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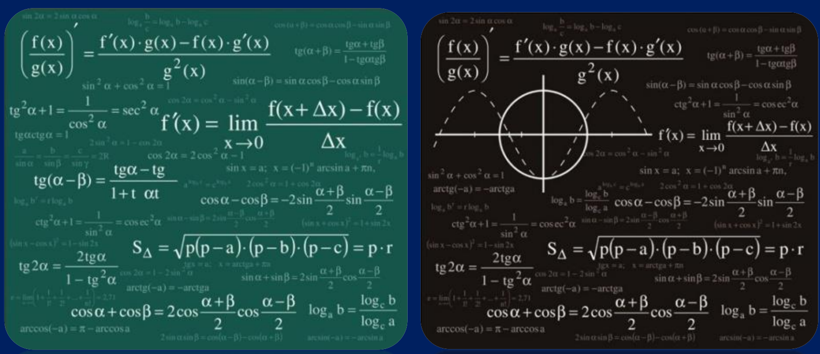

Articole:

1.Some solutions from some problems from Octogon Mathematical Magazine

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PP.21286. If $a,b,c\in(0,1)$, then $\sum\frac{a^2+b^2}{1+ab}\le\frac{3(a^2+b^2+c^2)}{a+b+c}$.

Solution. Since $a,b\in(0,1)$, we have $(1-a)(1-b)\ge 0\Leftrightarrow 1+ab\ge a+b$.

Therefore, it suffices to prove the inequality

$\sum\frac{a^2+b^2}{a+b}\le\frac{3(a^2+b^2+c^2)}{a+b+c}$, which is Problem O:803 from G.M.-B, No. 11/1995, proposed by Ion Bursuc.

Here is a demonstration:

$$\left(\sum\frac{a^2+b^2}{a+b}\right)-\frac{3(a^2+b^2+c^2)}{a+b+c}=\sum\left(\frac{a^2+b^2}{a+b}-\frac{a^2+b^2+c^2}{a+b+c}\right)=$$

$$=\frac{1}{a+b+c}\sum\frac{a^2c+b^2c-ac^2-bc^2}{a+b}=\frac{1}{a+b+c}\sum\frac{ac(a-c)+bc(b-c)}{a+b}=$$

$$=\frac{1}{a+b+c}\left(\sum\frac{ac(a-c)}{a+b}+\sum\frac{bc(b-c)}{a+b}\right)=\frac{1}{a+b+c}\left(\sum\frac{ac(a-c)}{a+b}+\sum\frac{ac(c-a)}{b+c}\right)=$$

$$=\frac{1}{a+b+c}\sum\frac{ac(a-c)(b+c-a-b)}{(a+b)(b+c)}=-\frac{1}{a+b+c}\sum\frac{ac(a-c)^2}{(a+b)(b+c)}\le 0.$$

The proof is complete.

PP.21291. If $a,b,c>0$, then $\left(\sum a\right)\left(\sum a^2\right)\left(\sum a^3\right)\le 9\sum a^6$. (correction)

Solution. By Chebyshev' s inequality, we obtain:

$$9\sum a^6 = 9\sum a^3 \cdot a^3 \geq 9 \cdot \frac{1}{3}\left(\sum a^3\right)\left(\sum a^3\right) = 3 \cdot \left(\sum a^3\right)\left(\sum a^2 \cdot a\right) \geq$$

$\geq 3 \cdot \left(\sum a^3\right) \cdot \frac{1}{3}\left(\sum a^2\right)\left(\sum a\right) = \left(\sum a\right)\left(\sum a^2\right)\left(\sum a^3\right)$, and the proof is complete.

PP.21299. If $a,b,c>0$ then $\sum(2b+c)\sqrt{a^2+ac+c^2} \geq 3\sqrt{3}\sum ab$.

Solution. Because $a^2+ac+c^2 = \frac{3(a+c)^2}{4}+\frac{(a-c)^2}{4}$, we have that:

$a^2+ac+c^2 \geq \frac{3(a+c)^2}{4}$. Using also the inequality $\sum a^2 \geq \sum ab$, we obtain

$$\sum(2b+c)\sqrt{a^2+ac+c^2} \geq \frac{\sqrt{3}}{2}\sum(2b+c)(a+c) = \frac{\sqrt{3}}{2}\sum(2ab+ac+2bc+c^2) =$$

$= \frac{\sqrt{3}}{2}\left(2\sum ab+\sum ab+2\sum ab+\sum a^2\right) \geq \frac{\sqrt{3}}{2} \cdot 6\sum ab = 3\sqrt{3}\sum ab$, and we are done.

PP.21301. If $a,b,c>0$, then:

$$(a+2b+c)(b+2c+a)(c+2a+b) \geq \frac{16}{9}(a+b+c)\left(\sum a^2+3\sum ab\right).$$

Solution. Undoing brackets we obtain:

$$9(2\sum a^3+27\sum a^2b+7\sum ab^2+16abc) \geq 16(\sum a^3+\sum a^2b+\sum ab^2+3\sum a^2b+$$

$$+3\sum ab^2+9abc) \Leftrightarrow 2\sum a^3 \geq \sum a^2b+\sum ab^2 .$$

The last inequality yields by Muirhead's inequality because $(3,0,0) \succ (2,1,0)$, i.e.

$\sum\limits_{sym} a^3 \geq \sum\limits_{sym} a^2b$.

Remark. Other method to prove $2\sum a^3 \geq \sum a^2b+\sum ab^2$ is by AM-GM inequality, i.e.

by adding the following inequalities:

$$a^3+a^3+b^3 \ge 3a^2b;$$

$$b^3+b^3+c^3 \ge 3b^2c;$$

$$c^3+c^3+a^3 \ge 3c^2a;$$

$$a^3+b^3+b^3 \ge 3ab^2;$$

$$b^3+c^3+c^3 \ge 3bc^2;$$

$$c^3+a^3+a^3 \ge 3ca^2.$$

The proof is complete.

PP.21302. If $a,b,c>0$, and $2\sum ab = 3+\sum a$ then $\sum \frac{1}{a^2+b+1} \le 1$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a^2+b+1)(1+b+c^2) \ge (a+b+c)^2$, and then

$$\sum \frac{1}{a^2+b+1} \le \sum \frac{1+b+c^2}{(a^2+b+1)(1+b+c^2)} \le \frac{\sum(1+b+c^2)}{(a+b+c)^2} = \frac{3+\sum a+\sum a^2}{(a+b+c)^2} =$$

$= \frac{2\sum ab+\sum a^2}{(a+b+c)^2} = \frac{(a+b+c)^2}{(a+b+c)^2} = 1$, and the proof is complete**.**

PP.21303. If $a,b,c>0$, then $\sum \frac{a^2+2bc}{(a^2+b+1)(c^2+b+1)} \le 1$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a^2+b+1)(c^2+b+1) \ge (a+b+c)^2$, so

$\sum \frac{a^2+2bc}{(a^2+b+1)(c^2+b+1)} \le \frac{\sum(a^2+2bc)}{(a+b+c)^2} = \frac{(a+b+c)^2}{(a+b+c)^2} = 1$, and we are done.

PP.21304. If $a,b,c>0$, then $\sum \frac{a^3+(2a+3b+3c)bc}{(a^2+b+1)(c^2+b+1)} \le a+b+c$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a^2+b+1)(c^2+b+1) \ge (a+b+c)^2$, and then

$$\sum \frac{a^3+(2a+3b+3c)bc}{(a^2+b+1)(c^2+b+1)} \le \frac{1}{(a+b+c)^2}\left(\sum a^3+3\sum a^2b+3\sum ab^2+6abc\right)=$$

$=\frac{(a+b+c)^3}{(a+b+c)^2}=a+b+c$, and we are done.

