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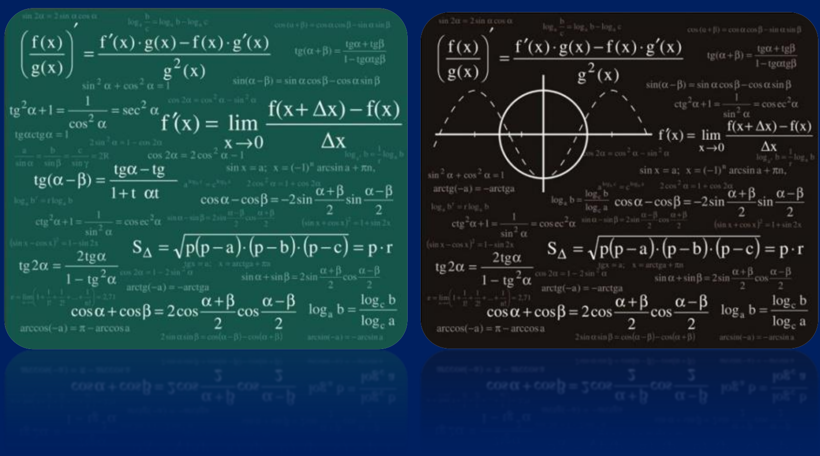

COORDONATOR: ANDREI OCTAVIAN DOBRE

REDACTORI PRINCIPALI ŞI SUSŢINĂTOR PERMANENŢI AI REVISTEI

NECULAI STANCIU, ROXANA MIHAELA STANCIU ŞI NELA CICEU

Articole:

1. Other problems from the Octogon Mathematical Magazine

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PP. 21959. If $a_k > 0$ $(k = 1,2,..., n)$, then $\sum_{cyclic} \frac{a_1 a_2}{a_1 + a_2}\left(\sqrt{\frac{a_1}{a_2}} + \sqrt{\frac{a_2}{a_1}}\right)^2 \geq 2\sum_{k=1}^{n} a_k$.

Solution. Because $\left(\sqrt{\frac{a_1}{a_2}} + \sqrt{\frac{a_2}{a_1}}\right)^2 = \frac{(a_1 + a_2)^2}{a_1 a_2}$ we obtain

$$\sum_{cyclic} \frac{a_1 a_2}{a_1 + a_2}\left(\sqrt{\frac{a_1}{a_2}} + \sqrt{\frac{a_2}{a_1}}\right)^2 = \sum_{cyclic}(a_1 + a_2) = 2\sum_{k=1}^{n} a_k$$, and we are done.

PP. 21960. If $a_k > 0$ $(k = 1,2,..., n)$, then $\sum_{cyclic} \frac{1}{a_1 + a_2}\left(\sqrt{\frac{a_1}{a_2}} + \sqrt{\frac{a_2}{a_1}}\right)^2 \geq 2\sum_{k=1}^{n} \frac{1}{a_k}$.

Solution. We have

$$\sum_{cyclic} \frac{1}{a_1 + a_2}\left(\sqrt{\frac{a_1}{a_2}} + \sqrt{\frac{a_2}{a_1}}\right)^2 = \sum_{cyclic} \frac{1}{a_1 + a_2} \cdot \frac{(a + a_2)^2}{a_1 a_2} =$$

$$= \sum_{cyclic} \frac{a_1 + a_2}{a_1 a_2} = \sum_{cyclic}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) = 2\sum_{k=1}^{n} \frac{1}{a_k}$$, and we are done.

PP. 21993. If $x, y, z, t > 0$, then $\left(\frac{x^2}{y} + \frac{y^2}{x}\right)\left(\frac{z^2}{t} + \frac{t^2}{z}\right) \geq 2(xz + yt)$.

Solution. By symmetry we can assume $x \geq y$ and $z \leq t$.

Since $\frac{x^2}{y}+\frac{y^2}{x} \geq \frac{(x+y)^2}{x+y}=x+y$ and $\frac{z^2}{t}+\frac{t^2}{z} \geq z+t$ it suffices to prove that

$(x+y)(z+t) \geq 2(xz+yt) \Leftrightarrow xt-xz+yz-yt \geq 0$

$\Leftrightarrow x(t-z)-y(t-z) \geq 0 \Leftrightarrow (x-y)(t-z) \geq 0$, and we are done.

PP. 22007. If $a,b,c>0$, then $\sum \frac{ac(b+c)}{\sqrt{(a^2+b^2)(a^2+c^2)}} \leq a+b+c$.

Solution. Applying C-B-S inequality we have

$$(a^2+b^2)(c^2+a^2) \geq (ac+ab)^2.$$

Then

$$\sum \frac{ac(b+c)}{\sqrt{(a^2+b^2)(a^2+c^2)}} \leq \sum \frac{ac(b+c)}{a(b+c)} = \sum a = a+b+c.$$

The proof is complete.

PP. 22059. If $x,y,z>0$, then $\left(\sum x^2\right)^2 + xyz\sum x \geq \left(\sum xy\right)^2 + \sum x^2y^2$.

Solution. The given inequality is written successively

$\left(\sum x^2\right)^2 + xyz\sum x \geq \left(\sum xy\right)^2 + \sum x^2y^2$

$\Leftrightarrow \sum x^4 + 2\sum x^2y^2 + xyz\sum x \geq \sum x^2y^2 + 2xyz\sum x + \sum x^2y^2$

$\Leftrightarrow \sum x^4 \geq xyz\sum x$, which yields by applying the inequality

$\sum a^2 \geq \sum ab$ for two times (first for x^2, y^2, z^2 and second for xy, yz, zx).

The proof is complete.

PP. 22072. In all triangle ABC holds $\sum \frac{m_b - m_c}{(m_a+m_b)(2\sin^2 A + 2\sin^2 B - \sin^2 C)} \geq 0$.

Solution. We have $2\sin^2 A + 2\sin^2 B - \sin^2 C = \frac{1}{4R^2}(2a^2+2b^2-c^2) = \frac{m_c^2}{R^2}$.

