_{b_{k+i+1}^t}-a_{b_k^t}\right)\right]=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[a_1+\left(b_{k+i+1}^t-1\right)r-a_1-\left(b_k^t-1\right)r\right]\right\}=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[b_{k+i+1}^t-b_k^t\right]r\right\}=$$

$$=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}\left[b_1^t+(k+i)r_t-b_1^t-(k-1)r_t\right]r\right\}=\sum_{k=1}^{n}\left\{\sum_{t=1}^{s}(i+1)r_tr\right\}=$$

$$=\sum_{k=1}^{n}r(i+1)(r_1+r_2+...+r_s)=nr(i+1)\cdot(r_1+r_2+...+r_s)$$

$$S=\frac{1}{nr(r_1+r_2+...+r_s)}\cdot\frac{nr(i+1)(r_1+r_2+...+r_s)}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_k^t}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^t}\right)}$$

$$=\frac{i+1}{\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_k^t}\right)\cdot\sum_{k=1}^{n}\left(\sum_{t=1}^{s}a_{b_{k+i+1}^t}\right)},\forall i\in N,\forall n\in N^*,s\in N^*.$$

2. Fie $(a_n)_{n\in N^*}$ - un şir de numere reale în progresie aritmetică de raţie r şi şirurile :

- $(b_m)_{m\in N^*}$ şir de numere naturale în progresie aritmetică de raţie r_1
- $(c_p)_{p\in N^*}$ şir de numere naturale în progresie aritmetică de raţie r_2
- $(d_s)_{s\in N^*}$ şir de numere reale în progresie aritmetică de raţie r_3

Demonstraţi egalitatea:

$$\frac{1}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)}+$$

$$+\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+2}}+a_{c_{k+2}}+a_{d_{k+2}}\right)}+$$

$$+\ldots+\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+i}}+a_{c_{k+i}}+a_{d_{k+i}}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)}$$

$$=\frac{i+1}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)},\forall i\in N,\forall n\in N^*$$

Rezolvare:

$$-\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)+\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)=$$

$$=a_{b_{n+1}}-a_{b_1}+a_{c_{n+1}}-a_{c_1}+a_{d_{n+1}}-a_{d_1}=$$

$$=a_1+(b_{n+1}-1)r-a_1-(b_1-1)r+a_1+(c_{n+1}-1)r-a_1-(c_1-1)r+a_1+(d_{n+1}-1)r-$$
$$-a_1-(d_1-1)r=d_{n+1}r-d_1r+c_{n+1}r-c_1r+b_{n+1}r-b_1r=(d_1+nr_3)r-d_1r+(c_1+nr_2)r-c_1r+$$

$$+(b_1+nr_1)r-b_1r=nr_3r+nr_2r+nr_1r=nr(r_1+r_2+r_3)$$

$$\frac{1}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)}=$$

$$=\frac{1}{nr(r_1+r_2+r_3)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)}-\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)}\right]$$

$$\sum_{k=1}^{n}\left(a_{b_{k+2}}+a_{c_{k+2}}+a_{d_{k+2}}\right)-\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)=$$

$$=a_{b_{n+2}}-a_{b_2}+a_{c_{n+2}}-a_{c_2}+a_{d_{n+2}}-a_{d_2}=$$

$$=a_1+\left(b_{n+2}-1\right)r-a_1-\left(b_2-1\right)r+a_1+\left(c_{n+2}-1\right)r-a_1-\left(c_2-1\right)r+$$

$$+a_1+\left(d_{n+2}-1\right)r-a_1-\left(d_2-1\right)r=\left(b_{n+2}-b_2\right)r+\left(c_{n+2}-c_2\right)r+\left(d_{n+2}-d_2\right)r=$$

$$=\left[b_1+(n+1)r_1-b_1-r_1\right]+\left[c_1+(n+1)r_2-c_1-r_2\right]+\left[d_1+(n+1)r_3-d_1-r_3\right]=$$

$$=nr_1r+nr_2r+nr_3r=nr\left(r_1+r_2+r_3\right)$$

$$\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+2}}+a_{c_{k+2}}+a_{d_{k+2}}\right)}=$$

$$=\frac{1}{nr\left(r_1+r_2+r_3\right)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+1}}+a_{c_{k+1}}+a_{d_{k+1}}\right)}-\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+2}}+a_{c_{k+2}}+a_{d_{k+2}}\right)}\right]$$

$$\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)-\sum_{k=1}^{n}\left(a_{b_{k+i}}+a_{c_{k+i}}+a_{d_{k+i}}\right)=$$