PP.21305. If $a,b,c>0$, and $\sum a=\sum ab$ then $\sum \frac{1}{a+b+1} \le 1$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a+b+1)(a+b+c^2) \ge (a+b+c)^2$, and then

$$\sum \frac{1}{a+b+1} \le \sum \frac{a+b+c^2}{(a+b+1)(a+b+c^2)} \le \frac{\sum(a+b+c^2)}{(a+b+c)^2}=\frac{\sum a^2+2\sum a}{(a+b+c)^2}=$$

$=\frac{\sum a^2+2\sum ab}{(a+b+c)^2}=\frac{(a+b+c)^2}{(a+b+c)^2}=1$, and the proof is complete**.**

PP.21306. If $a,b,c>0$, then $\sum \frac{1}{a+b+c^2} \le \frac{3+2\sum a}{(\sum a)^2}$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a+b+1)(a+b+c^2) \ge (a+b+c)^2$, and then

$\sum \frac{1}{a+b+c^2}=\sum \frac{a+b+1}{(a+b+c^2)(a+b+1)} \le \frac{\sum(a+b+1)}{(a+b+c)^2}=\frac{3+2\sum a}{(\sum a)^2}$, and we are done.

PP.21307. If $a,b,c>0$, then $\sum \frac{a^2+2bc}{(a+b+1)(a+b+c^2)} \le 1$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a+b+1)(a+b+c^2)\geq(a+b+c)^2$, and then

$\sum\frac{a^2+2bc}{(a+b+1)(a+b+c^2)}\leq\frac{\sum(a^2+2bc)}{(a+b+c)^2}=\frac{(a+b+c)^2}{(a+b+c)^2}=1$, and we are done.

PP.21308. If $a,b,c>0$, then $\sum\frac{a^3+(2a+3b+3c)bc}{(a+b+1)(a+b+c^2)}\leq a+b+c$.

Solution. By Cauchy-Buniakovski-Schwarz inequality we have:

$(a+b+1)(a+b+c^2)\geq(a+b+c)^2$, and then proceed as in solution of PP.21304.

PP.21309. In all triangle ABC holds $\sum\sqrt[3]{ab(a+b-c)}\leq a+b+c$.

Solution. By AM-GM inequality we obtain

$\sum\sqrt[3]{ab(a+b-c)}\leq\frac{a+b+a+b-c+b+c+b+c-a+c+a+c+a-b}{3}=a+b+c$, and the proof is complete.

PP.21311. Solve in $(0,\infty)$ the following system:

$$\begin{cases}\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}=y+z+t+\frac{4(x-y)^2}{x+y+z}\\ \frac{y^2}{z}+\frac{z^2}{t}+\frac{t^2}{y}=z+t+x+\frac{4(y-z)^2}{y+z+t}\\ \frac{z^2}{t}+\frac{t^2}{x}+\frac{x^2}{z}=t+x+y+\frac{4(z-t)^2}{z+t+x}\\ \frac{t^2}{x}+\frac{x^2}{y}+\frac{y^2}{t}=x+y+z+\frac{4(t-x)^2}{t+x+y}\end{cases}.$$

Solution. At Balkan Mathematical Olympiad, 2005, was proposed the following inequality:

(*) $\qquad \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z+\frac{4(x-y)^2}{x+y+z}$.

We have $\frac{x^2}{y}=2x-y+\frac{(x-y)^2}{y}$, and then by Bergström's inequality we obtain:

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}=2x-y+2y-z+2z-x+\frac{(x-y)^2}{y}+\frac{(z-y)^2}{z}+\frac{(x-z)^2}{x}=$$

$$=x+y+z+\frac{(x-y)^2}{y}+\frac{(z-y)^2}{z}+\frac{(x-z)^2}{x}\geq x+y+z+\frac{(x-y+z-y+x-z)^2}{x+y+z}=$$

$$=x+y+z+\frac{4(x-y)^2}{x+y+z}$$, so (*) is proved.

Adding up the equations of the system we obtain:

$$\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}-x-y-z-\frac{4(x-y)^2}{x+y+z}\right)+\left(\frac{y^2}{z}+\frac{z^2}{t}+\frac{t^2}{y}-z-t-y-\frac{4(y-z)^2}{y+z+t}\right)+$$

$$+\left(\frac{z^2}{t}+\frac{t^2}{x}+\frac{x^2}{z}-t-x-z-\frac{4(z-t)^2}{z+t+x}\right)+\left(\frac{t^2}{x}+\frac{x^2}{y}+\frac{y^2}{t}-x-y-t-\frac{4(t-x)^2}{x+y+t}\right)=0.$$

Yields that we must to have equality in all four inequalities of type (*), i.e. the solutions of the system are (a,a,a,a) with $a\in(0,\infty)$.

PP.21312. If $a,b,c>0$, then $3\sum\frac{a}{b}\geq 7+\frac{2\sum a^2}{\sum ab}$.

Solution. Using Bergström's inequality and well-known $\sum a^2\geq\sum ab$, we obtain

$$3\sum\frac{a}{b}=3\sum\frac{a^2}{ab}\geq\frac{3\left(\sum a\right)^2}{\sum ab}=\frac{6\sum ab+3\sum a^2}{\sum ab}=6+\frac{\sum a^2+2\sum a^2}{\sum ab}\geq$$

$$\geq 6+\frac{\sum ab+2\sum a^2}{\sum ab}=7+\frac{2\sum a^2}{\sum ab}.$$

PP.21313. If $a,b,c>0$, then $\sum a^2\geq\sum ab+\frac{1}{\sqrt{3}}\sum|a-b||a-c|$.

Solution. Analogously as in solution of PP.20892, let $a\leq b\leq c$ and let $x,y\geq 0$ such that

$b = a + x,\ c = a + x + y$. The inequality from the statement becomes

$$x^2 + xy + y^2 \geq \frac{1}{\sqrt{3}}\left(x(x+y) + xy + y(x+y)\right)$$

$\Leftrightarrow (\sqrt{3}-1)x^2 + (\sqrt{3}-3)xy + (\sqrt{3}-1)y^2 \geq 0$, which is true because the discriminant of the equation

$(\sqrt{3}-1)t^2 + (\sqrt{3}-3)t + \sqrt{3} - 1 \geq 0$ is $\Delta = (\sqrt{3}-3)^2 - 4(\sqrt{3}-1)^2 = 2\sqrt{3} - 4 < 0$, and we are done.

PP.21318. Let ABC be a triangle. Prove that:

1) $\prod \frac{a^2+b^2-c^2}{(a+b-c)^2} \leq 1$;

2) $3r^2 + 4Rr + 4R^2 \geq s^2$, are equivalent.

Solution. We prove (1) and (2).

1)In RMT, No. 1/2005, Titu Zvonaru proved that:

$$(a+b+c)^2(a+b-c)^2(a-b+c)^2(-a+b+c)^2 \geq$$

$\geq 3(a^2+b^2+c^2)(a^2+b^2-c^2)(a^2-b^2+c^2)(-a^2+b^2+c^2)$, i.e.

$\prod \frac{a^2+b^2-c^2}{(a+b-c)^2} \leq \frac{(a+b+c)^2}{3(a^2+b^2+c^2)}$, and because $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ yields that (1) is true.

2)The inequality $s^2 \leq 3r^2 + 4Rr + 4R^2$ is the item 5.8 from Bottema.

We let the readers to prove the equivalence.

PP.21319. Let ABC be a triangle. Prove that

1) $\prod(a+b) + \prod(a+b-c) \geq 9abc$

2) $s^2 + 5r^2 \geq 16Rr$ are equivalent.