Denoting $x=m_a, y=m_b, z=m_c, x,y,z>0$, the inequality to prove becomes successively

$\frac{y-z}{z^2(x+y)} + \frac{z-x}{x^2(y+z)} + \frac{x-y}{y^2(z+x)} \geq 0$

$\Leftrightarrow x^2y^2(y^2-z^2)(z+x) + y^2z^2(z^2-x^2)(x+y) + z^2x^2(x^2-y^2)(y+z) \geq 0$

$\Leftrightarrow x^2y^4z + x^3y^4 - x^2y^2z^3 - x^3y^2z^2 + xy^2z^4 + y^3z^4 - x^3y^2z^2 - x^2y^3z^2 +$

$+ x^4yz^2 + x^4z^3 - x^2y^3z^2 - x^2y^2z^3 \geq 0$

$\Leftrightarrow x^2z(y^2-xz)^2+xy^2(z^2-xy)^2+yz^2(x^2-yz)^2\geq 0$, true and we are done.

PP. 22082. If $a,b,c>0$, then $\frac{3}{2}+\sum\frac{ab}{a^2+b^2}\geq\frac{\left(\sum a\right)^2}{\sum a^2}$.

Solution. Applying the inequality of Harald Brergström we obtain

$$\frac{3}{2}+\sum\frac{ab}{a^2+b^2}=\sum\left(\frac{1}{2}+\frac{ab}{a^2+b^2}\right)=\frac{1}{2}\sum\frac{(a+b)^2}{a^2+b^2}\geq\frac{1}{2}\cdot\frac{\left(\sum(a+b)\right)^2}{\sum(a^2+b^2)}=$$

$$=\frac{4\left(\sum a\right)^2}{2\cdot 2\sum a^2}=\frac{\left(\sum a\right)^2}{\sum a^2}.$$

The proof is complete.

PP. 22133. If F_n denote the nth Fibonacci number, then

$$\frac{F_{n+2}}{F_{n-1}^2}+\frac{1}{F_n}+\frac{1}{F_{n+1}}\geq\frac{9}{F_{n+2}}\quad\text{for all } n\geq 1.$$

Solution. Since

$$\frac{1}{F_n}+\frac{1}{F_{n+1}}\geq\frac{4}{F_n+F_{n+1}}=\frac{4}{F_{n+2}},\text{ it remains to show that}$$

$$\frac{F_{n+2}}{F_{n-1}^2}+\frac{4}{F_{n+2}}\geq\frac{9}{F_{n+2}}\Leftrightarrow\frac{F_{n+2}}{F_{n-1}^2}\geq\frac{5}{F_{n+2}}\Leftrightarrow F_{n+2}^2\geq 5F_{n-1}^2,\ (1).$$

We prove (1) by mathematical induction

For $n=1$, we have $9>5\cdot 1$, true, for $n=2$ we have $25>5\cdot 1$, true.

We assume that $F_{n+2}^2\geq 5F_{n-1}^2$ and we must to prove that $F_{n+3}^2\geq 5F_n^2$.

We have

$$F_{n+3}^2=(F_{n+2}+F_{n+1})^2=(F_{n+1}+F_n+F_n+F_{n-1})^2=(F_n+F_{n-1}+F_n+F_n+F_{n-1})^2=$$

$$=(3F_n+2F_{n-1})^2\geq 9F_n^2>5F_n^2,\text{ and we are done.}$$

PP. 22137. If $x,y>0$ and $x\neq y$, then

$$\left(\frac{(x+y)xy}{(x-y)^2}+x+y\right)\left(\frac{x+y}{(x-y)^2}+\frac{1}{x}+\frac{1}{y}\right)\geq\frac{81xy}{(x+y)^2}.$$

Solution. After some algebra the given inequality is successively equivalent to

$$(x+y)^2(x^2-xy+y^2)\geq 9xy(x-y)^2\Leftrightarrow(x+y)(x^3+y^3)\geq 9xy(x-y)^2$$

$\Leftrightarrow x^4-8x^3y+18x^2y^2-8xy^3+y^4\geq 0\Leftrightarrow(x^2+y^2-4xy)^2\geq 0$, and we are done.

PP. 22159. In all triangle ABC holds $\sum \frac{(c+\sqrt{ab})^2}{(b+c-a)(a+c)} \leq \frac{2(R+r)}{r}$.

Solution. By C-B-S inequality we obtain $(c+\sqrt{ab})^2 \leq (c+a)(c+b)$.

Because $\sum \frac{a}{s-a} = \frac{2(2R-r)}{r}$, we have

$$\sum \frac{(c+\sqrt{ab})^2}{(b+c-a)(a+c)} \leq \sum \frac{b+c}{b+c-a} = \sum \frac{b+c-a+a}{b+c-a} = 3 + \frac{1}{2}\sum \frac{a}{s-a} =$$

$$= 3 + \frac{2R-r}{r} = \frac{2(R+r)}{r}.$$

PP. 22180. If $a,b,c>0$, then

$$\left(\sum a^2b^2\right)\left(\sum \frac{1}{a^2b^2}\right)\left(\sum a^3\right)\left(\sum \frac{1}{a^3}\right) \geq \left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)\left(\sum a\right)\left(\sum \frac{1}{a}\right).$$

Solution. Using the inequality $x^2+y^2+z^2 \geq xy+yz+zx$ we have

$$\left(\sum a^2b^2\right)\left(\sum \frac{1}{a^2b^2}\right) = 3 + \frac{a^2}{c^2} + \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{a^2}{b^2} \geq$$

$$\geq 3 + \frac{a}{c}\cdot\frac{b}{a} + \frac{b}{a}\cdot\frac{c}{b} + \frac{c}{b}\cdot\frac{a}{c} + \frac{b}{c}\cdot\frac{c}{a} + \frac{c}{a}\cdot\frac{a}{b} + \frac{a}{b}\cdot\frac{b}{c} \geq 3 + \sum \frac{a}{b} + \sum \frac{a}{c} = \left(\sum a\right)\left(\sum \frac{1}{a}\right),\text{ i.e.}$$

$$\left(\sum a^2b^2\right)\left(\sum \frac{1}{a^2b^2}\right) \geq \left(\sum a\right)\left(\sum \frac{1}{a}\right) \qquad (1)$$