$$=a_{b_{n+i+1}}+a_{c_{n+i+1}}+a_{d_{n+i+1}}-a_{b_{i+1}}-a_{c_{i+1}}-a_{d_{i+1}}=$$

$$=a_1+\left(b_{n+i+1}-1\right)r-a_1-\left(b_{i+1}-1\right)r+a_1+\left(c_{n+i+1}-1\right)r-a_1-\left(c_{i+1}-1\right)r+$$

$$+a_1-\left(d_{n+i+1}-1\right)r-a_1-\left(d_{i+1}-1\right)r=\left(b_{n+i+1}-b_{i+1}\right)r+\left(c_{n+i+1}-c_{i+1}\right)r+\left(d_{n+i+1}-d_{i+1}\right)r=$$

$$=\left[b_1+(n+i)r_1-b_1-ir_1\right]r+\left[c_1+(n+i)r_2-c_1-ir_2\right]r+$$

$$+\left[d_1+(n+i)r_3-d_1-ir_3\right]r=nr_1r+nr_2r+nr_3r=nr\left(r_1+r_2+r_3\right)$$

$$\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+i}}+a_{c_{k+i}}+a_{d_{k+i}}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)}=$$

$$=\frac{1}{nr(r_1+r_2+r_3)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+i}}+a_{c_{k+i}}+a_{d_{k+i}}\right)}-\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)}\right]$$

$$S=\frac{1}{nr(r_1+r_2+r_3)}\cdot\left[\frac{1}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)}-\frac{1}{\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)}\right]$$

$$\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)-\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)=$$

$$=\sum_{k=1}^{n}\left[\left(a_{b_{k+i+1}}-a_{b_k}\right)+\left(a_{c_{k+i+1}}-a_{c_k}\right)+\left(a_{d_{k+i+1}}-a_{d_k}\right)\right]=$$

$$=\sum_{k=1}^{n}\left[(b_{k+i+1}-b_k)r+(c_{k+i+1}-c_k)r+(d_{k+i+1}-d_k)r\right]=$$

$$=\sum_{k=1}^{n}\left\{\left[b_1+(k+i)r_1-b_1-(k-1)r_1\right]r+\left[c_1+(k+i)r_2-c_1-(k-1)r_2\right]r+(i+1)r_3r\right\}=$$

$$=\sum_{k=1}^{n}\left[(i+1)r_1r+(i+1)r_2r+(i+1)r_3r\right]=n(i+1)(r_1+r_2+r_3)$$

$$S=\frac{1}{nr(r_1+r_2+r_3)}\cdot\frac{n(i+1)(r_1+r_2+r_3)}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)}$$

$$=\frac{i+1}{\sum_{k=1}^{n}\left(a_{b_k}+a_{c_k}+a_{d_k}\right)\cdot\sum_{k=1}^{n}\left(a_{b_{k+i+1}}+a_{c_{k+i+1}}+a_{d_{k+i+1}}\right)},\forall i\in N,\forall n\in N^*$$

3. Fie $(a_n)n\geq 1$un şir de numere reale în progresie aritmetică de raţie r şi $(b_m)_{m\geq 1}\in N$ un şir de numere naturale în progresie aritmetică de raţie q. Demonstraţi egalitate:

$$\frac{1}{\sum_{k=1}^{n} a_{b_k} \cdot \sum_{k=1}^{n} a_{b_{k+1}}} + \frac{1}{\sum_{k=1}^{n} a_{b_{k+1}} \cdot \sum_{k=1}^{n} a_{b_{k+2}}} + \ldots + \frac{1}{\sum_{k=1}^{n} a_{b_{k+m}} \cdot \sum_{k=1}^{n} a_{b_{k+m+1}}} =$$

$$= \frac{m+1}{\sum_{k=1}^{n} a_{b_k} \cdot \sum_{k=1}^{n} a_{b_{k+m+1}}}; \forall m \in N, \forall n \in N^*$$