Solution. We have

$$\prod(a+b) = (a+b+c)(ab+bc+ac) - abc =$$

$= 2s(s^2 + r^2 + 4Rr) - 4Rrs$, and

$$\prod(a+b-c) = 8 \cdot \frac{s(s-a)(s-b)(s-c)}{s} = 8sr^2.$$

So, $\prod(a+b) + \prod(a+b-c) \geq 9abc$

$\Leftrightarrow 2s(s^2 + r^2 + 4Rr) - 4Rrs + 8sr^2 \geq 36Rrs$

$\Leftrightarrow s^2 + r^2 + 4Rr - 2Rr + 4r^2 \geq 18Rr$

$\Leftrightarrow s^2 + 5r^2 \geq 16Rr$.

Note. $s^2 + 5r^2 \geq 16Rr$ is the item 5.8 from Bottema.

PP.21321. If $a,b,c \in [0,1]$, then $\sum(1+a)^2(1+b)^2 + \sum(1-a)^2(1-b)^2 \geq 2\sum(1+ab)^2$.

Solution. We prove that the inequality from the statement for $a,b,c \in R$.

We have:

$$(1+a)^2(1+b)^2 + (1-a)^2(1-b)^2 \geq 2(1+ab)^2 \qquad (*)$$

$\Leftrightarrow 2(a+b)^2$. Writing other two inequalities similar with (*) and adding up we obtain the inequality from the statement.

PP.21331. If $a,b,c > 0$, then $\sum \frac{b}{a^2 - b^2 + c^2} \geq \frac{a^2 + b^2 + c^2}{abc}$.

Solution. Sure, because the denominators to be positive must that a,b,c to be the length sides of an acute triangle. By Harald Bergström's inequality we obtain that:

$$\sum \frac{b}{a^2 - b^2 + c^2} = \sum \frac{b^2}{a^2 b - b^3 + bc^2} \geq \frac{\left(\sum a\right)^2}{\sum a^2 b + \sum ab^2 - \sum a^3}.$$

Then it suffices to show that

$$\frac{\left(\sum a\right)^2}{\sum a^2b+\sum ab^2-\sum a^3}\geq\frac{\sum a^2}{abc}\Leftrightarrow abc\left(\sum a\right)^2\geq\left(\sum a^2\right)\left(\sum a^2b+\sum ab^2-\sum a^3\right)$$

$$\Leftrightarrow\sum a^3bc+2\sum a^2b^2c\geq\sum a^4b+\sum ab^4+\sum a^3b^2+\sum a^2b^3+\sum a^2b^2c+$$

$$+\sum a^2b^2c-\sum a^5-\sum a^3b^2-\sum a^2b^3$$

$$\Leftrightarrow\sum a^5+\sum a^3bc-\sum a^4b-\sum ab^4\geq 0$$

$\Leftrightarrow\sum a^3(a^2+bc-ab-ac)\geq 0\Leftrightarrow\sum a^3(a-b)(a-c)\geq 0$, which is the inequality of Schur, so is true. The proof is complete.

PP.21381. Solve in Z the equation $1+x+x^2+x^3+x^4=y^4$.

Solution.

If $x>0$, we have $x^4<1+x+x^2+x^3+x^4=y^4$ and

$y^4=1+x+x^2+x^3+x^4<(x+1)^4\Leftrightarrow 3x^3+5x^2+3x>0$, true, so $x^4<y^4<(x+1)^4$ and we not obtain solutions.

If $x<-1$ we have the inequalities:

$x^4>1+x+x^2+x^3+x^4\Leftrightarrow(1+x)(1+x^2)<0$, true.

$(x+1)^4<1+x+x^2+x^3+x^4\Leftrightarrow x(3x^2+5x+3)<0$, true because $3x^2+5x+3>0$ (has the discriminant negativ). Therefore, $(x+1)^4<y^4<x^4$, and we not obtain solutions. It remains to check $x=0$, and $x=-1$, which are solutions.

In conclusion we have the solutions $(x,y)\in\{(-1,-1);(-1,1);(0,-1),(0,1)\}$, and we are done.

PP.21409. In all triangles ABC holds:

$$1)\sum\sqrt{\frac{h_ah_b}{(r-h_a)(r-h_b)}}\geq\frac{9}{2};\ 2)\sum\sqrt{\frac{r_ar_b}{(r-r_a)(r-r_b)}}\geq\frac{9}{2}.$$

Solution. Let F be the area of the triangle ABC.

We have: $\frac{h_a h_b}{(r-h_a)(r-h_b)}=\frac{\frac{4F^2}{ab}}{\left(\frac{F}{s}-\frac{2F}{a}\right)\left(\frac{F}{s}-\frac{2F}{b}\right)}=\frac{4}{ab}\cdot\frac{s^2ab}{(2s-a)(2s-b)}=$

$=\frac{4s^2}{(b+c)(a+c)}$.

Using the inequality $x^2+y^2+z^2\geq xy+yz+zx$ and than Cauchy-Buniakovski-Schwarz, we obtain that:

$$\sum\sqrt{\frac{h_a h_b}{(r-h_a)(r-h_b)}}=(a+b+c)\sum\frac{1}{\sqrt{(b+c)(a+c)}}=$$

$$=\frac{1}{2}[(a+b)+(b+c)+(c+a)]\sum\frac{1}{\sqrt{(b+c)(c+a)}}\geq$$

$$\geq\frac{1}{2}\left(\sum\sqrt{(a+b)(b+c)}\right)\left(\sum\frac{1}{\sqrt{(a+b)(b+c)}}\right)\geq\frac{9}{2}.$$

2) $\frac{r_a r_b}{(r-r_a)(r-r_b)}=\frac{\frac{F^2}{(s-a)(s-b)}}{\left(\frac{F}{s}-\frac{F}{s-a}\right)\left(\frac{F}{s}-\frac{F}{s-b}\right)}=\frac{1}{(s-a)(s-b)}\cdot\frac{s^2(s-a)(s-b)}{ab}=\frac{s^2}{ab}$,

and we proceed like above thus we obtain:

$$\sum\sqrt{\frac{r_a r_b}{(r-r_a)(r-r_b)}}=\frac{1}{2}(a+b+c)\sum\frac{1}{\sqrt{ab}}\geq\frac{1}{2}\left(\sum\sqrt{ab}\right)\left(\sum\frac{1}{\sqrt{ab}}\right)\geq\frac{9}{2}.$$

The proof is complete.

PP.21410. Let be $a,b>0$ such that its arithmetic, geometric and harmonic means are the sides of triangle $ABC(\angle A=90^0,\angle B<\angle C)$. Prove that:

1) $\sin B=\cos^2 B$;

2) $\cos C=\sin^2 C$;

3) $30^0 < \angle B < 45^0$.

Solution. If $a = b$, all means are equal and the triangle is equilateral. Because

$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2}$, we have: $BC = \frac{a+b}{2}$, $AC = \frac{2ab}{a+b}$, $AB = \sqrt{ab}$.

1) $\sin B = \frac{\frac{2ab}{a+b}}{\frac{a+b}{2}} = \frac{4ab}{(a+b)^2}$; $\cos B = \frac{\sqrt{ab}}{\frac{a+b}{2}} = \frac{2\sqrt{ab}}{a+b}$, and easily yields that $\sin B = \cos^2 B$.

2)Because $\sin B = \cos C$ and $\cos B = \sin C$ yields that $\cos C = \sin^2 C$.

3) $\angle B + \angle C = 90^0$ and $\angle B < \angle C$, we deduce immediately that $\angle B < 45^0$.

We prove that $\angle B > 36^0$. We have:

$\sin B = \cos^2 B \Leftrightarrow \sin^2 B + \sin B - 1 = 0$, so $\sin B = \frac{-1+\sqrt{5}}{2}$.