Also we have

$$\left(\sum a^3\right)\left(\sum \frac{1}{a^3}\right) = 3 + \sum \frac{a^3+b^3}{c^3} \text{ and } \left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right) = 3 + \sum \frac{a^2+b^2}{c^2}.$$

We can write

$$\sum \frac{a^3+b^3}{c^3} - \sum \frac{a^2+b^2}{c^2} = \sum \frac{a^3-a^2c+b^3-b^2c}{c^3} = \sum \frac{a^2(a-c)}{c^3} + \sum \frac{b^2(b-c)}{c^3} =$$

$$= \sum \frac{a^2(a-c)}{c^3} + \sum \frac{c^2(c-a)}{a^3} = \sum \frac{a^5(a-c)+c^5(c-a)}{a^3c^3} =$$

$$= \sum \frac{(a-c)(a^5-c^5)}{a^3c^3} \geq 0, \text{ which yields that}$$

$$\left(\sum a^3\right)\left(\sum \frac{1}{a^3}\right) \geq \left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right) \qquad (2)$$

Multiplying (1) and (2) we obtain the desired result.

PP. 22188. In all triangle ABC holds $\prod(a+b+2c) \le \frac{2s^3(s^2+r^2+2rR)^2}{27R^2r^2}$.

Solution. By AM-GM inequality we have $\prod(a+b+2c) \le \left(\frac{4(a+b+c)}{3}\right)^3 = \frac{64\cdot 8s^3}{27}$.

Since $s^2+r^2+2Rr = s^2+r^2+4Rr-2Rr = ab+bc+ca-\frac{abc}{2s} =$

$$= \frac{(ab+bc+ca)(a+b+c)-abc}{2s} = \frac{(a+b)(b+c)(c+a)}{2s},$$

we have to prove that

$$\frac{64\cdot 8s^3}{27} \le \frac{2s^3\cdot\frac{(a+b)^2(b+c)^2(c+a)^2}{4s^2}}{27R^2r^2} \Leftrightarrow 64R^2r^2\cdot 4 \le \frac{(a+b)^2(b+c)^2((c+a)^2}{4s^2}$$

$\Leftrightarrow (4Rrs)^2\cdot 64 \le (a+b)^2(b+c)^2(c+a)^2 \Leftrightarrow (a+b)(b+c)(c+a) \ge 8abc$,

i.e. Cesaro's inequality. The proof is complete.

PP. 22193. In all triangle ABC holds $\left(\sum\sqrt[3]{m_a^2m_b}\right)^2 \le 4s^2-3r^2-12Rr$.

Solution. We use AM-GM inequality, the inequality $4m_am_b \le 2c^2+ab$ and well-knows formulas

$$\sum m_a^2 = \frac{3}{4}\sum a^2,\ \sum a^2 = 2(s^2-r^2-4Rr), \sum ab = s^2+r^2+4Rr,$$

we obtain

$$\left(\sum\sqrt[3]{m_a^2m_b}\right)^2 \le \left(\sum\frac{m_a+m_a+m_b}{3}\right)^2 = \left(\sum m_a\right)^2 =$$

$$= \sum m_a^2 + \frac{1}{2}\sum 4m_am_b \le \frac{3}{4}\sum a^2 + \frac{1}{2}\sum(2c^2+ab) =$$

$$= \frac{3}{2}(s^2-r^2-4Rr) + \frac{1}{2}(4s^2-4r^2-16Rr+s^2+r^2+4Rr) =$$

$$= 4s^2-3r^2-12Rr,$$

and we are done.

PP. 22197. In all triangle ABC holds

1) $\sum m_am_b \le \frac{1}{4}(5s^2-3r^2-12Rr)$;

2) $\left(\sum m_a\right)^2 \le 4s^2-3r^2-12Rr$.

Solution. We have the inequality $4m_b m_c \le 2a^2 + bc$.
Indeed

$$4m_b m_c \le 2a^2 + bc \Leftrightarrow 16 \cdot \frac{2a^2+2c^2-b^2}{4} \cdot \frac{2a^2+2b^2-c^2}{4} \le (2a^2+bc)^2$$

$\Leftrightarrow (b-c)^2(b+c+a)(b+c-a) \ge 0$, true.
Hence,

$$\sum m_a m_b \le \frac{1}{4}\sum(2c^2+ab) = \frac{1}{4}\left(4(s^2-r^2-4Rr)+s^2+r^2+4Rr\right) =$$
$$= \frac{1}{4}(5s^2-3r^2-12Rr).$$

2)Since $\sum m_a^2 = \frac{3}{4}\sum a^2$, we obtain

$$\left(\sum m_a\right)^2 = \sum m_a^2 + 2\sum m_a m_b = \frac{3}{2}(s^2-r^2-4Rr) + \frac{1}{2}(5s^2-3r^2-12Rr) =$$
$$= 4s^2-3r^2-12Rr,$$

and the proof is complete.

PP. 22203. In all triangle ABC holds $\sum a(b^2+c^2+4rr_a) \ge 9abc$.

Solution. We denote with F the area of ΔABC and we have

$$\sum a(b^2+c^2+4rr_a) = \sum a\left(b^2+c^2+4\cdot\frac{F^2}{s(s-a)}\right) =$$
$$= \sum a(b^2+c^2+4(s-b)(s-c)) = \sum a(b^2+c^2+a^2-(b-c)^2) =$$
$$= \sum a(a^2+2bc) = \sum a^3 + 6abc \ge 3abc + 6abc = 9abc,$$ and we are done.