Rezolvare:

$$\sum_{k=1}^{n} a_{b_k} = a_{b_1} + a_{b_2} + \ldots + a_{b_n}$$

$$\sum_{k=1}^{n} a_{b_{k+1}} = a_{b_2} + a_{b_3} + \ldots + a_{b_{n+1}}$$

$$\sum_{k=1}^{n} a_{b_{k+1}} - \sum_{k=1}^{n} a_{b_k} = a_{b_{n+1}} - a_{b_1} =$$

$$= a_1 + (b_{n+1} - 1)r - a_1 - (b_1 - 1)r = b_{n+1}r - b_1 r = (b_1 + nq)r - b_1 r = nqr$$

$$\frac{1}{\sum_{k=1}^{n} a_{b_k} \cdot \sum_{k=1}^{n} a_{b_{k+1}}} = \frac{1}{nqr} \cdot \left(\frac{1}{\sum_{k=1}^{n} a_{b_k}} - \frac{1}{\sum_{k=1}^{n} a_{b_{k+1}}} \right)$$

$$\sum_{k=1}^{n} a_{b_{k+2}} = a_{b_3} + a_{b_4} + \ldots. + a_{b_{n+2}}$$

$$\sum_{k=1}^{n} a_{b_{k+1}} = a_{b_2} + a_{b_3} + \ldots + a_{b_{n+1}}$$

$$\sum_{k=1}^{n} a_{b_{k+2}} - \sum_{k=1}^{n} a_{b_{k+1}} = a_{b_{n+2}} - a_{b_2} =$$

$$= a_1 + (b_{n+2} - 1)r - a_1 - (b_2 - 1)r = (b_{n+2} - b_2)r = [b_1 + (n+1)q]r - (b_1 + q)r =$$

$$= b_1 r - nqr + qr - b_1 r - qr = nqr$$

$$\frac{1}{\sum_{k=1}^{n} a_{b_{k+1}} \cdot \sum_{k=1}^{n} a_{b_{k+2}}} = \frac{1}{nqr} \cdot \left(\frac{1}{\sum_{k=1}^{n} a_{b_{k+1}}} - \frac{1}{\sum_{k=1}^{n} a_{b_{k+2}}} \right)$$

$$\sum_{k=1}^{n} a_{b_{k+m+1}} - \sum_{k=1}^{n} a_{b_{k+m}} = a_{b_{m+n+1}} - a_{b_{m+1}} =$$

$$= a_1 + (b_{m+n+1} - 1)r - a_1 - (b_{m+1} - 1)r = b_{m+n+1}r - b_{m+1}r =$$

$$= [b_1 + (m+n)q]r - (b_1 + mq)r = nqr$$

$$\frac{1}{\sum_{k=1}^{n} a_{b_{k+m}} \cdot \sum_{k=1}^{n} a_{b_{k+m+1}}} = \frac{1}{nqr} \bullet \left(\frac{1}{\sum_{k=1}^{n} a_{b_{k+m}}} - \frac{1}{\sum_{k=1}^{n} a_{b_{k+m+1}}} \right)$$

$$S = \frac{1}{nqr} \cdot \left(\frac{1}{\sum_{k=1}^{n} a_{b_k}} - \frac{1}{\sum_{k=1}^{n} a_{b_{k+m+1}}} \right)$$

$$\sum_{k=1}^{n} a_{b_{k+m+1}} - \sum_{k=1}^{n} a_{b_k} = \left(a_{b_{m+2}} - a_{b_1}\right) + \left(a_{b_{m+3}} - a_{b_2}\right) + \ldots + \left(a_{b_{m+n+1}} - a_{b_n}\right)$$

$$= [a_1 + (b_{m+2} - 1)r - a_1 - (b_1 - 1)r] + [a_1 + (b_{m+3} - 1)r - a_1 - (b_2 - 1)r] + \ldots.$$

$$+ [a_1 + (b_{m+n+1} - 1)r - a_1 - (b_n - 1)r] =$$

$$= (b_{m+2}r - b_1 r) + (b_{m+3} - b_2)r + \ldots. + (b_{m+n+1} - b_n)r =$$

$$= \{[b_1 + (m+1)q]r - b_1 r\} + \{[b_1 + (m+2)q]r - b_1 r - qr\} + \ldots +$$

$$+ [b_1 + (m+n)q - b_1 - (n-1)q]r = \underbrace{(m+1)qr + (m+1)qr + \ldots + (m+1)rq}_{de\ n\ ori} = (m+1)nqr$$

$$S = \frac{1}{nqr} \cdot \frac{(m+1)nqr}{\sum_{k=1}^{n} a_{b_k} \cdot \sum_{k=1}^{n} a_{b_{k+m+1}}} = \frac{m+1}{\sum_{k=1}^{n} a_{b_k} \cdot \sum_{k=1}^{n} a_{b_{k+m+1}}}.$$

4.Câteva aplicatii la formula radicalilor compuși

Prof.Roman Rozalia Ioana

Școala Gimnazială Preluca Veche, Maramureș

1. Formula radicalilor compuși

Fie ași b două numere reale, $b \geq 0$. Are loc relația:

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}, \text{ unde } c^2 = a^2 - b$$

Demonstrație:

Rezolvăm pe rând cei doi membri ai egalității.