Using the fact $\cos 36^0 = \frac{1+\sqrt{5}}{4}$ and $\sin 36^0 = \sqrt{\frac{10-2\sqrt{5}}{16}} = \sqrt{\frac{5-\sqrt{5}}{8}}$ we have:

$$\angle B > 36^0 \Leftrightarrow \sin \angle B > \sin 36^0 \Leftrightarrow \frac{\sqrt{5}-1}{2} > \sqrt{\frac{5-\sqrt{5}}{8}}$$

$$\Leftrightarrow \frac{6-2\sqrt{5}}{4} > \frac{5-\sqrt{5}}{8} \Leftrightarrow 7 > 3\sqrt{5} \Leftrightarrow 49 > 45, \text{ true.}$$

The proof is complete.

PP.21417. In all triangle ABC holds $\sum \frac{1}{ctg\frac{A}{2} + ctg\frac{B}{2}} \leq \frac{\sqrt{3}}{2}$.

Solution. Since $ctg\frac{A}{2} = \frac{s-a}{2}$, $ctg\frac{B}{2} = \frac{s-b}{2}$, we have $ctg\frac{A}{2} + ctg\frac{B}{2} = \frac{c}{r}$, so we must to prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}$, but this inequality is the item 5.22 and 5.23 from Bottema.

PP.21418. If $a,b,c>0$, then $\sum\frac{a^3}{a+b}\geq\frac{1}{2}\sum a^2\geq\sum\frac{ab^2}{a+b}$.

Solution. For the left inequality we use the inequality of Harald Bergström and we have that:

$$\sum\frac{a^3}{a+b}=\sum\frac{a^4}{a^2+ab}\geq\frac{\left(\sum a^2\right)^2}{\sum a^2+\sum ab}\geq\frac{\left(\sum a^2\right)^2}{2\sum a^2}=\frac{1}{2}\sum a^2 .$$

The inequality from the right is written as follows:

$$-\sum\frac{ab^2}{a+b}\geq-\frac{1}{2}\sum a^2\Leftrightarrow\sum a^2-\sum\frac{ab^2}{a+b}\geq\sum a^2-\frac{1}{2}\sum a^2$$

$\Leftrightarrow\sum\left(b^2-\frac{ab^2}{a+b}\right)\geq\frac{1}{2}\sum a^2\Leftrightarrow\sum\frac{b^3}{a+b}\geq\frac{1}{2}\sum a^2$. So, we must to prove

$\sum\frac{b^3}{a+b}\geq\frac{1}{2}\sum a^2$. Indeed, by Harald Bergström's inequality we obtain:

$\sum\frac{b^3}{a+b}=\sum\frac{b^4}{ab+b^2}\geq\frac{\left(\sum a^2\right)^2}{2\sum a^2}=\frac{1}{2}\sum a^2$, and we are done.

PP.21421. If $a,b,c>0$, then $3\left(\sum a^3\right)\left(\sum a^4\right)\geq abc\left(\sum a\right)^2\left(\sum a^2\right)$.

Solution. We use the AM-GM inequality and well-known $3\sum x^2\geq\left(\sum x\right)^2$.

We have: $3\sum a^4\geq\left(\sum a^2\right)^2$, $3\sum a^2\geq\left(\sum a\right)^2$ and $\sum a^3\geq 3abc$, and by multiplying we obtain the desired result. The solution is complete.

PP.21431. In all triangle ABC holds $4R^2+6Rr\geq s^2+r^2$.

Solution 1. By the item 5.2 from Bottema we have $4s^2\leq 16R^2+22Rr$, so it suffices to prove that

$16R^2+24Rr\geq 16R^2+22Rr+4r^2\Leftrightarrow R\geq 2r$, true.

Solution 2. By the item 5.8 from Bottema we have $s^2 \le 4R^2 + 4Rr + 3r^2$, so it suffices to prove that:

$4R^2 + 6Rr \ge 4R^2 + 4Rr + 3r^2 + r^2 \Leftrightarrow R \ge 2r$, true, and the solution is complete.

PP.21432. If $a,b,c > 0$, then $\frac{3}{2}\sum\frac{1}{a} + \frac{9}{a+b+c} \ge 5\sum\frac{1}{a+b}$.

Solution. After some algebra the inequality from the statement becomes successively:

$$\frac{3\sum ab}{2abc} + \frac{9}{\sum a} \ge \frac{5\sum a^2 + 15\sum ab}{\sum a\sum ab - abc} \Leftrightarrow \frac{3\sum a\sum ab + 18abc}{2abc\sum a} \ge \frac{5\sum a^2 + 15\sum ab}{\sum a\sum ab - abc}$$

$$\Leftrightarrow 3(\sum a)^2(\sum ab)^2 - 3abc\sum a\sum ab + 18abc\sum a\sum ab - 18a^2b^2c^2 \ge$$

$$\ge 10abc\sum a\sum a^2 + 30abc\sum a\sum ab$$

$$\Leftrightarrow 3(\sum a)^2(\sum ab)^2 \ge 15abc\sum a\sum ab + 10abc\sum a\sum a^2 + 18a^2b^2c^2$$

$$\Leftrightarrow 3\sum a^4b^2 + 3\sum a^2b^4 + 45a^2b^2c^2 + 6\sum a^4bc + 6\sum a^3b^3 + 24\sum a^3b^2c + 24\sum a^3bc^2 \ge$$

$$\ge 15\sum a^3b^2c + 15\sum a^3bc^2 + 45a^2b^2c^2 + 10\sum a^3b^2c + 10\sum a^3bc^2 + 10\sum a^4bc + 18a^2b^2c^2$$

$\Leftrightarrow 3\sum a^4b^2 + 3\sum a^2b^4 + 6\sum a^3b^3 \ge 4\sum a^4bc + \sum a^3b^2c + \sum a^3bc^2 + 18a^2b^2c^2$, which yields by adding the following inequalities:

$$6\sum a^3b^3 \ge 18a^2b^2c^2$$
$$2\sum a^4b^2 + 2\sum a^2b^4 \ge 4a^4bc$$
$$\sum a^4b^2 + \sum a^2b^4 \ge \sum a^3b^2c + \sum a^3bc^2 \qquad (1)$$

Remark. Two demonstrations for (1) was given in the solution of PP.21165.

The proof is complete.

PP.21435. If $a_k > 0 \ (k = 1,2,...,n)$, then $\sum_{cyclic}\frac{a_1^2}{a_1 + a_2} \ge \frac{1}{2}\sum_{k=1}^{n} a_k \ge \sum_{cyclic}\frac{a_1a_2^2}{a_1^2 + a_2^2}$.

Solution. For the first inequality we apply the inequality of Harald Bergström:

$$\sum_{cyclic}\frac{a_1^2}{a_1+a_2}\geq\frac{\left(\sum_{cyclic}a_1\right)^2}{\sum_{cyclic}(a_1+a_2)}=\frac{\left(\sum_{k=1}^{n}a_k\right)^2}{2\sum_{k=1}^{n}a_k}=\frac{1}{2}\sum_{k=1}^{n}a_k .$$

For the second inequality, we use the inequality:

$\frac{a_1a_2^2}{a_1^2+a_2^2}\leq\frac{a_2}{2}\Leftrightarrow(a_1-a_2)^2\geq 0$, which yields that:

$\sum_{cyclic}\frac{a_1a_2^2}{a_1^2+a_2^2}\leq\sum_{cyclic}\frac{a_2}{2}=\frac{1}{2}\sum_{k=1}^{n}a_k$, and the proof is complete.

PP.21439. If $x,y,z>0$, then $\sum\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}\leq 5(x^2+y^2+z^2)$.

Solution. After some algebra we get

$$\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}=\frac{5}{3}(x^2-xy+y^2).$$

Using the inequality $\sum xy\leq\sum x^2$, we obtain

$$\sum\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}=\frac{5}{3}\sum(x^2+xy+y^2)=\frac{5}{3}\left(2\sum x^2+\sum xy\right)\leq$$

$\leq\frac{5}{3}\left(2\sum x^2+\sum x^2\right)=5\sum x^2$, and we are done.