PP. 22237. Let K be the symmedian point of triangle ABC. Prove that

$$\sum \frac{AK}{bc} \le \frac{3}{\sqrt{a^2+b^2+c^2}}.$$

Solution. Let s_a, m_a be the symmedian respectively the median from A.
We have

$$s_a = \frac{2bc}{b^2+c^2} m_a \text{ and } \sum m_a^2 = \frac{3}{4}\sum a^2.$$

Let $D = AK \cap BC$.
By Van Aubel theorem we obtain

$$\frac{AK}{KD} = \frac{b^2+c^2}{a^2} \Leftrightarrow \frac{AK}{s_a} = \frac{b^2+c^2}{a^2+b^2+c^2} \Rightarrow$$
$$\Rightarrow AK = \frac{b^2+c^2}{a^2+b^2+c^2} s_a = \frac{2bc}{a^2+b^2+c^2} m_a.$$

Therefore,

$$\sum \frac{AK}{bc} = \frac{2}{a^2+b^2+c^2}\sum m_a,$$

and using the inequality $x+y+z \le \sqrt{3(x^2+y^2+z^2)}$ we obtain

$$\sum \frac{AK}{bc} \le \frac{2}{a^2+b^2+c^2}\sqrt{3\sum m_a^2} =$$

$$= \frac{2}{a^2+b^2+c^2}\cdot\frac{3}{2}\sqrt{a^2+b^2+c^2} = \frac{3}{\sqrt{a^2+b^2+c^2}}.$$

We have equality iff ΔABC is equilateral.

PP. 22261. In all triangle ABC holds $\sum_{cyc}\sin\frac{A}{2} \le \sqrt{2+\frac{8R}{r}}$.

Solution. We shall prove the inequality $\sum_{cyc}\sin\frac{A}{2} \le \sqrt{2+\frac{R}{8r}}$.

Using the well-known inequality $\sum_{cyc}\sin\frac{A}{2} \le \frac{3}{2}$ it suffices to show that

$\frac{9}{4} \le 2+\frac{R}{8r} \Leftrightarrow \frac{1}{4} \le \frac{R}{8r} \Leftrightarrow 2r \le R$, true, and we are done.

PP. 22266. In all triangle ABC holds $\sum\sqrt{\frac{r_a}{s-a}} \le \frac{4R+r}{\sqrt{sr}}$.

Solution. If F is the area of ΔABC we have

$$\sum\sqrt{\frac{r_a}{s-a}} = \sum\sqrt{\frac{F}{(s-a)^2}} = \sqrt{sr}\cdot\sum\frac{1}{s-a} = \sqrt{sr}\cdot\frac{4R+r}{sr} = \frac{4R+r}{\sqrt{sr}},$$

and we are done.

PP. 22273. If $a,b,c>0$, then $\frac{3}{2}\sum a^2(b+c) \le \left(\sum a\right)\left(\sum a^2\right)$.

Solution. The given inequality is written successively

$$3\sum a^2(b+c) \le 2\sum a^3 + 2\sum a^2(b+c) \Leftrightarrow 2\sum a^3 \ge \sum a^2(b+c)$$

$\Leftrightarrow \sum_{sym} a^3 \ge \sum_{sym} a^2 b$, whih is true by Muirhead's inequality (because $[3,0,0] \succ [2,1,0]$), or by AM-GM inequality.

PP. 22300. In all triangle ABC holds $\sum\frac{1}{(r_a+r_b)\sin A\sin B}\geq\frac{9R}{s^2}$.

Solution. Let F be the area of ΔABC. We have

$$r_a+r_b=\frac{F}{s-a}+\frac{F}{s-b}=\frac{cF}{(s-a)(s-b)}.$$

Since

$$\sum(s-a)(s-b)=3s^2-2s\sum ab=3s^2-4s^2+s^2+r^2+4Rr=r(r+4R),$$

we obtain

$$\sum\frac{1}{(r_a+r_b)\sin A\sin B}=\sum\frac{(s-a)(s-b)}{cF}\cdot\frac{4R^2}{ab}=\frac{4R^2}{abcF}\sum(s-a)(s-b)=$$

$$=\frac{4R^2}{4Rrs\cdot rs}\cdot r(r+4R)=\frac{R(r+4R)}{rs^2}.$$

So given inequality is written

$$\frac{R(r+4R)}{rs^2}\geq\frac{9R}{s^2}\Leftrightarrow r+4R\geq 9r\Leftrightarrow R\geq 2r\text{, true.}$$

We have equality iff ΔABC is equilateral.
The proof is complete.

PP. 22301. In all triangle ABC holds

1) $\sum\frac{(r_a+r_b)(r_b+r_c)}{ac}=1+\frac{4R}{r}$

2) $\sum\frac{r_a+r_b}{c}=\frac{s}{r}$.

Solution. We have $r_a+r_b=\frac{F}{s-a}+\frac{F}{s-b}=\frac{cF}{(s-a)(s-b)}$ where F is the area of ΔABC.

Also we use the identities $\sum\frac{1}{s-a}=\frac{4R+r}{sr}$, $\sum\frac{1}{(s-a)(s-b)}=\frac{1}{r^2}$.

1) $\sum\frac{(r_a+r_b)(r_b+r_c)}{ac}=\frac{F^2}{(s-a)(s-b)(s-c)}\sum\frac{1}{s-b}=s\cdot\frac{4R+r}{sr}=1+\frac{4R}{r}$.

2) $\sum\frac{r_a+r_b}{c}=\sum\frac{F}{(s-a)(s-b)}=F\cdot\frac{1}{r^2}=\frac{s}{r}$.

We are done.

PP. 22339. If $x,y,z>0$ then

$$\left(\frac{x}{y}+\sqrt{\frac{y}{z}}+\sqrt[3]{\frac{z}{x}}\right)\left(\frac{y}{z}+\sqrt{\frac{z}{x}}+\sqrt[3]{\frac{x}{y}}\right)\left(\frac{z}{x}+\sqrt{\frac{x}{y}}+\sqrt[3]{\frac{y}{z}}\right)>\frac{27}{28}.$$

Solution. By Hölder inequality we obtain that

$$\left(\frac{x}{y}+\sqrt{\frac{y}{z}}+\sqrt[3]{\frac{z}{x}}\right)\left(\frac{y}{z}+\sqrt{\frac{z}{x}}+\sqrt[3]{\frac{x}{y}}\right)\left(\frac{z}{x}+\sqrt{\frac{x}{y}}+\sqrt[3]{\frac{y}{z}}\right)\geq$$

$$\geq\left(\sqrt[3]{\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}}+\sqrt[3]{\sqrt{\frac{y}{z}\cdot\frac{z}{x}\cdot\frac{x}{y}}}+\sqrt[3]{\sqrt[3]{\frac{z}{x}\cdot\frac{x}{y}\cdot\frac{y}{z}}}\right)^3=27$$, and we are done.