Notăm $t = \sqrt{a + \sqrt{b}}, t \geq 0 \Leftrightarrow$

$t^2 = a + \sqrt{b}$, unde $t \geq 0 \quad (1)$

Notăm $s = \sqrt{\frac{a+c}{2}} + \sqrt{\frac{a-c}{2}}, s \geq 0 \Leftrightarrow$

$$s = \frac{\sqrt{2(a+c)}}{2} + \frac{\sqrt{2(a-c)}}{2} \Leftrightarrow$$

$$2s = \sqrt{2(a+c)} + \sqrt{2(a-c)} \Leftrightarrow$$

$$(2s)^2 = \left(\sqrt{2(a+c)} + \sqrt{2(a-c)}\right)^2 \Leftrightarrow$$

$$4s^2 = |2(a+c)| + 2\sqrt{2(a+c)} \cdot \sqrt{2(a-c)} + |2(a-c)| \Leftrightarrow$$

$$4s^2 = 2(a+c) + 2(a-c) + 4\sqrt{(a+c)(a-c)} \Leftrightarrow$$

$$4s^2 = 4a + 4\sqrt{(a^2 - c^2)} \Leftrightarrow$$

$$s^2 = a + \sqrt{(a^2 - c^2)}$$

Din relația din ipoteză știm că $c^2 = a^2 - b \Longrightarrow b = a^2 - c^2$.

Deci, relația de mai sus se poate rescrie astfel:

$s^2 = a + \sqrt{b} \quad (2)$

Din egalitățile (1) și (2) obținem că $t^2 = s^2$.

Cum $t, s \geq 0 \Longrightarrow t = s$, adică:

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}} \text{ (q.e.d.)}$$

2. Aplicații rezolvate

Enunțuri:

1. Arătați că $\sqrt{5-2\sqrt{6}}-\sqrt{5+2\sqrt{6}}+\sqrt{16+8\sqrt{3}}$ este număr natural.

2. Calculați:

$$S_1=\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}+\frac{1}{\sqrt{9+2\sqrt{4\cdot 5}}}$$

$$S_2=\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\ldots+\frac{1}{\sqrt{2n+1+2\sqrt{n(n+1)}}}$$

3. Arătați că $n \vdots 101$, unde:

$$\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\ldots+\frac{1}{\sqrt{2n+1+2\sqrt{n(n+1)}}}=99.$$

4. Aflați $n\in\mathbb{N}^*$ astfel încât

$$N=\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\ldots+\frac{1}{\sqrt{2n+1+2\sqrt{n(n+1)}}}$$

să fie număr natural mai mic decât 10.

5. Fie $N=\left[\sqrt{3+2\sqrt{2}}+\sqrt{\left(3+2\sqrt{2}\right)^{-1}}\right]^{-2}+\left[\sqrt{5+2\sqrt{6}}-\sqrt{\left(5+2\sqrt{6}\right)^{-1}}\right]^{-2}+$

$$+\left[\sqrt{7+2\sqrt{12}}+\sqrt{\left(7+2\sqrt{12}\right)^{-1}}\right]^{-2}+\left[\sqrt{9+2\sqrt{20}}-\sqrt{\left(9+2\sqrt{20}\right)^{-1}}\right]^{-2}+$$

$$+\left[\sqrt{11+2\sqrt{30}}+\sqrt{\left(11+2\sqrt{30}\right)^{-1}}\right]^{-2}+$$

$$\left[\sqrt{13+2\sqrt{42}}-\sqrt{\left(13+2\sqrt{42}\right)^{-1}}\right]^{-2}$$

. Arătați că numărul N este rațional.

6. Arătați că pentru orice valoare reală pozitivă a lui a, valoarea expresiei

$$E = \frac{\sqrt{3a+1+\sqrt{6a+1}}}{2} - \frac{1+\sqrt{6a+1}}{2\sqrt{2}}.$$

este constantă.