PP.21440. Prove that for all $n\in N$ the equation $x^2+y^2+z^2=25^n$ has solution in Z .

Solution. The theorem of three squares (see for e.g. [1]) says that a natural number m is written as $m=a^2+b^2+c^2, a,b,c\in Z$ if and only if $m\neq 4^i(8k+7)$, $i,k\geq 0$.

Because $25^n=(24+1)^n=8k+1$, we deduce that $25^n\neq 4^i(8k+7)$, so 25^n can written as a sum of three integers squares, and we are done.

References:

[1] Panaitopol, L., Aplicaţii ale teoremei celor trei pătrate, RMT, No. 1/2002.

PP.21441. If $x, y, z \in C$ then:

$$\frac{\left((x+y+z)^3-x^3-y^3-z^3\right)\left((x+y+z)^7-x^7-y^7-z^7\right)}{\left((x+y+z)^5-x^5-y^5-z^5\right)}=\frac{21}{25}\left(1+\frac{xyz\sum x}{(\sum x^2+\sum xy)^2}\right).$$

Solution. We have that:

$$(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x);$$

$(x+y+z)^5-x^5-y^5-z^5=5(x+y)(y+z)(z+x)(\sum x^2+\sum xy)$ (see the solution of PP.21445).

To find the decomposition of $(x+y+z)^7-x^7-y^7-z^7$ we use fundamental symmetric sums. So, we must to find a, b, c, d such that:

$$(x+y+z)^7-x^7-y^7-z^7=(x+y)(y+z)(z+x)[a\sum x^4+b(\sum x^3y+\sum xy^3)+$$

$$+c\sum x^2y^2+d\sum x^2yz] \qquad (1)$$

We take $z=0$, and we have:

$(x+y)^7-x^7-y^7=xy(x+y)[a(x^4+y^4)+b(x^3y+xy^3)+cx^2y^2]$, and we deduce

$a=7, b=14, c=21$ (see the solution of PP.21165).

Setting in (1) $x=y=z=1$ we obtain $d=35$, so:

$$(x+y+z)^7-x^7-y^7-z^7=7(x+y)(y+z)(z+x)(\sum x^4+2\sum x^3y+2\sum xy^3+$$

$$+3\sum x^2y^2+5xyz\sum x).$$

We obtain that:

$$\frac{\left((x+y+z)^3-x^3-y^3-z^3\right)\left((x+y+z)^7-x^7-y^7-z^7\right)}{\left((x+y+z)^5-x^5-y^5-z^5\right)}=$$

$$=\frac{21}{5}\cdot\frac{\sum x^4+2\sum x^3y+2\sum xy^3+3\sum x^2y^2+5xyz\sum x}{(\sum x^2+\sum xy)^2}=$$

$$=\frac{21}{5}\cdot\frac{(\sum x^2)^2-2\sum x^2y^2+(\sum xy)^2-2xyz\sum x+2\sum x^2\sum xy-2xyz\sum x+2\sum x^2y^2+5xyz\sum x}{(\sum x^2+\sum xy)^2}$$

$$=\frac{21}{5}\cdot\frac{(\sum x^2+\sum xy)^2+xyz\sum x}{(\sum x^2+\sum xy)^2}=\frac{21}{5}\left(1+\frac{xyz\sum x}{\left(\sum x^2+\sum xy\right)^2}\right)$$, and the proof is complete.

PP.21442. If $x, y, z > 0$, then $\prod_{cyclic}\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}\geq\frac{125}{27}\left(\sum xy\right)^3$.

Solution. After some algebra we obtain:

$$\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}=\frac{5}{3}(x^2+xy+y^2).$$

Thus, it suffices to show that:

$\prod_{cyclic}(x^2+xy+y^2)\geq\left(\sum xy\right)^3$, i.e. the inequality (*) of the solution of PP.21165.

The proof is complete.

PP.21443. If $x, y > 0$, then: $5\left((x+y)^3-x^3-y^3\right);\left((x+y)^5-x^5-y^5\right)\sqrt{21};$

$5\left((x+y)^7-x^7-y^7\right)$ are in geometrical progression.

Solution. We showed in the solutions of PP.21165 and PP.21442 (see also the solution of PP.21439), that:

$$\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}=\frac{5}{3}(x^2+xy+y^2)$$ and

$$\frac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}=\frac{7}{5}(x^2+xy+y^2).$$

Therefore:

$$\frac{\left((x+y)^5-x^5-y^5\right)\sqrt{21}}{5\left((x+y)^3-x^3-y^3\right)}=\frac{5}{3}\cdot\frac{\sqrt{21}}{5}\cdot(x^2+xy+y^2)=\sqrt{\frac{7}{3}}(x^2+xy+y^2)$$ and

$$\frac{5\left((x+y)^7-x^7-y^7\right)}{\sqrt{21}\left((x+y)^5-x^5-y^5\right)}=\frac{5}{\sqrt{21}}\cdot\frac{7}{5}(x^2+xy+y^2)=\sqrt{\frac{7}{3}}\left(x^2+xy+y^2\right),$$ which yields to conclusion.

PP.21445. If $x_k>0\ (k=1,2,...,n)$, then:

$$\sum_{cyclic}\frac{(x_1+x_2+x_3)^5-x_1^5-x_2^5-x_3^5}{(x_1+x_2+x_3)^3-x_1^3-x_2^3-x_3^3}\le 10\sum_{k=1}^{n}x_k^2.$$

Solution. The expression $(x+y+z)^5-x^5-y^5-z^5$ is divisible by $(x+y)(y+z)(z+x)$.

To find the decomposition of $(x+y+z)^5-x^5-y^5-z^5$ we use fundamental symmetric sums. So, we must to find a and b such that:

$$(x+y+z)^5-x^5-y^5-z^5=(x+y)(y+z)(z+x)[a(\sum x)^2+b\sum xy].$$

For $x=y=1, z=0\Rightarrow 4a+b=15$ and for $x=y=z=1\Rightarrow 3a+b=10$.

Yields $a=5, b=-5$, therefore:

$$(x+y+z)^5-x^5-y^5-z^5=5(x+y)(y+z)(z+x)[(\sum x)^2-\sum xy]=.$$

$$=5(x+y)(y+z)(z+x)(\sum x^2+\sum xy).$$

Since $(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)$, we obtain that:

$$\sum_{cyclic}\frac{(x_1+x_2+x_3)^5-x_1^5-x_2^5-x_3^5}{(x_1+x_2+x_3)^3-x_1^3-x_2^3-x_3^3}=\frac{5}{3}\sum_{cyclic}(x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_3x_1)\le$$

$$\le\frac{5}{3}\sum_{cyclic}2(x_1^2+x_2^2+x_3^2)=10\sum_{k=1}^{n}x_k^2,$$ i.e. the desired result.