PP. 22343. If $a_k>0\ (k=1,2,...,n)$, then $\sum_{cyclic}\frac{(3a_1+a_2)\sqrt{a_2}}{\sqrt{a_1+a_2}}\leq 2\sqrt{2}\sum_{k=1}^{n}a_k$.

Solution. We have the inequality $\frac{(3x+y)\sqrt{y}}{\sqrt{x+y}}\leq\sqrt{2}(x+y)$, $x,y>0$.

Indeed, we have

$$\frac{(3x+y)\sqrt{y}}{\sqrt{x+y}}\leq\sqrt{2}(x+y)\Leftrightarrow y(3x+y)^2\leq 2(x+y)^3\Leftrightarrow$$

$\Leftrightarrow 9x^2y+6xy^2+y^3\leq 2x^3+6x^2y+6xy^2+2y^3\Leftrightarrow 2x^3+y^3\geq 3x^2y$, which yields by AM-GM inequality $x^3+x^3+y^3\geq 3\sqrt[3]{x^6y^3}=3x^2y$.

We obtain

$$\sum_{cyclic}\frac{(3a_1+a_2)\sqrt{a_2}}{\sqrt{a_1+a_2}}\leq\sum_{cyclic}\sqrt{2}(a_1+a_2)=2\sqrt{2}\sum_{k=1}^{n}a_k$$, and the proof is complete.

PP. 22344. If $x,y>0$, then

1) $2\leq\sqrt{\frac{x+y}{2x}}+\sqrt{\frac{2x}{x+y}}\leq\frac{x+y}{\sqrt{xy}}$.

2) $4\leq\left(\sqrt{\frac{x+y}{2x}}+\sqrt{\frac{2x}{x+y}}\right)\left(\sqrt{\frac{x+y}{2y}}+\sqrt{\frac{2y}{x+y}}\right)\leq\frac{(x+y)^2}{xy}$.

Solution. 1)The inequality $\sqrt{\frac{x+y}{2x}}+\sqrt{\frac{2x}{x+y}}\geq 2$ yields immediately by $AM-GM$ inequality.

By AM-HM inequality we have

$$\sqrt{\frac{2x}{x+y}}=\sqrt{\frac{1}{y}\cdot\frac{2xy}{x+y}}\leq\sqrt{\frac{x+y}{2y}}$$, and then it suffices to prove that

$$\sqrt{\frac{x+y}{2x}}+\sqrt{\frac{x+y}{2y}}\le\frac{x+y}{\sqrt{xy}}\Leftrightarrow\frac{1}{\sqrt{2x}}+\frac{1}{\sqrt{2y}}\le\frac{\sqrt{x+y}}{\sqrt{xy}}\Leftrightarrow\sqrt{x}+\sqrt{y}\le\sqrt{2(x+y)},$$

which is AM-GM inequality for numbers $\sqrt{x}$ and $\sqrt{y}$.

2)By 1) we have also that

$$2\le\sqrt{\frac{x+y}{2y}}+\sqrt{\frac{2y}{x+y}}\le\frac{x+y}{\sqrt{xy}},$$

and by multiplying we obtain the desired inequality.

The proof is complete.

2.CÂTEVA PROBLEME DE LICEU TRATATE METODIC

Profesori : Gheorghe Alexe , Colegiul National "Gheorghe Munteanu Murgoci " , Braila
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1)Se considera polinomul $P(x)=x^3-x-1$.

a)Studiati ireductibilitatea polinomului P in C[x] ,R[x] si Q[x].Descrieti care sunt dificultatile ce le-ar putea avea un elev in rezolvarea acestei probleme.

b)Notam cu a,b si c radacinile polinomului $P\in C[x]$. Sa se calculeze $a^2+b^2+c^2$ si $a^3+b^3+c^3$.

c)Un elev afirma:

Deoarece $a^2+b^2+c^2>0$ rezulta ca toate radacinile lui P sunt reale .

Explicati in ce consta greseala din rationamentul acestui elev . Propuneti o modalitate de actiune (la clasa) pentru corectarea acestei greseli.

2) a) Fie $x\in(-1,1)$. Determinati $\lim_{n\to\infty} nx^n$.

b) Calculati $\lim_{n\to\infty}\int_0^1\frac{x^n}{x^2+1}dx$ si $\lim_{n\to\infty}\int_0^1\frac{nx^n}{x^2+1}dx$. Comentati din punct de vedere metodic rezultatele obtinute .

c) Fie $f:[0,1]\to R$, o functie derivabila cu derivate continua . Utilizand eventual metoda integrarii prin parti aratati ca $\lim_{n\to\infty}\int_0^1 nx^n f(x)dx=f(1)$.

d) Observam ca subpunctul c) este o generalizare a subpunctului b) , construiti un alt exemplu de analiza matematica pentru a pune in evidenta trecerea de la particular la general .

3) a) Sa se demonstreze ca orice triunghi ABC din plan are loc inegalitatea
$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$. Cand se realizeaza cazul de egalitate ?
b) Este adevarat urmatorul rezultat :
Fie ABC un triunghi si P un punct interior triunghiului . Fie $A^{|}$, $B^{|}$ si $C^{|}$ picioarele perpendicularelor din P pe laturile triunghiului . Atunci
$PA^{|}+PB^{|}+PC^{|}\le\frac{1}{2}(PA+PB+PC)$.
Folosind (eventual) rezultatul de mai sus , sa se rezolve urmatoarea problema :
Fie ABC un triunghi si P un punct interior triunghiului . Sa se arate ca cel putin unul dintre unghiurile $\sphericalangle PAB$, $\sphericalangle PAC$ si $\sphericalangle PCB$ are masura mai mica sau egala cu 30^0 .
c) Se considera urmatorul enunt : In orice triunghi medianele sun concurente . Sa se demonstreze in cel putin doua moduri acest enunt (de exemplu sintetic , analitic , vectorial etc) .Sa se explice din punct de vedere metodic care este diferenta dintre tehnicile folosite .Care dintre tehnicile folosite va este mai comoda in predare ? Care credeti ca este mai potrivita , mai utila sau pe placul elevilor ?