Soluții:

1. Pentru a arăta că $\sqrt{5-2\sqrt{6}} - \sqrt{5+2\sqrt{6}} + \sqrt{16+8\sqrt{3}}$ este număr natural, aplicăm formula radicalilor compuși pentru fiecare termen în parte:

$$\sqrt{5-2\sqrt{6}} = \sqrt{5-\sqrt{24}} = \sqrt{\frac{5+1}{2}} - \sqrt{\frac{5-1}{2}} = \sqrt{3}-\sqrt{2}$$

$$\sqrt{5+2\sqrt{6}} = \sqrt{5+\sqrt{24}} = \sqrt{\frac{5+1}{2}} + \sqrt{\frac{5-1}{2}} = \sqrt{3}+\sqrt{2}$$

$$\sqrt{16+8\sqrt{3}} = 2\sqrt{4+\sqrt{12}} = 2\left(\sqrt{\frac{4+2}{2}} + \sqrt{\frac{4-2}{2}}\right) = 2\left(\sqrt{3}+1\right)$$

Obținem că:

$$\sqrt{5-2\sqrt{6}} - \sqrt{5+2\sqrt{6}} + \sqrt{16+8\sqrt{3}} =$$

$$\sqrt{3}-\sqrt{2}-\left(\sqrt{3}+\sqrt{2}\right)+2\left(\sqrt{3}+1\right) = 2 \in \mathbb{N}$$

2. Calculăm S_1 aplicând formula radicalilor compuși pentrunumitorii fiecărui termen:

$$S_1 = \frac{1}{\sqrt{3+2\sqrt{2}}} + \frac{1}{\sqrt{5+2\sqrt{6}}} + \frac{1}{\sqrt{7+2\sqrt{12}}} + \frac{1}{\sqrt{9+2\sqrt{4\cdot 5}}}$$

$$\sqrt{3+2\sqrt{2}} = \sqrt{3+\sqrt{8}} = \sqrt{\frac{3+1}{2}} + \sqrt{\frac{3-1}{2}} = \sqrt{2}+1$$

$$\sqrt{5+2\sqrt{6}} = \sqrt{5+\sqrt{24}} = \sqrt{\frac{5+1}{2}} + \sqrt{\frac{5-1}{2}} = \sqrt{3}+\sqrt{2}$$

$$\sqrt{7+2\sqrt{12}}=\sqrt{7+\sqrt{48}}=\sqrt{\frac{7+1}{2}}+\sqrt{\frac{7-1}{2}}=\sqrt{4}+\sqrt{3}$$

$$\sqrt{9+2\sqrt{4\cdot 5}}=\sqrt{9+\sqrt{80}}=\sqrt{\frac{9+1}{2}}+\sqrt{\frac{9-1}{2}}=\sqrt{5}+\sqrt{4}$$

$$S_1=\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{4}}$$

$$S_1=\frac{\sqrt{2}-1}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{\sqrt{4}-\sqrt{3}}{4-3}+\frac{\sqrt{5}-\sqrt{4}}{5-4}$$

$$S_1=\sqrt{2}-1+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\sqrt{5}-\sqrt{4}$$

$$S_1=\sqrt{5}-1$$

Pentru a calcula S_2, se procedează în mod similar, obținându-se:

$$S_2=\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$S_2=\frac{\sqrt{2}-1}{2-1}+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{\sqrt{4}-\sqrt{3}}{4-3}+\cdots+\frac{\sqrt{n+1}-\sqrt{n}}{n+1-n}$$

$$S_2=\sqrt{2}-1+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\cdots+\sqrt{n+1}-\sqrt{n}$$

$$S_2=\sqrt{n+1}-1$$

Reținem acest rezultat încât va fi util în rezolvarea următoarelor cerințe:

$$\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\dots+\frac{1}{\sqrt{2n+1+2\sqrt{n(n+1)}}}=\sqrt{n+1}-1$$

3. Folosim rezultatul obținut cu ajutorul formulei radicalilor compuși în calculul sumei S_2 de mai sus:

$$\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\dots+\frac{1}{\sqrt{2n+1+2\sqrt{n(n+1)}}}$$
$$=\sqrt{n+1}-1$$

Se obține:

$$\sqrt{n+1}-1=99 \Leftrightarrow$$

$$\sqrt{n+1}=100 \Leftrightarrow$$

$n+1=10000 \Leftrightarrow$

$n=9999 \vdots 101$

4. Se folosește rezultatul obținut în calculul sumei S_2:

$$N=\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\dots+\frac{1}{\sqrt{2n+1+2\sqrt{n(n+1)}}}$$
$$=\sqrt{n+1}-1$$

Se obține:

$\sqrt{n+1}-1\in\{0;1;2;3;4;5;6;7;8;9\}$

$\sqrt{n+1}\in\{1;2;3;4;5;6;7;8;9;10\}$

$n+1\in\{1;4;9;16;25;36;49;64;81;100\}$

$n\in\{0;3;8;15;24;35;48;63;80;99\}$

5. *A*plicând formula radicalilor compuși pentru fiecare radical dublu în parte, se obține:

$$N=\left(\sqrt{2}+1+\frac{1}{\sqrt{2}+1}\right)^{-2}+\left(\sqrt{3}+\sqrt{2}-\frac{1}{\sqrt{3}+\sqrt{2}}\right)^{-2}$$
$$+\left(\sqrt{4}+\sqrt{3}+\frac{1}{\sqrt{4}+\sqrt{3}}\right)^{-2}++\left(\sqrt{5}+\sqrt{4}-\frac{1}{\sqrt{5}+\sqrt{4}}\right)^{-2}$$
$$+\left(\sqrt{6}+\sqrt{5}+\frac{1}{\sqrt{6}+\sqrt{5}}\right)^{-2}+\left(\sqrt{7}+\sqrt{6}-\frac{1}{\sqrt{7}+\sqrt{6}}\right)^{-2}$$
$$N=(\sqrt{2}+1+\sqrt{2}-1)^{-2}+(\sqrt{3}+\sqrt{2}-\sqrt{3}+\sqrt{2})^{-2}$$
$$+(\sqrt{4}+\sqrt{3}+\sqrt{4}-\sqrt{3})^{-2}++(\sqrt{5}+\sqrt{4}-\sqrt{5}+\sqrt{4})^{-2}$$
$$+(\sqrt{6}+\sqrt{5}+\sqrt{6}-\sqrt{5})^{-2}+(\sqrt{7}+\sqrt{6}-\sqrt{7}+\sqrt{6})^{-2}$$
$$N=\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\sqrt{4}}\right)^2+\left(\frac{1}{2\sqrt{4}}\right)^2+\left(\frac{1}{2\sqrt{6}}\right)^2+\left(\frac{1}{2\sqrt{6}}\right)^2$$
$$N=\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{24}+\frac{1}{24}$$
$$N=\frac{1}{4}+\frac{1}{8}+\frac{1}{12}=\frac{11}{24}\in\mathbb{Q}$$

6. $\sqrt{3a+1+\sqrt{6a+1}}=\sqrt{\frac{3a+1+3a}{2}}+\sqrt{\frac{3a+1-3a}{2}}=\frac{\sqrt{6a+1}}{\sqrt{2}}+\frac{1}{\sqrt{2}}$

$$E=\frac{\sqrt{6a+1}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}-\frac{1+\sqrt{6a+1}}{2\sqrt{2}}=\frac{1}{\sqrt{2}}-\text{constantă}$$

3. Probleme propuse

1. Să se arate că:

$$\left(\frac{6+4\sqrt{2}}{\sqrt{2}+\sqrt{6+4\sqrt{2}}}+\frac{6-4\sqrt{2}}{\sqrt{2}-\sqrt{6-4\sqrt{2}}}\right)^2=8$$

2. Aflați $x, y \in \mathbb{R}$ astfel încât:

$$x\sqrt{7+2\sqrt{10}}+y\sqrt{7-2\sqrt{10}}=\sqrt{47+2\sqrt{90}}.$$

3. Arătați că $\sqrt{4+2\sqrt{3}}+\sqrt{11-8\sqrt{4-2\sqrt{3}}}\in\mathbb{N}$.

4. Aflați numerele întregi a pentru care

$$\frac{\sqrt{19+8\sqrt{3}}-\sqrt{8+2\sqrt{15}}+\sqrt{6-2\sqrt{5}}}{a-2}\in\mathbb{Z}.$$

5. Aflați $n\in\mathbb{N}^*$ astfel încât să aibă loc egalitatea:

$$\frac{1}{\sqrt{3+2\sqrt{2}}}+\frac{1}{\sqrt{5+2\sqrt{6}}}+\frac{1}{\sqrt{7+2\sqrt{12}}}\ldots+\frac{1}{\sqrt{2n-1+2\sqrt{n(n-1)}}}=49.$$

Bibliografie

[1] Fianu M., Perianu M., Săvulescu D., Matematică pentru clasa a VIII-a, editura Clubul Matematicienilor, București, 2012

[2] Morar, Dana-Maria, Modele de teste pentru Evaluarea Națională: Matematică,Editua Delfin, București, 2013

[3] Ganga, Mircea, Matematică, clasa a IX-a, ediția a 2-a, Editura Mathpress, Ploiești, 2000