2. O integrală remarcabilă

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Să se calculeze integrala:

$$\int_a^b (x-a)^m (b-x)^n \, dx.$$

Rezolvare:

Notăm cu $\mathrm{I}_{m,n}$ integrala dată, pe care o integrăm prin părţi:

$$\int_a^b (x-a)^m (b-x)^n \, dx = -\frac{1}{n+1}(x-a)^m (b-x)^{n+1}\Big|_a^b + \frac{m}{n+1}\int_a^b (x-a)^{m-1} (b-x)^{n+1} \, dx,$$

deci

$$\mathrm{I}_{m,n} = \frac{m}{n+1}\mathrm{I}_{m-1,n+1}, \forall m,n \in \mathbb{N}.$$

Atunci $\mathrm{I}_{m,n} = \frac{m}{n+1}\cdot\frac{m-1}{n+2}\cdots\cdots\frac{1}{n+m}\mathrm{I}_{0,m+n}$, şi deoarece

$$\mathrm{I}_{0,m+n} = \int_a^b (b-x)^{m+n} \, dx = \frac{(b-a)^{m+n+1}}{m+n+1},$$ deducem

$$\mathrm{I}_{m,n} = \frac{m}{n+1}\cdot\frac{m-1}{n+2}\cdots\cdots\frac{1}{n+m}\cdot\frac{(b-a)^{m+n+1}}{m+n+1} = \frac{n!m!(b-a)^{m+n+1}}{(n+m+1)!} \qquad (1).$$

Consecinţe:

Consecinţa 1:

$$\int_0^1 x^m (1-x)^n \, dx = \frac{n!m!}{(n+m+1)!}.$$

Consecinţa 2:

$$\int_0^1 \left(1-x^2\right)^n dx = \frac{(n!)^2\, 2^{2n}}{(2n+1)!}\ .$$

Într-adevăr, utilizând relaţia de mai sus avem :

$$\int_0^1 \left(1-x^2\right)^n dx = \frac{1}{2}\int_{-1}^1 \left(1-x^2\right)^n = \frac{1}{2}\int_{-1}^1 (x+1)^n (1-x)^n dx = \frac{1}{2}\cdot\frac{(n!)^2\, 2^{2n+1}}{(2n+1)!}.$$

Consecinţa 3: Şirul $a_n = \int_a^b \left(1-x^2\right)^n dx \ \ (n\geq 1)$ este strict descrescător.

Ţinem seama că $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{2}\cdot\dfrac{\left[(n+1)!\right]^2 2^{2n+3}}{(2n+3)!}\cdot\dfrac{2(2n+1)!}{(n!)^2\, 2^{2n+1}} = \dfrac{2(n+1)}{2n+3} < 1$

Aplicaţii:

1. Să se calculeze integrala $\lim_{n\to\infty}\left(\int_1^2\left((x-1)(2-x)\right)^n dx\right)$ (exerciţiu propus în varianta 77 de bacalaureat M_1-2009).

Ţinând seama de rezultatele de mai sus obţinem:

$$\int_1^2\left((x-1)(2-x)\right)^n dx=\frac{(n!)^2}{(2n+1)!},$$ de unde

aplicând teorema cleştelui pentru şirul $a_n=\frac{(n!)^2}{(2n+1)!}$, obţinem

$$\frac{a_{n+1}}{a_n}=\frac{\left[(n+1)!\right]^2}{(2n+3)!}\cdot\frac{(2n+1)!}{(n!)^2}=\frac{n+1}{2(2n+3)}\to\frac{1}{4}<1,$$

deci

$$\lim_{n\to\infty}\left(\int_1^2\left((x-1)(2-x)\right)^n dx\right)=\lim_{n\to\infty}a_n=0.$$

sau observând uşor că $x\in[1,2]$ deducem că:

$$0\le(x-1)(2-x)\le\frac{1}{4}.$$

Ridicând la puterea n, integrând inegalitatea obţinută şi folosind monotonia integralei, rezultă că:

$0\le\int_1^2\left((x-1)(2-x)\right)^n dx\le\left(\frac{1}{4}\right)^n$,şi conform criteriului cleştelui avem

$$\lim_{n\to\infty}\left(\int_1^2\left((x-1)(2-x)\right)^n dx\right)=0$$

2. Să se calculeze:

$\lim_{n\to\infty}\left(\int_a^b(x-a)^n(b-x)^n\,dx\right)^{\frac{1}{n}}$, dacă $a<b$ (exerciţiu propus la admiterea în învăţământul superior-profilul matematică-1984).

Calculăm mai întâi integrala din interiorul limilei.

Aceasta se rezolvă utilizând substituţia: $t=\frac{a+b}{2}-x,\ dt=-dx$

pentru $x=a$ obţinem t=$\frac{b-a}{2}$ iar pentru $x=b$ obţinem t=$\frac{a-b}{2}$

deci,

$$I_n=\int_{\frac{b-a}{2}}^{\frac{a-b}{2}}\left(\frac{b-a}{2}-t\right)^n\left(\frac{b-a}{2}+t\right)^n(-1)\,dt$$

Cu notaţia $c=\frac{b-a}{2}$ avem $I_n=\int_{-c}^{c}\left(c^2-t^2\right)^n dt=2\int_0^c\left(c^2-t^2\right)^n dt.$

Integrăm prin părţi pentru a stabili o relaţie de recurenţă între termenii şirului $\left(I_n\right)_{n\geq 1}$.

Fie $u(t)=\left(c^2-t^2\right)^n$, de unde $u'(t)=-2nt(c^2-t^2)^{n-1}$

şi $v'(t)=1,$ de unde $v(t)=t,$ atunci

$$I_n=2t\left(c^2-t^2\right)^n\Big|_0^c+4n\int_0^c t^2(c^2-t^2)^{n-1}dt=-4n\int_0^c(c^2-t^2-c^2)\left(c^2-t^2\right)^{n-1}dt=$$

$=-4n\int_0^c(c^2-t^2)^n dt+4nc^2\int_0^c(c^2-t^2)^{n-1}dt,$ deci $I_n=-2nI_n+2nc^2I_{n-1},$ adică

$I_n=\frac{2n}{2n+1}c^2I_{n-1}=\frac{2n}{2n+1}\left(\frac{b-a}{2}\right)^2 I_{n-1},$ pentru $n\geq 2$. Deoarece $I_1=\int_a^b\left(x-a\right)\left(b-x\right)dx=\frac{\left(b-a\right)^3}{6},$

obţinem pentru $n\geq 2$

$$I_n=\frac{2n}{2n+1}\left(\frac{b-a}{2}\right)^2 I_{n-1}=\frac{2n}{2n+1}\left(\frac{b-a}{2}\right)^2\cdot\frac{2\left(n-1\right)}{2n-1}\left(\frac{b-a}{2}\right)^2 I_{n-2}=......=$$

$$=\frac{2n}{2n+1}\left(\frac{b-a}{2}\right)^2\cdot\frac{2\left(n-1\right)}{2n-1}\left(\frac{b-a}{2}\right)^2........\frac{2\cdot 2}{2\cdot 2+1}\left(\frac{b-a}{2}\right)^2\cdot I_1=$$

$$=\frac{2^{n-1}\cdot\left(2\cdot 3\cdot.....\cdot n\right)}{5\cdot 7\cdot.....\left(2n+1\right)}\cdot\left(\frac{b-a}{2}\right)^{2n-2}\cdot\frac{\left(b-a\right)^3}{6}=\frac{\left(b-a\right)^{2n+1}\cdot n!}{2^n\cdot 3\cdot 5\cdot....\cdot\left(2n+1\right)!},\text{ deci}$$

$$I_n=\left(b-a\right)^{2n+1}\frac{\left(n!\right)^2}{\left(2n+1\right)!}.$$

Sau din relaţia (1) pentru m=n deducem că $\mathrm{I}_{n,n}=\int_a^b\left(x-a\right)^n\left(b-x\right)^n dx=\frac{\left(n!\right)^2\left(b-a\right)^{2n+1}}{\left(2n+1\right)!}.$

Avem de calculat $\lim\limits_{n\to\infty}\sqrt[n]{\mathrm{I}_{n,n}}$.