Solutii:

1)a) Studiem numarul de radacini reale ale polinomului P(x) cu sirul lui Rolle .

x	$-\infty$	$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	∞
$P^{\vert}(x)$		0	0	
P(x)	$-\infty$	$\frac{9+2\sqrt{3}}{9}$	$\frac{9-2\sqrt{3}}{9}$	∞

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$P^{|}(x)=3x^2-1$, $P^{|}(x)=3x^2-1=0$, $x\in\{-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\}$, $P(\frac{\sqrt{3}}{3})=\frac{9-2\sqrt{3}}{9}$, $P(-\frac{\sqrt{3}}{3})=\frac{9+2\sqrt{3}}{9}$

. Deci polinomul P(x) are o singura radacina reala $x_1\in(-\infty,-\frac{\sqrt{3}}{3})$, $x_2,x_3\in C\setminus R$.

Descompunerea in factori ireductibili a lui P(x) in C[x] este $P(x)=(x-x_1)(x-x_2)(x-x_3)$,
$P(x)=(x-x_1)(x-u-iv)(x-u+iv)=(x-x_1)[(x-u)^2+v^2]$ .Radacini rationale nu admite deoarece
$P(1)=1\neq 0$ si $P(-1)=1\neq 0$. Radacinile rationale la un polinom in Z[x] se cauta printer fractiile $\frac{p}{q}$, p divisor pentru termenul liber , iar q divisor pentru coeficientul termenului de gradul cel mai mare , adica p,q| 1, $p,q\in\{-1,1\}$, am aratat ca -1 si 1 nu sunt radacini ale lui P . Deci polinomul P(x) este ireductibil in Q[x] si reductibil in R[x] si C[x] .Elevul ar putea intimpina dificultati in problema stabilirea numarului de radacini reale ale lui f (aplicarea sirului lui Rolle).Unii pot gresi afirmand ca radacinile a, b si c sunt reale deoarece $a^2+b^2+c^2=2>0$.

b) Aplic relatiile lui Viete $a+b+c=0$, $ab+bc+ca=-1$ si $abc=-1$.
Calculez $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ac)=0+2=2$. Punem conditia ca a , b si sunt radacini ale polinomului P(x) . Adica $a^3-a+1=0$, $b^3-b+1=0$, $c^3-c+1=0$. Le adun si obtin
$a^3+b^3+c^3-(a+b+c)+3=0$, $a^3+b^3+c^3=-3$.

c) Avem exemplul $a=3$, $b=i$ si $c=-i$ atunci $a^2+b^2+c^2=9-1-1=7>0$. Avem o radacina reala si doua complexe conjugate . Exista si numere pur complexe pentru care suma patratelor lor sa fie pozitiva . Daca am fi avut $a^2+b^2+c^2<0$ atunci ar fi rezultat ca a , b si c nu sunt toate reale (adica una este reala si celelalte sunt complexe conjugate) deoarece daca a,b si c sunt reale rezulta ca $a^2+b^2+c^2\geq 0$ .Profesorul propune elevilor sa gaseasca si alte exemple .

2) a) Avem cazul de nedeterminare $\lim_{n\to\infty} nx^n=\infty\cdot 0$. Aplic Criteriul lui Cesaro-Stolz .

$\lim_{n\to\infty} nx^n=\lim_{n\to\infty}\frac{n}{x^{-n}}=\lim_{n\to\infty}\frac{n+1-n}{x^{-n-1}-x^{-n}}=\lim_{n\to\infty}\frac{1}{x^{-n}(x^{-1}-1)}=\lim_{n\to\infty}\frac{x^n}{x^{-1}-1}=\frac{0}{x^{-1}-1}=0$, daca $x\neq 0$.

Daca x=0 atunci $\lim_{n\to\infty} nx^n=0$.

b) Daca $x\in[0,1]$, atunci $0\leq\frac{x^n}{x^2+1}\leq x^n$, rezulta $0\leq\int_0^1\frac{x^n}{x^2+1}dx\leq\int_0^1 x^n dx=\frac{1}{n+1}\to 0$.

$\lim_{n\to\infty}\int_0^1\frac{x^n}{x^2+1}dx=0$.

b) $\int_0^1\frac{nx^n}{x^2+1}dx=\frac{n}{n+1}\int_0^1\frac{(x^{n+1})^{|}}{x^2+1}dx=\frac{n}{n+1}[\frac{1}{2}-\int_0^1 x^{n+1}(\frac{1}{x^2+1})^{|}dx]=\frac{n}{n+1}[\frac{1}{2}+2\int_0^1\frac{x^{n+2}}{(x^2+1)^2}dx]$

Am aplicat integrarea prin parti . Pentru $x\in[0,1]$ avem $0\leq\frac{x^{n+2}}{(x^2+1)^2}\leq x^{n+2}$ deoarece $x^{n+2}[(x^2+1)^2-1]\geq 0$, $(\forall)x\in[0,1]$, $x^{n+2}(x^4+2x^2)\geq 0$, $(\forall)x\in[0,1]$. Integrez de la 0 la 1 in inegaliate si obtin $0\leq\int_0^1\frac{x^{n+2}}{(x^2+1)^2}dx\leq\int_0^1 x^{n+2}dx=\frac{1}{n+3}\to 0$. Deci $\lim_{n\to\infty}\int_0^1\frac{x^{n+2}}{(x^2+1)^2}dx=0$.

$$\lim_{n\to\infty}\int_0^1\frac{nx^n}{x^2+1}dx=\lim_{n\to\infty}\frac{n}{n+1}[\frac{1}{2}+2\int_0^1\frac{x^{n+2}}{(x^2+1)^2}dx]=\frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1}+\lim_{n\to\infty}\frac{2n}{n+1}\cdot\lim_{n\to\infty}\int_0^1\frac{x^{n+2}}{(x^2+1)^2}dx=\frac{1}{2}+2\cdot 0=\frac{1}{2}$$

c)

$$\int_0^1 nx^n f(x)dx=\frac{n}{n+1}\int_0^1 f(x)\cdot(x^{n+1})^{|}dx=\frac{n}{n+1}[f(1)-\int_0^1 f^{|}(x)\cdot x^{n+1}dx]=\frac{nf(1)}{n+1}-\frac{n}{n+1}\int_0^1 f^{|}(x)x^{n+1}dx$$

$f^{|}:[0,1]\to\mathrm{R}$, este continua , dupa teorema lui Waistrass este marginita si isi atinge marginile pe [0,1] rezulta ca exista m si M numere reale cu $m\leq f^{|}(x)\leq M$, $mx^{n+1}\leq f^{|}(x)x^{n+1}\leq Mx^{n+1}$,

$0 \leftarrow \frac{m}{n+2} = \int_0^1 mx^{n+1}dx \le \int_0^1 f^{|}(x)x^{n+1}dx \le \int_0^1 Mx^{n+1}dx = \frac{M}{n+2} \to 0$. Deci $\lim_{n\to\infty}\int_0^1 f^{|}(x)x^{n+1}dx = 0$.

$\lim_{n\to\infty}\int_0^1 nx^n f(x)dx = \lim_{n\to\infty}\frac{nf(1)}{n+1} - \lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\int_0^1 f^{|}(x)x^{n+1}dx = f(1) - 1\cdot 0 = f(1)$

d) $\lim_{n\to\infty}\int_0^1 x^n f(x)dx = \lim_{n\to\infty}\frac{1}{n+1}\int_0^1 (x^{n+1})^{|} f(x)dx = \lim_{n\to\infty}\frac{1}{n+1}[f(1) - \int_0^1 x^{n+1}f^{|}(x)dx]$

$\lim_{n\to\infty}\int_0^1 x^n f(x)dx = \lim_{n\to\infty}\frac{f(1)}{n+1} - \lim_{n\to\infty}\frac{1}{n+1}\int_0^1 x^{n+1}f^{|}(x)dx = 0 - 0 = 0$ este o generalizare pentru $\lim_{n\to\infty}\int_0^1 \frac{x^n}{x^2+1}dx = 0$ iar $\lim_{n\to\infty}\int_0^1 nx^n f(x)dx = f(1)$ este o generalizare pentru $\lim_{n\to\infty}\int_0^1 \frac{nx^n}{x^2+1}dx$ unde $f(x) = \frac{1}{x^2+1}$ unde , $f:[0,1] \to R$, este o functie derivabila cu derivate continua.

3)a) Metoda 1 : Folosesc formulele $\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$,

$\sin\frac{B}{2} = \sqrt{\frac{(p-a)(p-c)}{ac}}$, $\sin\frac{C}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}}$, deci

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}\sqrt{\frac{(p-a)(p-c)}{ac}}\sqrt{\frac{(p-a)(p-b)}{ab}} = \frac{(p-a)(p-b)(p-c)}{abc}$$

Folosesc inegalitatea mediilor $\sqrt{xy} \le \frac{x+y}{2}$, $xy \le (\frac{x+y}{2})^2$,

$(p-a)(p-b) \le (\frac{p-a+p-b}{2})^2 = (\frac{2p-a-b}{2})^2 = (\frac{a+b+c-a-b}{2})^2 = (\frac{c}{2})^2$ si analoagele

$(p-a)(p-c) \le (\frac{p-a+p-c}{2})^2 = (\frac{2p-a-c}{2})^2 = (\frac{a+b+c-a-c}{2})^2 = (\frac{b}{2})^2$

$(p-b)(p-c) \le (\frac{p-b+p-c}{2})^2 = (\frac{2p-b-c}{2})^2 = (\frac{a+b+c-b-c}{2})^2 = (\frac{a}{2})^2$. Le inmultim si rezulta $[(p-a)(p-b)(p-c)]^2 \le (\frac{abc}{8})^2$, $(p-a)(p-b)(p-c) \le \frac{abc}{8}$,

$\frac{(p-a)(p-b)(p-c)}{abc} \le \frac{1}{8}$, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$.

Metoda 2 : f $:[0,\frac{\pi}{2}) \to R$, $f(x) = \ln(\sin x)$, $f^{|}(x) = ctgx$, $f^{||}(x) = \frac{-1}{\sin^2 x} < 0$, $(\forall)x \in [0,\frac{\pi}{2})$

f concave pe $[0,\frac{\pi}{2})$. Aplic inegalitatea lui Jensen $f(\frac{x+y+z}{3}) \ge \frac{f(x)+f(y)+f(z)}{3}$, luam

$x=\frac{A}{2}$, $y=\frac{B}{2}$, $z=\frac{C}{2}$, $f(30^0)=f(\frac{A+B+C}{6})\geq\frac{f(\frac{A}{2})+f(\frac{B}{2})+f(\frac{C}{2})}{3}=\frac{\ln(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})}{3}$

$3\ln(\sin 30^0)=\ln(\frac{1}{2})^3\geq\ln(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})$, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$.

b)Presupun prin reducere la absurd ca $m(\sphericalangle PAB)>30^0$, $m(\sphericalangle PBC)>30^0$, $m(\sphericalangle PAC)>30^0$

$\sin:[0,\frac{\pi}{2}]\to[0,1]$, strict crescatoare pe $[0,\frac{\pi}{2}]$. Unghiurile $\sphericalangle PAB$, $\sphericalangle PBC$, $\sphericalangle PAC$ sunt ascutite , $30^0<m(\sphericalangle PAB)<90^0$, $30^0<m(\sphericalangle PBC)<90^0$, $30^0<m(\sphericalangle PAC)<90^0$.