Pentru aceasta notăm $\mathrm{a}_n=\frac{\left(n!\right)^2}{\left(2n+1\right)!}$ şi avem:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left[\left(n+1\right)!\right]^2}{\left(2n+3\right)!}\cdot\frac{\left(2n+1\right)!}{\left(n!\right)^2}=\lim_{n\to\infty}\frac{\left(n+1\right)^2}{\left(2n+2\right)\left(2n+3\right)}=\lim_{n\to\infty}\frac{n+1}{2\left(2n+3\right)}=\frac{1}{4}$$

Deci $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\frac{1}{4}$. Putem scrie:

$$\lim_{n\to\infty}\sqrt[n]{I_{n,n}}=\lim_{n\to\infty}\sqrt[n]{(b-a)^{2n+1}a_n}=\lim_{n\to\infty}\left[(b-a)^{2+\frac{1}{n}}\cdot\sqrt[n]{a_n}\right]=\frac{(b-a)^2}{4}.$$

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3. O DEMONSTRAŢIE VECTORIALĂ A TEOREMEI NEWTON-GAUSS

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Teorema Newton-Gauss sau teorema patrulaterului complet are următorul enunţ: *mijloacele diagonalelor unui patrulater complet sunt coliniare.* Nota de faţă propune o soluţie vectorială a acestei teoreme.

Considerăm patrulaterul complet $ABCDEF$ (vezi fig.1) şi M, N, respectiv P mijloacele diagonalelor (AC), (BD), respectiv (EF). Notăm $\frac{FD}{DC}=k_1$, $\frac{BC}{EB}=k_2$, $k_i>0$, $i=1,2$ şi $\vec{x}$ vectorul de poziţie al punctului X. Din teorema lui Menelaus aplicată în triunghiul EDC cu transversala $F-A-B$ avem $\frac{FD}{FC}\cdot\frac{CB}{BE}\cdot\frac{EA}{AD}=1$, de unde $\frac{EA}{AD}=\frac{k_1+1}{k_1k_2}$. Atunci:

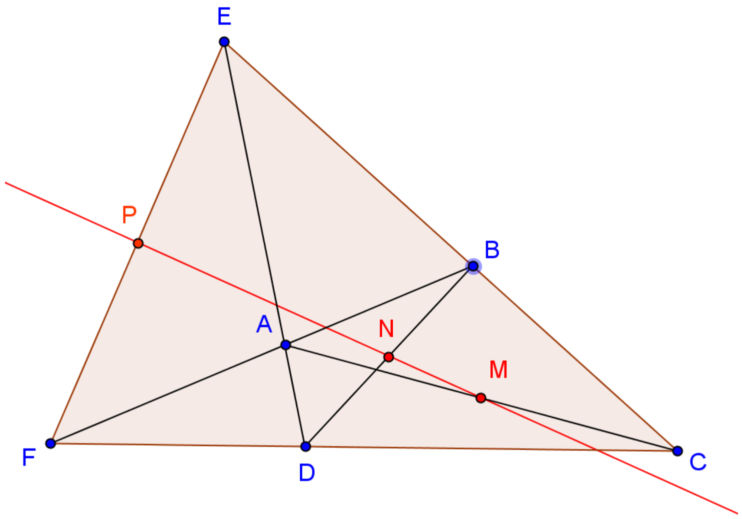

fig. 1

$$\vec{a}=\frac{\vec{e}+\frac{k_1+1}{k_1k_2}\vec{d}}{1+\frac{k_1+1}{k_1k_2}}=\frac{k_1k_2\vec{e}+(k_1+1)\cdot\frac{\vec{f}+k_1\vec{c}}{1+k_1}}{1+k_1+k_1k_2}=\frac{k_1k_2\vec{e}+\vec{f}+k_1\vec{c}}{1+k_1+k_1k_2}$$

$$\vec{m}=\frac{\vec{a}+\vec{c}}{2}=\frac{k_1k_2}{2(1+k_1+k_1k_2)}\vec{e}+\frac{1}{2(1+k_1+k_1k_2)}\vec{f}+\frac{1+2k_1+k_1k_2}{2(1+k_1+k_1k_2)}\vec{c}$$

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$$\vec{n}=\frac{\vec{d}+\vec{b}}{2}=\frac{\frac{\vec{f}+k_1\vec{c}}{1+k_1}+\frac{\vec{c}+k_2\vec{e}}{1+k_2}}{2}=\frac{k_2+k_1k_2}{2\left(1+k_1+k_2+k_1k_2\right)}\vec{e}+\frac{1+k_2}{2\left(1+k_1+k_2+k_1k_2\right)}\vec{f}+\frac{1+2k_1+k_1k_2}{2\left(1+k_1+k_2+k_1k_2\right)}\vec{c}$$

$$\vec{p}=\frac{\vec{e}+\vec{f}}{2}.$$

Notăm $\frac{MN}{NP}=k$ şi determinăm k astfel încât $\vec{n}=\frac{\vec{m}+k\vec{p}}{1+k}$. Se obţine egalitatea vectorială:

$$\frac{k_2+k_1k_2}{2\left(1+k_1+k_2+k_1k_2\right)}\vec{e}+\frac{1+k_2}{2\left(1+k_1+k_2+k_1k_2\right)}\vec{f}+\frac{1+2k_1+k_1k_2}{2\left(1+k_1+k_2+k_1k_2\right)}\vec{c}=$$

$$=\frac{k_1k_2+k\left(1+k_1+k_1k_2\right)}{2\left(1+k\right)\left(1+k_1+k_1k_2\right)}\vec{e}+\frac{1+k\left(1+k_1+k_1k_2\right)}{2\left(1+k\right)\left(1+k_1+k_1k_2\right)}\vec{f}+\frac{1+2k_1+k_1k_2}{2\left(1+k\right)\left(1+k_1+k_1k_2\right)}\vec{c}.$$

Din proporţionalitatea coeficienţilor rezultă $\frac{k_2+k_1k_2}{k_1k_2+k\left(1+k_1+k_1k_2\right)}=\frac{1+k_2}{1+k\left(1+k_1+k_1k_2\right)}=1$, de unde $k=\frac{k_2}{1+k_1+k_1k_2}$.

Comentariu. Demonstraţia de mai sus determină şi raportul în care se găsesc cele trei puncte, de unde putem deduce: *o condiţie necesară şi suficientă pentru ca N să fie mijlocul segmentului* $[MP]$ *este* $k_2=\frac{1+k_1}{1-k_1}$, $0<k_1<1$.

4. INEGALITĂŢI GEOMETRICE REZOLVATE CU AJUTORUL NUMERELOR COMPLEXE

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1. Să se arate că într-un patrulater convex există relaţia: $AC \cdot BD \le AB \cdot CD + AD \cdot BC$.
(Inegalitatea lui Ptolemeu)

Soluţie:

Fie z_A, z_B, z_C, z_D afixele punctelor A, B, C, D. Atunci

$$(z_C - z_A)\cdot(z_D - z_B) + (z_B - z_A)\cdot(z_C - z_D) +$$
$$+(z_D - z_A)\cdot(z_B - z_C) = 0 \Rightarrow$$
$$(z_C - z_A)\cdot(z_D - z_B) = (z_B - z_A)\cdot(z_D - z_C) +$$
$$+(z_D - z_A)\cdot(z_C - z_B)$$

Prin trecere la modul $\Rightarrow$

$$|z_C - z_A|\cdot|z_D - z_B| \le |z_B - z_A|\cdot|z_D - z_C| + |z_D - z_A|\cdot|z_C - z_B| \Rightarrow AC \cdot BD \le AB \cdot CD + AD \cdot BC .$$

2.Fie ABC un triunghi echilateral şi M un punct nesituat pe cercul ccircumscris. Să se arate că se poate forma un triunghi cu segmentele [MA], [MB], [MC] (teorema D. Pompei).

Soluţie:

Fie $A(a)$,$B(b)$,$C(c)$, $M(z)$ cele patru puncte în planul complex.
Are loc inegalitatea
$(z-a)(b-c)+(z-b)(c-a)+(z-c)(a-b)=0$ (se demonstrează prin calcul direct)
De aici
$(z-a)(b-c)=-(z-b)(c-a)-(z-c)(a-b)$
Luând modului aici avem
$|z-a||b-c|=|(z-b)(c-a)+(z-c)(a-b)|\le |(z-b)(c-a)|+|(z-c)(a-b)|$
Din AB=BC=AC rezultă că $|a-b|=|b-c|=|c-a|$
înmulţim prin $|b-c|$ si rezultă
$|z-a|\le|z-b|+|z-c|$.
În această inegalitate avem egalitate dacă M aparţine cercului circumscris triunghiului ABC , caz în care patrulaterul ABMC este inscriptibil şi are loc teorema lui Ptolemeu
$AM \cdot BC = AB \cdot MC + AC \cdot MB, adică\ AM = MC + MB$
Cum M nu aparţine cercului în inegalitate nu avem egalitate.

Din simetria relaíţiei (1) se deduc inegalităţile MB<MA+MC, MC<MA+MB ceea ce arată că segmentele [MA],[MB],[MC] determină un triunghi.

3 . *Fie un triunghi ABC, A_1,B_1,C_1mijloacele laturilor (BC),(AC), respectiv (AC) si H ortocentrul triunghiului. Atunci $HA_1\cdot BC\leq HB_1\cdot BC+HC_1\cdot AB$.*

Soluţie:

Se consideră ca origine centrul O al cercului circumscris triunghiului ABC si M=H ortocentru triunghiului ABC.
Afixul H in acest caz este z=a+b+c şi relaţia $|z-a||b-c|\leq |(z-b)(c-a)|+|(z-c)(a-b)|$devine $|b+c||b-c|\leq |c+a|c-a|+|a+b||a-b|$ (1)
Dacă ţinem cont că afixele punctelor A_1,B_1,C_1sunt $\frac{b+c}{2},\frac{c+a}{2},\frac{b+a}{2}$ atunci relaţia (1) devine $HA_1\cdot BC\leq HB_1\cdot AC+HC_1\cdot AB$ unde H este ortocentrul triunghiului ABC.

4. *Dacă M este un punct din planul triunghiuli ABC, atunci* $AM^2\sin A+BM^2\sin B+CM^2\sin C\geq 2S$*, unde S este aria triunghiului.*

Soluţie:

Daca x,y,z$\in\mathbb{C}$, atunci se poate demonstra uşor urmatoarea egalitate
$x^2(y-z)+y^2(z-x)+z^2(x-z)=(x-y)(x-z)(y-z)$
Aplicând inegalitatea modulului obţinem
$|x-y||z-x||y-z|\leq|x^2||y-z|+|y^2||z-x|+|z^2||x-y|$. (2)
Fie m afixul lui M si a,b,c, afixele punctelor. Inlocuind a,b,c în relaţia (2) şi simplificăm prin 2R obţinem $AM^2\sin A+BM^2\sin B+CM^2\sin C\geq 2S$

5. *Fie M un punct d în planul triunghiului ABC şi G centrul său de greutate, atunci avem inegalitatea* $AM^3\sin A+BM^3\sin B+CM^3\sin C\geq 6MG\cdot S$

Soluţie:

Daca x,y,z$\in\mathbb{C}$, atunci se poate demonstra uşor urmatoarea egalitate
$x^3(y-z)+y^3(z-x)+z^3(x-z)=(x-y)(x-z)(y-z)(x+y+z)$ de unde
$|x^3(y-z)|+|y^3(z-x)|+|z^3(x-z)|\geq|x-y||y-z||z-x||x+y+z|$ (3)
Fie a,b,c,m afixele punctelor A,B,C respectiv M si x=m-a, y=m-b, z=m-c
Înlocuind x,y,z in (3) şi ţinând cont ca afixul lui G este $\frac{a+b+c}{3}$, obţinem
$AM^3\sin A+BM^3\sin B+CM^3\sin C\geq 6MG\cdot S$

6. *Fie O_1 si O_2 mijloacele diagonalelor AC si BD ale unui patrulater ABCD si M intersectia diagonalelor. Atinci* $S_{ABCD}\geq 4\cdot S_{O_1MO_2}$

Soluţie:

Fie m,a,b,c,d afixele punctelor M,A,B,C,D. Exprimand diagonalele in functie de afixele varfurilor, avem $BD \cdot AC = (|m-a|+|m-c|)(|m-b|+|m-d|)$.
Tinand cont de inegalitatea modulelor obtinem:
$(|m-a|+|m-c|)(|m-b|+|m-d|) \geq 4|m-\frac{a+c}{2}||m-\frac{b+d}{2}|$, de unde deducem ca
$BD \cdot AC \geq 4MO_1 \cdot MO_2$
Inmultind relatia cu $\sin\alpha$ (α fiind unghiul dintre diagonale) obtinem inegalitatea din enunt.

7. *Fie ABCD, O_1, O_2 mijloacele diagonalelor AC si BD , un patrulater inscris înr-un cerc de rază R şi r raza cercului circumscris triunghiului O_1MO_2. Să se arate că R>2r.*

Soluţie:

Fie a,b,c,d afixele varfurilor A,B,C,D, atunci $|a-b||b-c||c-a|+|c-d||d-a||a-c|=4RS_{ABCD}$ de unde rezultă $2|c-a||d-b||\frac{b+d}{2}-\frac{a+c}{2}| \leq 4RS_{ABCD}$ si deci $AB \cdot CD \cdot OO_1 \leq 2RS_{ABCD}$
Ţinând cont că $S_{ABCD} = \frac{AC \cdot BD \cdot \sin\alpha}{2}, unde\, \alpha = m(\angle O_1, \mathrm{MO}_2)$
$obţinem\, 2r \leq R$

8.*Fie ABCD un paralelogram şi M un punct în planul său. Să se arate că*
$MA \cdot MC + MB \cdot MD \geq AB \cdot BC$.

Soluţie:

Fie *a, b, c, d* afixele vârfurilor *A, B, C, D* faţă de un reper arbitrar, *a* + *c* = *b* + *d*.

Avem: $MA \cdot MC + MB \cdot MD = |m-a| \cdot |m-c| + |m-b| \cdot |m-d| \geq |(m-a)(m-c)-(m-b)(m-d)| = |ac-bd| = |a-b| \cdot |c-b| = AB \cdot BC$.

9. *Dacă ABC şi MNP sunt două triunghiuri echilaterale din acelaşi plan la fel orientate, să se arate că se poate forma un triunghi cu segmentele AM, BN, CP.*

Soluţie::

$$\Delta ABC \sim \Delta MNP \Leftrightarrow \frac{z_A - z_B}{z_M - z_N} = \frac{z_A - z_C}{z_M - z_P} \Leftrightarrow (z_A - z_B)(z_M - z_P) = (z_M - z_N)(z_A - z_C) \Leftrightarrow$$

$z_M(z_C - z_B) + z_N(z_A - z_C) + z_P(z_B - z_A) = o$. Cum
$z_A(z_C - z_B) + z_B(z_A - z_C) + z_C(z_B - z_A) = o$, prin scăderea

celor două egalităţi $\Rightarrow$

$$(z_M - z_A)(z_C - z_B) + (z_N - z_B)(z_A - z_C) + (z_P - z_C)(z_B - z_A) = o \quad (*)$$

iar prin trecere la modul $\Rightarrow$

$$|(z_M - z_A)| \cdot |(z_C - z_B)| \leq |(z_N - z_B)| \cdot |(z_A - z_C)| + |(z_P - z_C)| \cdot |(z_B - z_A)| \Leftrightarrow$$

$AM \cdot BC \leq BN \cdot AC + CP \cdot AB$. Triunghiul ABC este echilateral (AB=AC=BC) $\Rightarrow$ $AM \leq BN + CP$. Din relaţia (*) pot fi scrise şi celelalte două inegalităţi ceea ce implică faptul că AM, BN şi CP pot fi laturile unui triunghi.

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