Rezulta $\sin(\sphericalangle PAB)>\frac{1}{2}$, $\sin(\sphericalangle PBC)>\frac{1}{2}$, $\sin(\sphericalangle PAC)>\frac{1}{2}$, deci $\sin(\sphericalangle PAB)=\frac{PC'}{PA}>\frac{1}{2}$,

$\sin(\sphericalangle PBC)=\frac{PA'}{PB}>\frac{1}{2}$, $\sin(\sphericalangle PAC)=\frac{PB'}{PC}>\frac{1}{2}$, $PA'+PB'+PC'>\frac{1}{2}(PA+PB+PC)$ fals , contrazice teorema lui Erdoss Mordell , data mai sus rezulta ca cel putin unul dintre unghiurile $\sphericalangle PAB$, $\sphericalangle PAC$ si $\sphericalangle PCB$ are masura mai mica sau egala cu 30^0.

c) Metoda 1 (Sintetic 1) : Folosim Reciproca Teoremei lui Ceva : $A'B=A'C$, $B'C=B'A$, $C'A=C'B$,

AA',BB',CC' sunt mediane in triunghiul ΔABC , $\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C'A}{C'B}=1\cdot1\cdot1=1$, rezulta ca

AA',BB',CC' sunt cocurente $AA'\cap BB'\cap CC'=\{G\}$, $\frac{GA}{GA'}=\frac{GB}{GB'}=\frac{GC}{GC'}=2$.

Metoda 2 (Sintetic 2): AA',BB' mediane , $AB'A'B$ trapez , $A'B'\parallel AB$, $A'B'=\frac{AB}{2}$ linie mijlocie in ΔABC. $\Delta GA'B'\approx\Delta GAB$, $\frac{GA'}{GA}=\frac{GB'}{GB}=\frac{B'A'}{BA}=\frac{1}{2}$, $AA'\cap BB'=\{G\}$. Analog

$ACA'C'$ trapez , $A'C'\parallel AC$, $A'C'=\frac{AC}{2}$ linie mijlocie in ΔABC , $\Delta G'A'C'\approx\Delta G'AC$,

$AA'\cap CC'=\{G'\}$, $\frac{G'A'}{G'A}=\frac{G'C'}{G'C}=\frac{C'A'}{CA}=\frac{1}{2}$, $\frac{G'A'}{G'A}=\frac{1}{2}=\frac{GA'}{GA}$, $G,G'\in(AA')$ rezulta $G=G'$, $AA'\cap BB'\cap CC'=\{G\}$, AA',BB',CC' sunt cocurente in G , centrul de greutate al triunghiului ΔABC.

Metoda 3 (cu afixe)

$A(z_A)$, $B(z_B)$, $C(z_C)$, $A'(\frac{z_B+z_C}{2})$, $B'(\frac{z_A+z_C}{2})$, $C'(\frac{z_B+z_A}{2})$, A',B',C' sunt mijloacele laturilor $[BC],[AC],[AB]$. Alegem un punct $G\in(AA')$ astfel ca $\frac{GA}{GA'}=2$,

$z_G=\frac{z_A+2z_{A'}}{1+2}=\frac{z_A+z_B+z_C}{3}$. Punctele B,G,B' sunt coliniare daca $\frac{z_G-z_B}{z_G-z_{B'}}\in R$,

$$\frac{z_G - z_B}{z_G - z_{B^|}} = \frac{\frac{z_A + z_B + z_C}{3} - z_B}{\frac{z_A + z_B + z_C}{3} - \frac{z_A + z_C}{2}} = \frac{2(z_A - 2z_B + z_C)}{-(z_A - 2z_B + z_C)} = -2 \in R$$, deci $B, G, B^|$ sunt coliniare .

Analog punctele $C, G, C^|$ sunt coliniare deoarece $\frac{z_G - z_C}{z_G - z_{C^|}} \in R$,

$$\frac{z_G - z_C}{z_G - z_{C^|}} = \frac{\frac{z_A + z_B + z_C}{3} - z_C}{\frac{z_A + z_B + z_C}{3} - \frac{z_A + z_B}{2}} = \frac{2(z_A - 2z_C + z_B)}{-(z_A - 2z_C + z_B)} = -2 \in R.$$ Rezulta $AA^| \cap BB^| \cap CC^| = \{G\}$,

deci medianele $AA^|, BB^|, CC^|$ sunt cocurente in G .

Metoda 4 (vectorial) :

$\overrightarrow{GB} + \overrightarrow{GC} = 2\overrightarrow{GA^|}$, din conditia $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$, $\overrightarrow{GA} + 2\overrightarrow{GA^|} = \vec{0}$, $\overrightarrow{GA} = 2\overrightarrow{A^|G}$ rezulta ca vectorii $\overrightarrow{GA}, \overrightarrow{A^|G}$ sunt coliniari , deci punctele $A, G, A^|$ sunt coliniare si $AG = 2A^|G$.

$\overrightarrow{GA} + \overrightarrow{GC} = 2\overrightarrow{GB^|}$, din conditia $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$, $\overrightarrow{GB} + 2\overrightarrow{GB^|} = \vec{0}$, $\overrightarrow{GB} = 2\overrightarrow{B^|G}$ rezulta ca vectorii $\overrightarrow{GB}, \overrightarrow{B^|G}$ sunt coliniari , deci punctele $B, G, B^|$ sunt coliniare si $BG = 2B^|G$.

$\overrightarrow{GB} + \overrightarrow{GA} = 2\overrightarrow{GC^|}$, din conditia $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$, $\overrightarrow{GC} + 2\overrightarrow{GC^|} = \vec{0}$, $\overrightarrow{GC} = 2\overrightarrow{C^|G}$ rezulta ca vectorii $\overrightarrow{GC}, \overrightarrow{C^|G}$ sunt coliniari , deci punctele $C, G, C^|$ sunt coliniare si $CG = 2C^|G$.

Rezulta $AA^| \cap BB^| \cap CC^| = \{G\}$, $AA^|, BB^|, CC^|$ centrul de greutate al triunghiului ΔABC , $\frac{GA}{GA^|} = \frac{GB}{GB^|} = \frac{GC}{GC^|} = 2$. La clasele 5-8 se pot folosi metodele 1 si 2 , iar la liceu , la clasa a 9-a metoda vectoriala si teorma lui Ceva iar la clasa a 10-a cea cu afixe .La clasele 5-8 cea mai comoda metoda ar fi cea cu teorema lui Ceva iar la liceu cea cu afixe .